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Structurally Dynamic Cellular Automata

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Abstract. A new kind of cellular automaton (CA) model is introduced in which binary *value-configurations* and the (conventionally quiescent) underlying *topological structure* are dynamically coupled. Topology alterations are defined by local transition rules analogous to the value functions studied in conventional CA models defined on fixed lattices. Preliminary investigation reveals a rich spectrum of evolutionary behavior, including growth, decay, periodicity, and a relaxation to dynamic states with a stable effective dimensionality. The model offers a unique arena in which to study the emergence of true *geometric self-organization* and may provide the simplest approach to modeling the dynamics of the many different varieties of random cellular structures found in nature.

1. Introduction

Cellular automata (CA) are discrete mathematical models for systems built up from a set of locally interacting components [1–4]. Among the many interesting behaviors observed in these models is a dynamic propensity for self-organization, in which highly ordered configurations at large times appear to be insensitive to any initial randomness. The role that the latent order of the underlying lattice geometry itself plays in shaping this globally ordered site-value behavior is presently unclear.

It is with the desire to isolate the intrinsic self-ordering ability of local processing rules in discrete systems that we introduce a *structurally dynamic cellular automaton* (SDCA) model system in which the topological structure of the underlying lattice is dynamically coupled to the local site value configuration. The coupling is defined to treat geometry and value configurations on an approximately equal footing: The structure is altered locally as a function of individual site neighborhood value-states and geometries while local site-connectivity supports site-value evolution precisely as in conventional nearest-neighbor CA-models defined on random lattices.

In addition to providing a natural framework for a CA-like analysis of the *generation*, *transmission*, and *interaction* of topological disturbances in

the lattice, the model allows for the study of true geometric self-organization and not merely geometric analogs such as organized space-time patterns of value-configurations, which clearly owe their existence to a regularity of structure of the background lattice. Immediate applications can be made to simulations of *crystal growth*, *pattern formation* of random cellular structures [5], and neural network models incorporating a *synaptic plasticity* in their dynamics [6]. More ambitiously, SDCA are seen to embody a necessary property of truly self-modifying systems: The events themselves are dynamically coupled to, and therefore continually modify, the spatial arena on which their transformations are defined. In view of the recent proposal that the topology and dimension of a discrete space-time may be self-generated dynamic constructs [7], such systems could prove to be of fundamental physical importance.

2. The model

2.1 Formalism

Conventional CA models are defined on particular lattice-networks, the sites of which are populated with discrete-valued dynamic elements evolving under certain local transition functions. Such a network with N sites is simply a general (undirected) graph G of size N and is completely defined by the $(N \times N)$ connectivity matrix

$$\ell_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are linked;} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Using the graph metric function $D(i,j) = min_{paths}[\#links(i,j;path)]$, we can write a general r-neighborhood CA value-transition rule f in the form

$$v_i^{t+1} = f[\{v_j^t\} \mid j \in S_r^G(i)], \tag{2.2}$$

where $S_r^G(i) = \{ j \mid D(i,j) \leq r\}$ is the radius, r, *graph sphere* about the site, i. Extending the conventional dynamic arena, consisting of $v_i^t \in \{0,1\}$, $i=1,\ldots,N$, to include the *vertex-interconnectivity* ℓ_{ij}^t, our task is to explore evolutionary properties of the more general system.

$$\begin{cases} v^{t+1} &= F_1[\{v^t\},\{\ell^t\}] \\ \ell^{t+1} &= F_2[\{v^t\},\{\ell^t\}], \end{cases} \tag{2.3}$$

in which the changing value states and geometries are explicitly coupled. The complete system at time, t, must now be specified by the state-vector

$$|G\rangle_t \quad = \quad |v_1^t,\ldots,v_N^t; \quad \{\ell_{ij}^t\}\rangle. \tag{2.4}$$

The time-evolution of $|G\rangle$ then proceeds according to the following transition rules: (i) value processors of the general form given above and familiar from CA simulations and (ii) link processors, which can be divided into

site *couplers*, linking previously unconnected vertices and site *decouplers*, which disconnect linked points. Because the topology can be altered only by either a deletion of existing links or an addition of links between pairs of vertices i and j with $D(i,j) = 2$, the dynamics is strictly local.

To be more precise, we first restrict the general value-transition rule F_1 to (maximally symmetric) *totalistic* (T) and *outer-totalistic* (OT) type. Since the underlying lattice is a fully dynamic object, $|G\rangle$ will, in general, tend toward having a complex local geometry with an unspecified local directionality. The most general rules which can therefore be applied are those which are completely invariant under all rotation and reflection symmetry transformations on local neighborhoods. $T(OT)$ value rules are then specified by listing particular *sums* $\{\alpha\}$ (*outer-sums* $\{\alpha_0\}, \{\alpha_1\}$ corresponding to center site values 0 and 1 respectively) for which the value of the center site becomes 1. Formally,

$$v_i^{t+1} = \phi_{\{\alpha\}}(\sum_j l_{ij}^t v_j^t, v_i^t) \tag{2.5}$$

where

$$\phi_{\{\alpha\}}(x,a) = \begin{cases} \sum_\alpha \delta(x+a,\alpha) & \longleftrightarrow T \\ a\sum_{\alpha_1}\delta(x,\alpha_1) + (1-a)\sum_{\alpha_0}\delta(x,\alpha_0) & \longleftrightarrow OT, \end{cases}$$

and $\delta(x,y) = 1$ *if and only if* $x = y$. Note that $\sum_j \ell_{ij}^t v_j^t$ sums the values of all sites j linked to i at time t. The action on the state $|G\rangle$ is represented by

$$\hat{\phi}_{\{\alpha\}}^i |v\rangle_t \;=\; |v_1^t, \ldots, v_i^{t+1} = \phi_{\{\alpha\}}(\sum \ell_{ij}^t v_j^t, v_i^t), \ldots, v_N^t\rangle \tag{2.6}$$

where we distinguish the *operator* $\hat{\phi}^i$ acting on the global value state from the actual local transition *function* ϕ which transforms each site value.

Local geometry altering rules are constructed by direct analogy: For any two selected sites i and j, we restrict attention to site values of vertices contained within a 1-*sphere* of either site, that is, to all $k \in S_1(i,j) = S_1(i) \cup S_1(j)$. Link operators, whose action on the state is represented by

$$\begin{aligned} \text{decouplers}: \quad \hat{\psi}_{\{\beta\}}^{ij} |\ell_{ij}^t\rangle &= |\ell_{11}^t, \ldots, \ell_{ij}^{t+1} = \psi^{ij}, \ldots, \ell_{NN}^t\rangle \\ \text{couplers}: \quad \hat{\omega}_{\{\varepsilon\}}^{ij} |\ell_{ij}^t\rangle &= |\ell_{11}^t \ldots, \ell_{ij}^{t+1} = \omega^{ij}, \ldots, \ell_{NN}^t\rangle, \end{aligned} \tag{2.7}$$

either link or unlink two sites i and j depending on whether the actual sum of values in $S_1(i,j)$ matches any of those given in the $\{\beta\}$ or $\{\varepsilon\}$ lists, which completely define decouplers and couplers respectively.

In order to construct classes of rules analogous to the two types of value-rules defined above, we partition the local neighborhood into three disjoint sets (figure 1): $S_1(i,j) = V_{ij} \cup A_{ij} \cup B_{ij}$, where

$$\begin{aligned} V_{ij} &= \{i,j\}, \\ A_{ij} &= \{k \mid k \in C_1(i) \cap C_1(j)\} \quad \text{where } C_1(i) = S_1(i) - \{i\}, \\ B_{ij} &= S_1(i) \cup S_1(j) - V_{ij} - A_{ij}. \end{aligned} \tag{2.8}$$

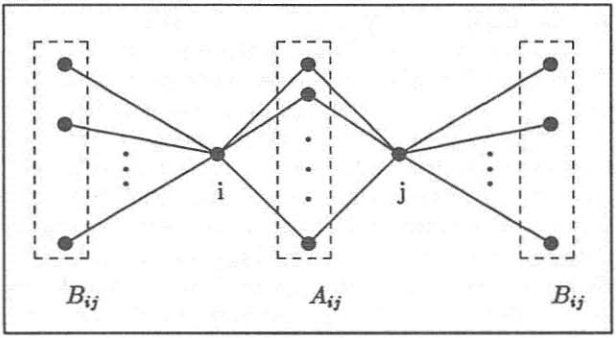

Figure 1: Neighborhood partitioning. In the same way as outer sites can be considered separately for value transitions, we may, for topology transitions, distinguish between those sites belonging to both i and j (A_{ij}) and those belonging to one of the two sites but not both ($\in B_{ij}$). In this way, we obtain the analogous totalistic, outer-totalistic, and an additional type called *restricted totalistic*.

The action of link operators is then conveniently expressed as a function of the sums within the individual partitions. Defining $\nu_{ij} = v_i + v_j$, $a_{ij} = \sum_{k \in A_{ij}} v_k$, and $b_{ij} = \sum_{k \in B_{ij}} v_k$, we get

$$\text{decouplers}: \quad \psi^{\mathrm{ij}}_{\{\beta\}} = \psi^{\mathrm{ij}}_{\{\beta\}}(\nu_{\mathrm{ij}}, \mathrm{a_{ij}}, \mathrm{b_{ij}}) \qquad \text{where,}$$

$$\psi^{ij}_{\{\beta\}}(x,y,z) = \begin{cases} \{1 - \sum_k \delta(x+y+z, \beta_k)\}\ell_{ij} & T \\ \{1 - \sum_k \delta(x, \beta_{1,k})\, \delta(y+z, \beta_{2,k})\}\ell_{ij} & OT \\ \{1 - \sum_k \delta(x, \beta_{1,k})\, \delta(y, \beta_{2,k})\, \delta(z, \beta_{3,k})\}\ell_{ij} & RT, \end{cases} \tag{2.9}$$

$$\text{and couplers}: \quad \omega^{\mathrm{ij}}_{\{\epsilon\}} = \omega^{\mathrm{ij}}_{\{\epsilon\}}(\nu_{\mathrm{ij}}, \mathrm{a_{ij}}, \mathrm{b_{ij}}) \qquad \text{where,}$$

$$\omega^{ij}_{\{\epsilon\}}(x,y,z) = \begin{cases} \delta(D_{ij},2) \sum_k \delta(x+y+z, \epsilon_k) & T \\ \delta(D_{ij},2) \sum_k \delta(x, \varepsilon_{1,k})\, \delta(y+z, \varepsilon_{2,k}) & OT \\ \delta(D_{ij},2) \sum_k \delta(x, \varepsilon_{1,k})\, \delta(y, \varepsilon_{2,k})\, \delta(z, \varepsilon_{3,k}) & RT. \end{cases} \tag{2.10}$$

In the above expressions, RT stands for *restricted totalistic* rules which maximally subdivide the local neighborhood. The inclusion of an ℓ_{ij} in the expressions for ψ assures that only those sites already linked can be decoupled and the $\delta(D_{ij}, 2)$ in the equations defining ω are put in to make sure that only sites separated by distance = 2 may be dynamically coupled.

The various type-specific sums appearing above are indexed with the following conventions: (i) T-rules are defined by the k overall sums of values in $S_1(i,j)$ for which the particular action is to be taken. Example: Define ψ by unlinking i and j if the total sum $= 1(=\beta_1), 3(=\beta_2)$, or $5(=\beta_3.)$ Equation 9 then states that $\ell^{t+1}_{ij} = 0$ if and only if $\ell^n_{ij} = 1$ and $(\nu^n_{ij} = 1,\ a^n_{ij} = 3,\ b^n_{ij} = 5)$. (ii) OT-rules are specified by giving k 2-tuples $(\beta_{1,k},\ \beta_{2,k})$, and $(\varepsilon_{1,k},\ \varepsilon_{2,k})$, where $\{1,k\}$ labels the sum $v_i + v_j$ and $\{2,k\}$ labels the corresponding outer

sum $= \sum_{s \in S_1(i,j)-\{i,j\}} v_s$. (Example: Link i and j if $v_i + v_j = 0$ and outer sum $= \{3, 4\}$, so that ω is defined by listing the two 2-tuples $(\varepsilon_{1,1} = 0, \varepsilon_{2,1} = 3)$ and $(\varepsilon_{1,2} = 0, \varepsilon_{2,2} = 4.)$
(iii) *RT*-rules are completely specified by giving the k 3-tuples of values $(x = v_i + v_j, y =$ sum in A, $z =$ sum in $B)$, for which the link operation between i and j is to be performed. (Example: Define ψ by unlinking i and j for the following values of partitioned sums: $(0,0,1)$, $(0,0,2)$, $(0,1,1)$, $(1,1,1)$; we then have that $(\beta_{1,1}=0,\ \beta_{2,1}=0,\ \beta_{3,1}=1)$, $(\beta_{1,2}=0,\ \beta_{2,2}=0,\ \beta_{3,2}=2)$, $(\beta_{1,3}=0,\ \beta_{2,3}=1,\ \beta_{3,3}=1)$, and $(\beta_{1,4}=0,\ \beta_{2,4}=1,\ \beta_{3,4}=1)$.

Global transition operators are obtained by applying individual value and link operators to all *sites* and *site-pairs* in the graph G:

$$\begin{aligned} \hat{\Phi}_{\{\alpha\}}|v\rangle &= \prod_i \hat{\phi}^i_{\{\alpha\}}|v\rangle, \\ \hat{\Psi}_{\{\beta\}}|\ell\rangle &= \prod_{n\langle ij\rangle} \hat{\psi}^{ij}_{\{\beta\}}|\ell\rangle, \quad \text{and} \\ \hat{\Omega}_{\{\varepsilon\}}|\ell\rangle &= \prod_{nn\langle ij\rangle} \hat{\omega}^{ij}_{\{\varepsilon\}}|\ell\rangle \end{aligned} \tag{2.11}$$

where the products for $\hat{\Psi}$ and $\hat{\Omega}$ need to be taken only over *nearest* and *next-nearest* pairs respectively. Given the full *value-topology* transition rule Γ, defined by

$$|G\rangle_{t+1} = (\hat{\Omega}\hat{\Psi}\hat{\Phi})|G\rangle_t = \Gamma|G\rangle_t, \tag{2.12}$$

the fundamental problem is to understand the generic behavior of accessible graphs, G, emerging from all possible initial structures and value configurations. We emphasize that the lattice fully participates in the dynamics and that, in general, no embedding is implied—it is the abstract connectivity itself whose evolution we are attempting to trace.

2.2 An example

The application of the rather cumbersome expressions defining transition rules is in practice extremely straightforward, as we demonstrate with the following example. Consider a graph G defined as a (3×3) lattice with some distribution of values $v = 1$ at time $= 1$ (see figure 2). We are interested in one global update of the system $|G\rangle_{t=1} \rightarrow |G\rangle_{t=2}$ with rules specified by

$$\begin{aligned} \text{(value)} \quad \Phi_{\{\alpha\}}: \quad \{\alpha\}_{totalistic} &= \{2\}, \\ \text{(topology)} \quad \Psi_{\{\beta\}}: \quad \{\beta\}_{outer-tot.} &= \{\beta_{1,1}=2, \beta_{2,1}=3\}, \\ \Omega_{\{\varepsilon\}}: \quad \{\varepsilon\}_{outer-tot.} &= \{(\varepsilon_{1,1}=1, \varepsilon_{2,1}=4), (\varepsilon_{1,2}=0, \varepsilon_{2,2}=3\}. \end{aligned} \tag{2.13}$$

We evolve the system by systematically sweeping through:

1. All *sites*, setting $v_i = 1$ only at those i for which the sum of the values at i and its neighbors is equal to 2 at $t=1$. By *neighbors* of any point

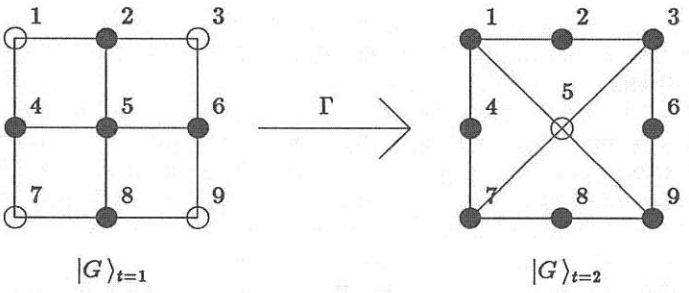

Figure 2: Sample dynamic update of a (3×3) lattice from $t = 1$ to $t = 2$, obeying a T-type value-rule with $v^{t=1} \rightarrow v^{t=2} = 1$ for local sum $\alpha = 2$, and OT-type link rules: (i) *link* for $\{\varepsilon_{1,1} = 1,\ \varepsilon_{2,1} = 4\}$ and $\{\varepsilon_{1,2} = 0,\ \varepsilon_{2,2} = 3\}$, (ii) *unlink* for $\{\beta_{1,1} = 2,\ \beta_{2,1} = 3\}$. *Circled* sites indicate $v = 0$; *Solid* sites have $v = 1$.

i, we will always mean the set of vertices linked to i: $(i,j) = (1,2)$, $(2,5)$, and $(8,9)$, for example, are all neighbors at $t=1$. Writing out a few value-changing terms explicitly, we find that

$$\begin{aligned} v_1^{t=2} &= \phi(v_1^{t=1} + v_2^{t=1} + v_4^{t=1}) = \phi(2) = 1, \quad \text{and} \\ v_5^{t=2} &= \phi(v_5^{t=1} + v_2^{t=1} + v_4^{t=1} + v_6^{t=1} + v_8^{t=1}) = \phi(5) = 0. \end{aligned} \tag{2.14}$$

2. All *linked pairs* of sites i and j, removing those links only if the 2-tuple $(a,b) = (2,3)$, where $a = v_i + v_j$ and b is the sum of values of the neighbors of i and j at $t=1$. For the points $i=2$ and $j=5$, for example, we have $(a,b) = (2,3)$, so that the link ℓ_{25} is no longer present in $|G\rangle_{t=2}$:

$$\begin{aligned} \ell_{25}^{t=2} &= \psi(v_2^{t=1} + v_5^{t=1},\ v_1^{t=1} + v_3^{t=1} + v_4^{t=1} + v_6^{t=1} + v_8^{t=1})\ \ell_{25}^{t=1} \\ &= \psi(2,\ 3)\ (1) \ = \ 0. \end{aligned} \tag{2.15}$$

3. All *next-nearest neighbors* i and j, linking them only if the 2-tuple $(a,b) \in \{(1,4),(0,3)\}$. By *next-nearest neighbor*, we mean those pairs which are themselves unlinked but which share at least one other linked neighbor: $(i,j) = (1,5)$, $(2,8)$ and $(7,9)$, for example, are all next-nearest neighbors at $t=1$. For $i=1$ and $j=5$, we find

$$\begin{aligned} \ell_{15}^{t=2} &= \omega(v_1^{t=1} + v_5^{t=1},\ v_2^{t=1} + v_4^{t=1} + v_6^{t=1} + v_8^{t=1})\ \delta(D_{15}, 2) \\ &= \omega(1,\ 4)\ (1) \ = \ 1. \end{aligned} \tag{2.16}$$

Notice that although $\ell_{79}^{t=1} = 0 \rightarrow \ell_{79}^{t=2} = 1$, it is hidden by overlap with the remaining links $\ell_{78}^{t=2} = 1$ and $\ell_{89}^{t=2} = 1$. For this reason, not all link changes can always be observed directly in the following figures.

Other sites and links are updated in precisely the same manner. Had the link-rules been of totalistic type, only one sum would have to be considered: the sum of the values of the points in question along with their neighbors' values. Had they been, instead, of restricted totalistic type, three sums would have to be considered: the sum of the values of the sites in question, the sum of the values of their common neighbors (neighborhood A in figure 1), and the sum of the values of the points that are neighbors of one of the considered points, but not of the other (neighborhood B in figure 1). The final state $|G\rangle_{t=2}$ emerges after the above process has been applied concurrently to all pairs, neighbors, and next-nearest neighbors in $|G\rangle_{t=1}$.

2.3 Comments

(a) As defined above, Γ consists of three operators acting simultaneously on the state $|G\rangle$. More generally, one may prescribe any of ten possible time-orderings to the operators Ω, Ψ, and Φ. That is, one may specify certain *intermediate* state dependencies, so that, for example, $\Gamma_1|G\rangle \equiv (\Omega\Psi)(\Phi|G\rangle)$ would in general be expected to yield results different from, say, $\Gamma_2|G\rangle \equiv \Omega(\Phi(\Psi|G\rangle))$. While this paper solely concerns the *synchronous* time ordering defined above, we do not expect the qualitative results to depend critically on this choice. Particular behaviors will be left to the sequel.

(b) A given rule Γ is completely defined by the set of sums $\{\alpha\}$, $\{\beta\}$, and $\{\varepsilon\}$. Alternatively, by generalizing Wolfram's encoding scheme for value rules [8], we can conveniently summarize a chosen transition rule by a vector-code

$$\vec{C} = (c[\phi],\ c[\psi],\ c[\omega])_{a,b}, \qquad \text{where} \tag{2.17}$$

$$c[\phi] = \begin{cases} \sum_\alpha 2^\alpha & \leftrightarrow T \\ \sum_{\alpha_0} 2^{2\alpha_0} + \sum_{\alpha_1} 2^{(2\alpha_1+1)} & \leftrightarrow OT, \end{cases} \tag{2.18}$$

$$c[\psi] = \begin{cases} \sum_k 2^{\beta_k} & \leftrightarrow T \\ \sum_k 2^{3\beta_{2,k}+\beta_{1,k}} & \leftrightarrow OT \\ \sum_k 2^{3(\beta_{2,k}+a\beta_{3,k})+\beta_{1,k}} & \leftrightarrow RT, \quad \text{and} \end{cases} \tag{2.19}$$

$$c[\omega] = \begin{cases} \sum_k 2^{\varepsilon_k} & \leftrightarrow T \\ \sum_k 2^{3\varepsilon_{2,k}+\varepsilon_{1,k}} & \leftrightarrow OT \\ \sum_k 2^{3(\varepsilon_{2,k}+b\varepsilon_{3,k})+\varepsilon_{1,k}} & \leftrightarrow RT, \end{cases} \tag{2.20}$$

where $a = max\{\beta_{2,k}\}+1$, $b = max\{\varepsilon_{2,k}\}+1$, and must be specified only for RT-type topology rules. The Γ in section (2.2) above, for example, can be summarized by $c[\phi] = 2^2 = 4$, $c[\psi] = 2^{3(3)+2} = 2048$, and $c[\omega] = 2^{3(4)+1} + 2^{3(3)} = 8704$. Note that Ψ and Ω are chosen always to be of the same type.

(c) Computer simulations of these systems require that some measures be taken to prevent possible memory overflows, such as would happen

Rule type	ϕ	ψ	ω
T	2^{d+1}	2^{2d}	2^{2d-1}
OT	2^{2d+2}	$2^{6(d-1)}$	$2^{3(2d-3)}$
RT	—	$2^{3(a+1)(2d-1)}$	$2^{3(a+1)(2d+1)}$

Table 1: Numbers of possible rules for each of the three types of transition rules. d = maximum allowable degree and a = maximum sum to be used from partition A_{ij}. Example: For $d = 5$, we have $N_\phi = 4096$, $N_\psi = 2^{24} \sim 2 \times 10^7$ and $N_\omega = 2^{21} \sim 2 \times 10^6$. We thus have $N_T = N_\phi N_\psi N_\omega \sim 10^{17}$ possible outer-totalistic type Γs.

in cases either of *pure coupling*, where links are continually added and none deleted, or in isolated regions of a graph where for a few sites more neighbors are added than are allowed by memory. We thus introduce *working* link transition rules

$$\tilde{\psi}^{ij} \equiv \begin{cases} \psi^{ij} & \longleftrightarrow d_i' \text{ or } d_j' > \delta \equiv d_{min} \\ 1 & \longleftrightarrow \text{else,} \end{cases} \tag{2.21}$$

$$\tilde{\omega}^{ij} \equiv \begin{cases} \omega^{ij} & \longleftrightarrow d_i' \text{ or } d_j' < \Delta \equiv d_{max} \\ 0 & \longleftrightarrow \text{else,} \end{cases} \tag{2.22}$$

where $d_i = degree(i)$ (i.e. number of neighbors of i). In other words, make a sweep of the lattice, temporarily storing the candidates to add and delete for each point. If, for any point i, the updated degree is greater than δ, then proceed with deleting the stored deletion-candidates; otherwise, do not delete. Similarly, provided that the updated degree is less than Δ, proceed with addition. Thus, it is sufficient that one of two points allow a dynamic link change between them for that change to be enacted. In the following, the complete constrained dynamics will be quoted as $\vec{C}^{[\delta,\Delta]}_{(a,b)}$. If constraints play no role in the actual evolution of specific examples, they will be left out of the definition.

(d) Because each dynamic update involves three separate types of processing, the number of possible rules is extraordinarily large (see table 1). Unlike pure value transitions, however, the fraction of the total number which yields interesting behavior (i.e., neither immediately explosive, where the number of links increases without bound, nor immediately degenerative, where an initial graph rapidly dwindles to a few isolated links) appears to be manageably smaller.

(e) We point out that, though it is the intrinsic geometrical patterning whose generic behavioral properties we are trying to deduce, one may approach these systems from an alternative point of view: Maintain the emphasis on unraveling the value configurational behavior, and interpret the presence of $[\Psi, \Omega]$ as background operators inducing non-local spatial connectivities. Whereas the systems defined above are

completely abstract entities, in that locality strictly defined by the link structure, the alternative scheme would be to embed the discrete networks in some specified manifold, and to study the effects of dynamically allocated non-local communication channels.

3. Emerging patterns and behaviors

We first consider patterns emerging from simple value seeds starting from ordered two-dimensional Euclidean lattices.[1] A *single non-zero* site may represent a small local disturbance which then propagates outward, restructuring the lattice. With appropriately chosen Γs, one can induce a rich spectrum of different time-evolution behaviors. These range from simple value evolutions only slightly perturbed by very few concurrent link changes to ones in which the initial geometry becomes radically altered.

Figure 3 shows the first five iterations of a system starting from a 4-neighbor lattice with a single non-zero site at its center. The link structure is given explicitly and the solid circles represent sites with $v = 1$. Notice how the link additions *follow* the emerging corrugated boundary surface of the value configuration. Remember that link additions are more than simple markers for appearances of particular local value configurations giving rise to a structural alteration—their presence directly influences all subsequent value development in their immediate vicinity.

Figure 4 (in which values have been suppressed for clarity) shows the continued development of this system. Though boundary effects begin to appear by $t = 25$, the characteristic manner in which this particular Γ restructures the initial graph is clear:

1. there is a high degree of geometrical organization (the symmetry of the initial state is trivially preserved by the totally symmetric Γ),

2. the lattice remains connected,

3. the distribution of link changes made throughout the lattice remains fairly uniform—i.e., there is an approximate uniformity in the probability of appearance of particular local value states which induce a structural change,

4. link-lengths do not get arbitrarily large. Thus, for a system embedded in the plane, communication channels remain approximately local; the global pattern emerges as a consequence of local ordering. Γs for which link-lengths get arbitrarily large are also easy to find.

Some other varieties of behavior are shown in figures 5 and 6. Figures 5a and b are representative of the class of link rules for which the induced

[1] The graphical representation of developing one-dimensional systems, in which link additions must be shown as arcs to avoid overlap with existing links, is needlessly confusing and is not considered.

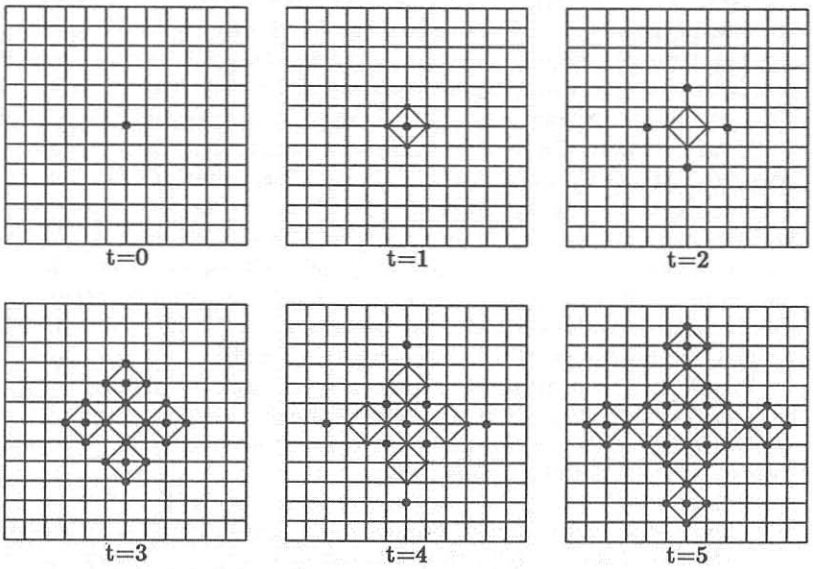

Figure 3: First five iterations of a system starting from a four-neighbor Euclidean lattice seeded with a single non-zero site at the center. The global transition rule Γ consists of *totalistic-value* and *restricted totalistic-topology* rules: $\vec{C} = (26, 69648, 32904)_{[3,3]}$ (defined in section 2.3, comment (b)). Solid sites have $v = 1$.

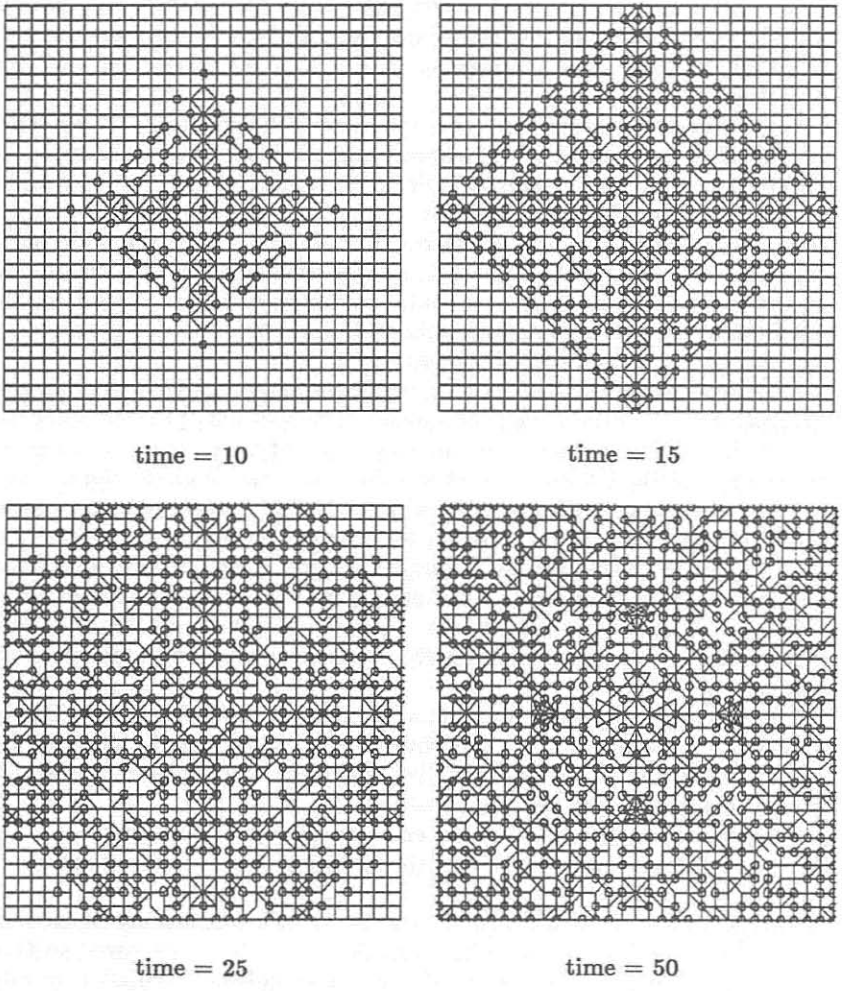

Figure 4: Several further time frames in the structural evolution of the same system whose first five iterations are shown in the preceding figure. The values have here been suppressed for clarity. The boundaries of the original lattice it do not extend beyond the region shown so that the development is strictly confined to a (31×31) graph.

structural change is only minimal: The system evolves essentially as a CA-system on a mildly perturbed underlying lattice. Other rules may have a much stronger effect on the lattice and can significantly alter the manner in which the pure value propagation proceeds in the absence of any link operators.

Figure 5c shows an example of a link rule which accelerates the outward propagation of the value configuration—compare the diameter of this pattern to that in the earlier figures, both shown at equal times. The outwardly oriented links which emerge from sites along the boundary surface become conduits by which non-zero values rapidly propagate. Had the underlying topology been suppressed in this figure and attention focused exclusively on the developing value state, we could have interpreted the result as showing an effective increase in information propagation speed, due to non-local connectivities (see comment (e) in section 2.3).

Figure 5d, on the other hand, gives an example in which the link dynamics lags behind the value development: The boundary proceeds outward essentially unaffected by changes in geometry which are confined to the interior parts of the lattice (at least at this early stage of its development).

Figure 6 shows snapshot views of a few systems undergoing a slightly more complex evolution. Figure 6b, for example, shows the the action of a Γ in which the outward value propagation rapidly deletes most links from the original lattice but leaves a complex—structurally stable—geometry at the origin of the initial disturbance. Figure 6c, on the other hand, gives a typical state of a system whose global connectivity becomes progressively more complicated.

A typical development of an initial state in which all sites are randomly assigned $v = 1$ with $prob = \frac{1}{2}$ is shown in figure 7. Notice the rapid development of complex local connectivity patterns, the appearance of which points to a geometrical self-organization.

In general, structural behaviors emerging from random value states under typical Γs can be grouped into the following representative types:

Type A which are characterized by a decay into structurally much simplified final states in which most links have been destroyed so that the graph G consists essentially of a large number of small local subgraphs.

Type B whose evolution leads to periodic but globally connected geometries. Systems of this type arise either because of specific class-2 Φs remaining unchanged by the coupling to the lattice or class-3 Φs coupling with $\{\Psi, \Omega\}$ in such a way as to induce a lattice structure supporting a periodic state.

Type C which exhibit a prolonged growth in complexity. Defining $\langle d \rangle \equiv$ *average degree* = *average number of neighbors per site* and an *effective dimensionality* $D_{effec} \equiv$ *ratio of number of next-nearest to nearest neighbors*, both increase in value in Type-C systems and appear to be

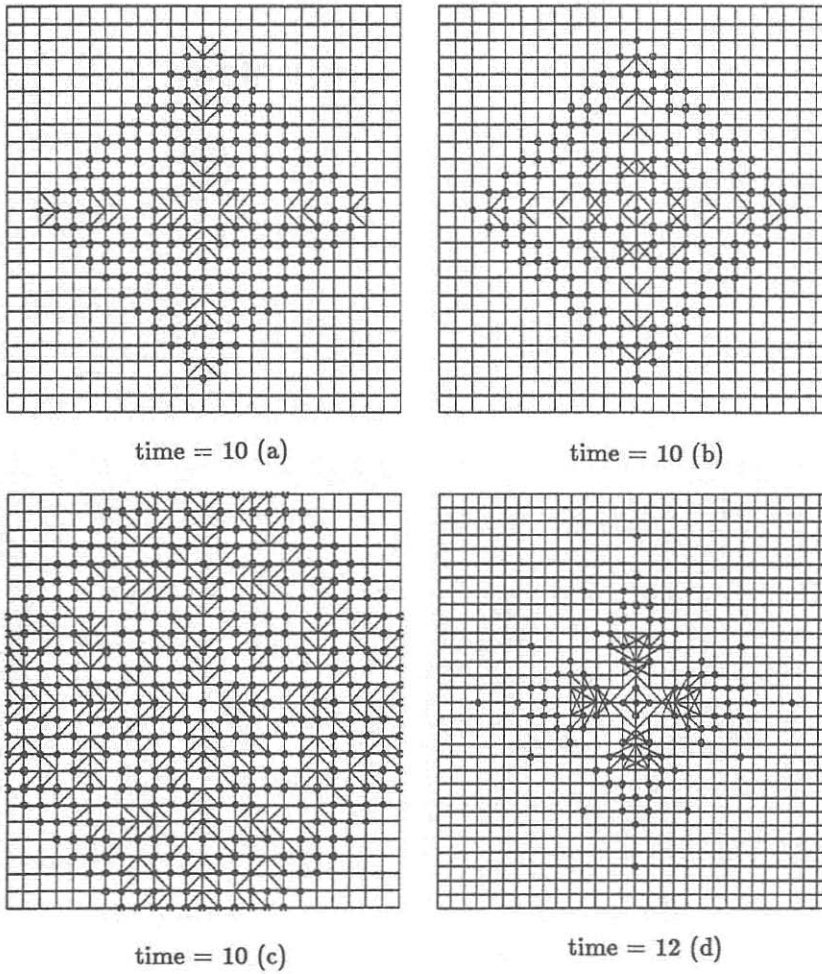

Figure 5: Snapshot views of four typical developing states starting from a single non-zero site at the center of a four-neighbor graph. Γs are defined by the following codes: (a) $OT-c[\phi]=1022$ and $RT-$ *coupler* $c[\omega]=16, b=1$, (b) $T-c[\phi]=22$ and $RT-$ *coupler* $c[\omega]=32, b=2$, (c) $OT-c[\phi]=1022$ and $(RT)-$ *coupler* $c[\omega]=8, b=1$, (d) T-value and (OT)-topology rules: $\vec{C}=(682, 19634061312, 133120)^{[2,8]}$.

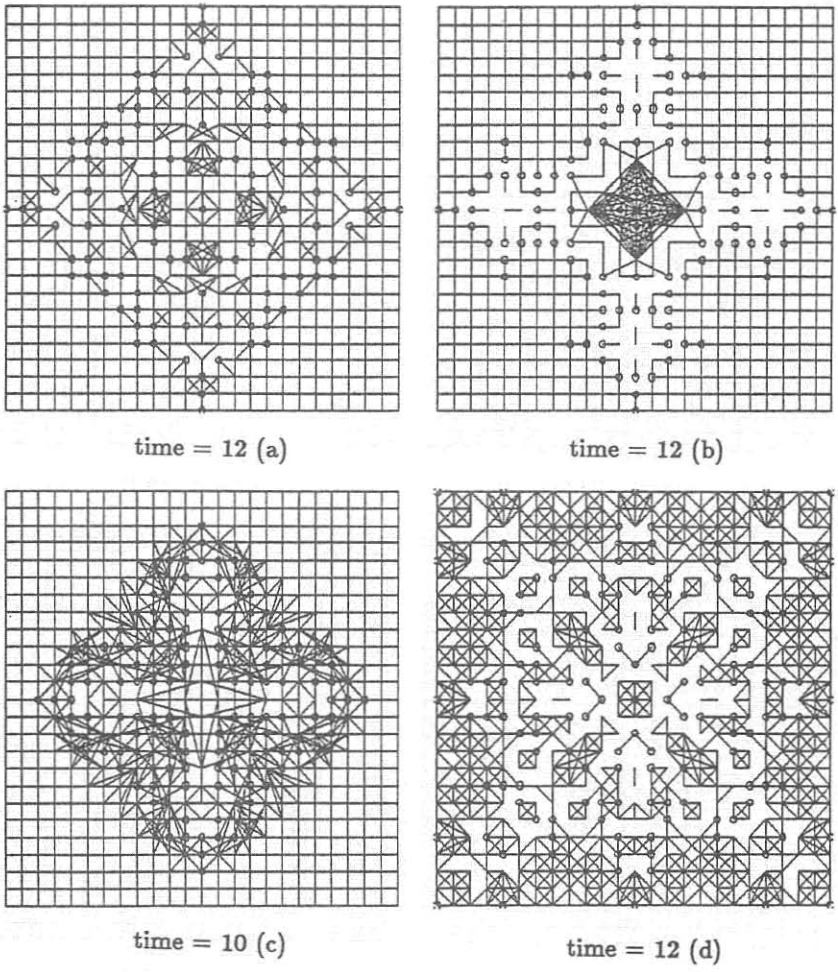

Figure 6: Four more examples of states emerging from simple initial value seeds; (a,b,c) start from four-neighbor graphs and (d) from an 8-neighbor graph (≡ four-neighbor with diagonals). Γs are defined by the following codes: (a) T-value and RT-topology $\vec{C} = (42, 69648, 32904)_{(3,3)}$, (b) T-value and OT-topology $\vec{C} = (42, 589952, 8192)^{[2,8]}$, (c) T-value and topology $\vec{C} = (42, 128, 4)^{[0,10]}$, (d) $T - c[\phi] = 682$ and RT-topology rules defined explicitly by $\Psi_{(104),(114),(124),(103),(113),(123)}$ and $\Omega_{(111),(215)}$.

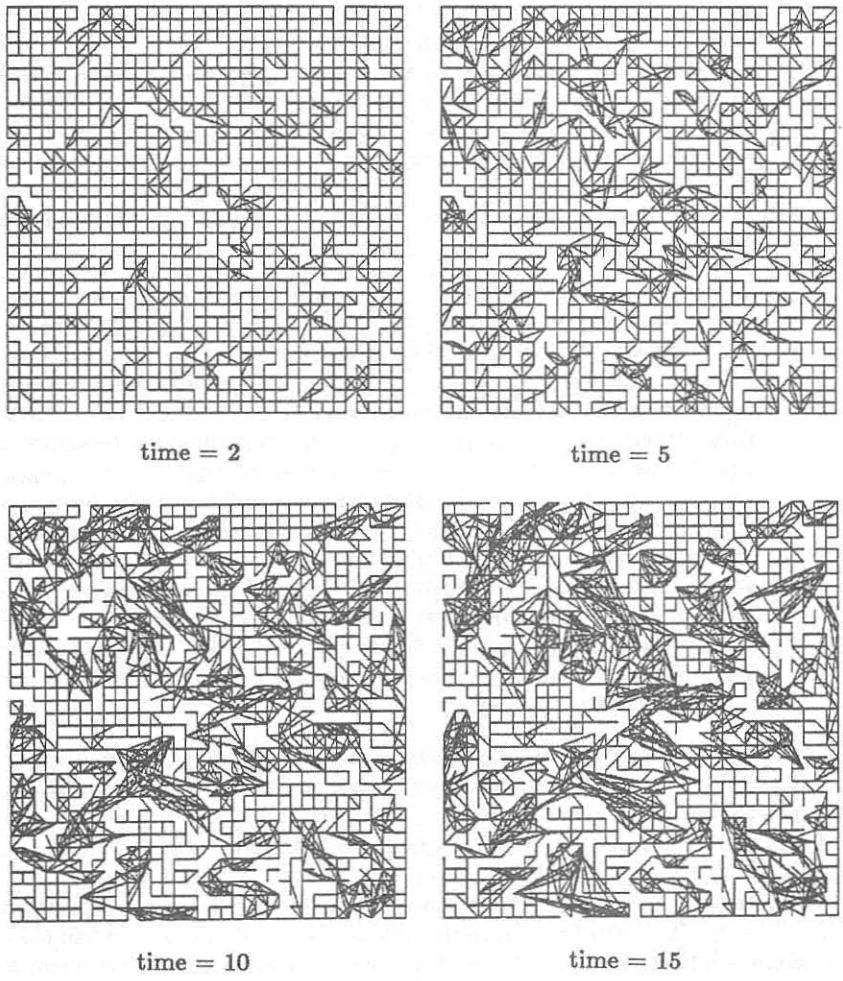

Figure 7: Evolution of a (35×35) lattice whose sites are initially randomly seeded with $v=1$ with $prob=\frac{1}{2}$. The development proceeds according to T-type value rule Φ and OT-type link rules specified by code $\vec{C} = (84, 36864, 2048)$. The constraints are $[\delta = 0, \Delta = 10]$. The appearance of localized substructures is a first sign of geometrical self-organization.

bounded only by imposition of an upper constraint Δ. Because the site-value density responds to the statistically changing local neighborhood structure, it is possible that what at first appears to be such an explosive behavior, in fact, eventually leads to a more sedate, if not entirely static, behavior for some larger $\langle d \rangle \gg \Delta$. Certainly, Φs yielding $\rho^{(v)}$s which remain fairly constant over a large range of $\langle d \rangle$ (such as the '$\oplus_2$'-rule), when coupled with link-rules which themselves become progressively *less* active with increasing $\langle d \rangle$, may induce evolutions leading to only mild changes within specific ranges of the local structural parameters.

Type D which are remarkable rules whose action seems to lead to a sustained global dynamics during which the value of D_{effec} remains almost constant. That is, particular combinations of Ψ and Ω can be found which effectively induce a *structural equilibrium*; though large numbers of link changes continue to be made so that the detailed structure of $|G\rangle$ is changing, the average ratio of the number of next-nearest to nearest neighbors stays approximately unchanged over long periods of time. Simulations have, unfortunately, been run for too short a time and on G containing too few sites to make a conclusive statement. It is conjectured, however, that as $N \to \infty$, there will always exist sufficiently fine-tuned Γs maintaining an average D_{effec} arbitrarily close to a desired value D^*. Type- behavior, of course, can always be artificially induced either by imposing severe $[\delta, \Delta = \delta]$ constraints, or, as is typically the case for Type-C Γs, by impeding growth with any desired Δ.

The evolution of systems starting from random initial value states is generally difficult to follow visually, particularly for Γs inducing many structural changes, and must therefore be studied somewhat indirectly: one charts the time-development with a record of selected statistical measures, a few of which are shown in the succeeding figures.

Examples of each of the four types of system developments are shown in figure 8. The initial structure in each case is a (35×35) four-neighbor Euclidean lattice, so that $D_{effec}^{t=0} \sim 2$. Figure 8a gives an example of Type-A behavior, in which a short period of initial growth is followed by a decay into an essentially disconnected state; the final state is characterized by $\langle d \rangle < 1$ and is stable. Figure 8b shows a system which starts from the same initial state but whose Γ leads to a periodic structural state. Just the right number of links have been deleted to permit regions with isolated activity to emerge. Figure 8c shows Type-C evolution in which D_{effec} steadily increases. The apparent leveling-off at the end is due both to a decreased overall activity level and the increasingly greater felt effects of a $\Delta = 10$ constraint. The system in figure 8d exhibits Type-D behavior, characterized by a continued development within a relatively narrow interval of D_{effec}. The evolution here is almost pure (i.e. unconstrained; see figure 9) and is not an artifact of strictly imposed dynamic conditions.

In figure 9, we show the detailed development of the same Type-D system defined in figure 8d. Notice how D_{effec} remains approximately constant despite the rather large variations in $\langle d \rangle$ in figure 9b. In figure 9d, which shows the fraction of all dynamic link changes which are constraint allowed, we see that the dynamics are close to being pure for the first 100 iterations, with about 90 percent of allowable transitions taking place thereafter. In figure 9c, which gives the total number of link changes occurring at step n, we see the activity level falling off; in fact, it soon settles to a more or less constant $\sim$ 20 changes. The long *valley* seen in figure 9b occurs frequently in evolutions obeying suspected Type-D Γs, and is a manifestation of the local conflict existing between Ψ and Ω.

Figure 10 exhibits the time development of a system whose effective dimensionality maintains an almost constant value, but at the cost of a much reduced purity of dynamics (figure 10c). Γ's tendency to increase local complexity is overwhelmed by the imposed constraint condition specified by $[\delta = 0, \Delta = 8]$. The activity, though stifled, remains at a high level (figure 10b), with the *total number of all link changes* and the *fraction of creations* relative to that total number oscillating within fixed intervals for the duration of the displayed behavior. To more fully understand the properties of this and other evolving geometries, it will become important to carefully define *structural correlation* measures; one can, for example, make a comparison of structural features with those existing in fully random graphs, which can be obtained via a simple mean-field analysis in which all local correlations are ignored [9].

In figure 11, we see the differences in evolution for systems obeying the same Γ but starting from a variety of initial structures. Since structure changes depend primarily on D_{effec}, a qualitative similarity in behavior should not be expected across a broad range of initial graphs. However, as long as most of the initial lattice survives the structural randomization without becoming disconnected, it is empirically found that many systems will then tend towards having similar asymptotic states. Four-neighbor graphs, for example, may evolve toward states with higher degree and/or D_{effec}, while a three-dimensional hypercubic lattice may reduce its dimensionality under the same Γ. More detailed explorations for systems starting from completely random states are presently being conducted.

4. Discussion

This paper has presented a preliminary phenomenological account of a new class of structurally dynamic cellular automata. What distinguishes this model from its precursors is the fact that the dynamics is all-inclusive: No *a priori* fixed spatial dynamic arena is presupposed. Instead, the events are themselves dynamically coupled to, and therefore continually modify, the space on which on their transformations are defined. This conceptual starting point may be of considerable theoretical interest to researchers studying behaviors of truly self-modifying systems. Although the emphasis

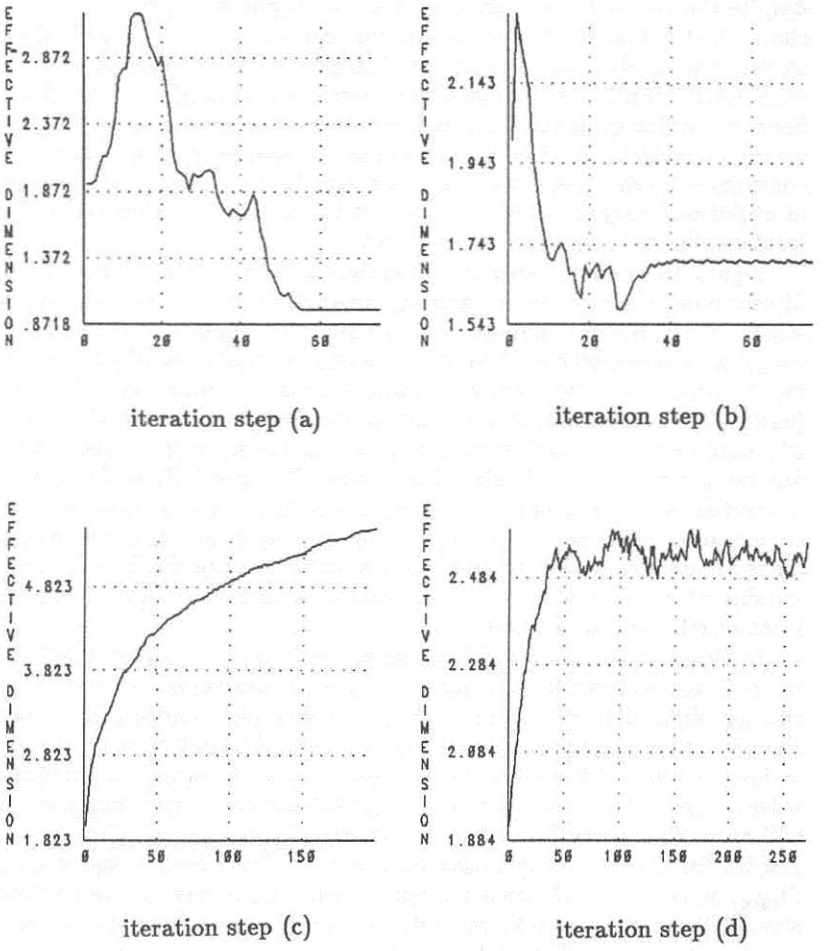

Figure 8: Time development of the *effective dimensionality* D_{effec} for the four general varieties of behavior: (a) T-type Γ defined by $\vec{C} = (42, 128, 4)$, (b) T-type Φ, and OT-type link rules given by $\vec{C} = (64, 9216, 1024)$, (c) T-type Φ and OT-link rules–$\vec{C} = (682, 512, 512)^{[0,10]}$ and (d) T-type Φ and RT-type link rules defined explicitly by $\vec{\beta} \in \{(0,1,1),(1,1,0),(1,2,1),(2,3,3),(2,4,3)\}$ and $\vec{\varepsilon} \in \{(1,2,0),(0,1,0,),(0,2,1),(2,2,4)\}$.

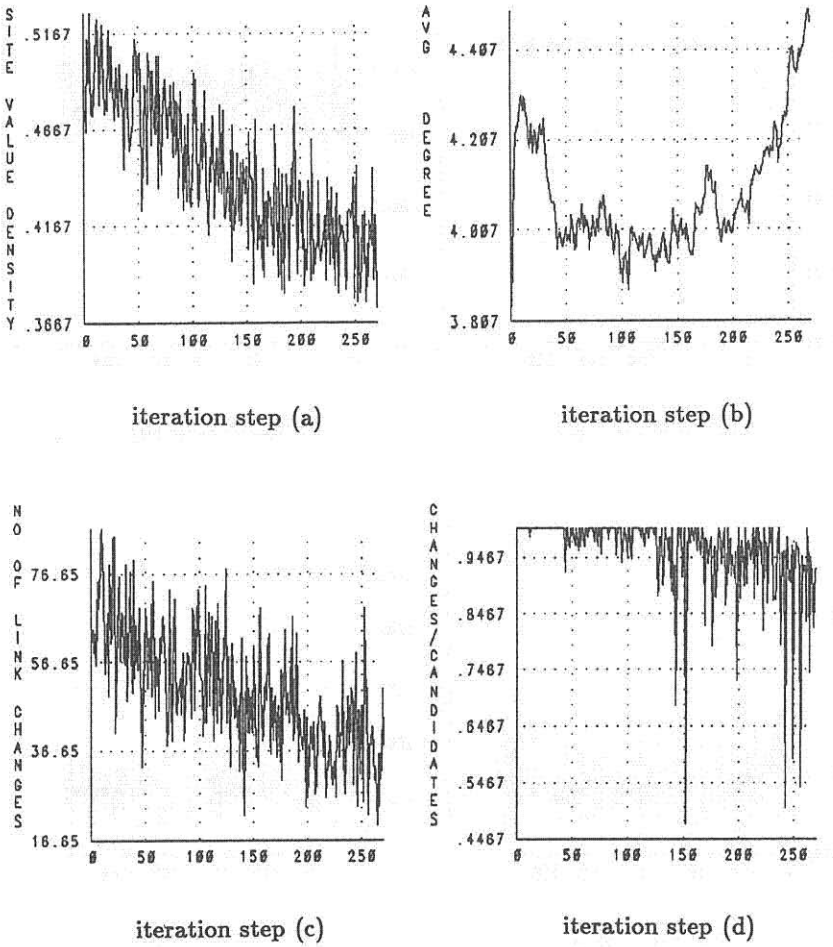

Figure 9: Four dynamic measures giving additional information about the evolution of the same Type-D system defined in figure 8d. Notice how the approximate constancy of D_{effec} seen in the earlier figure occurs despite the strong variation in $\langle d \rangle$. Also, as we get further into the development, both the number of changes and purity of dynamics decreases.

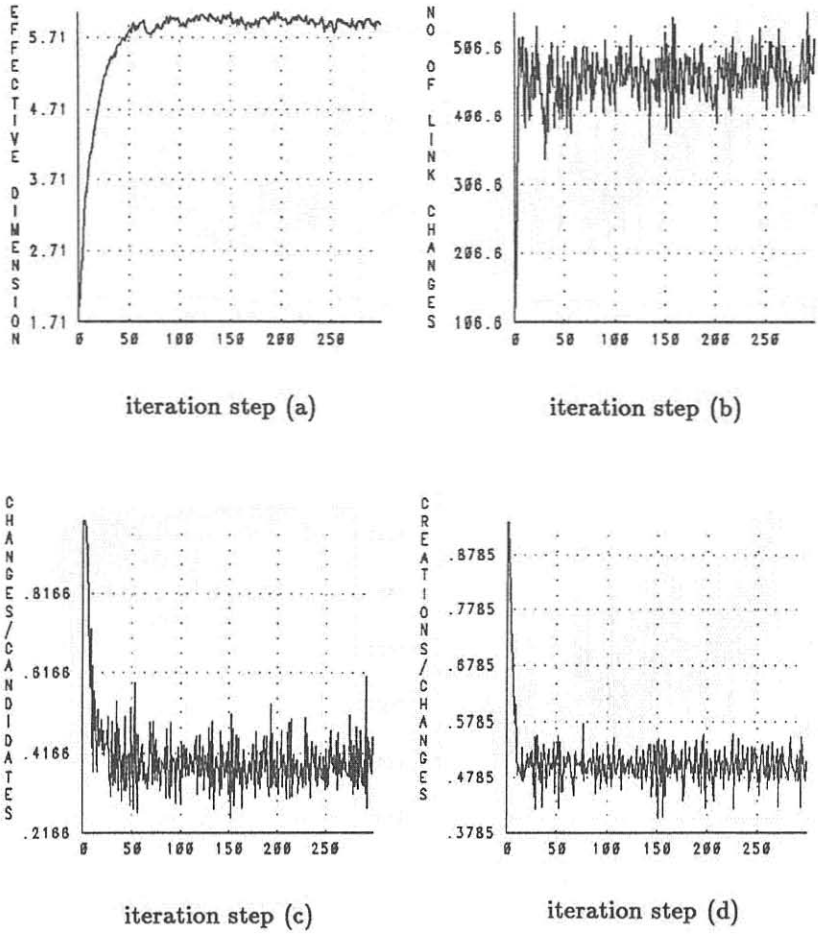

Figure 10: Four measures for a Type-C system evolving according to T-value and OT-link rules defined by $\vec{C} = (42, 524288, 65536)$. The constraint condition plays an important role and is given by $[\delta=0, \Delta=8]$. Although only about 40 percent of all allowable changes actually take place under this constraint (10c), the actual number of link changes (10d) remains at a fairly high level throughout. The fraction of link changes which are *link creations* $\sim \frac{1}{2}$, which is consistent with a stable D_{effec}.

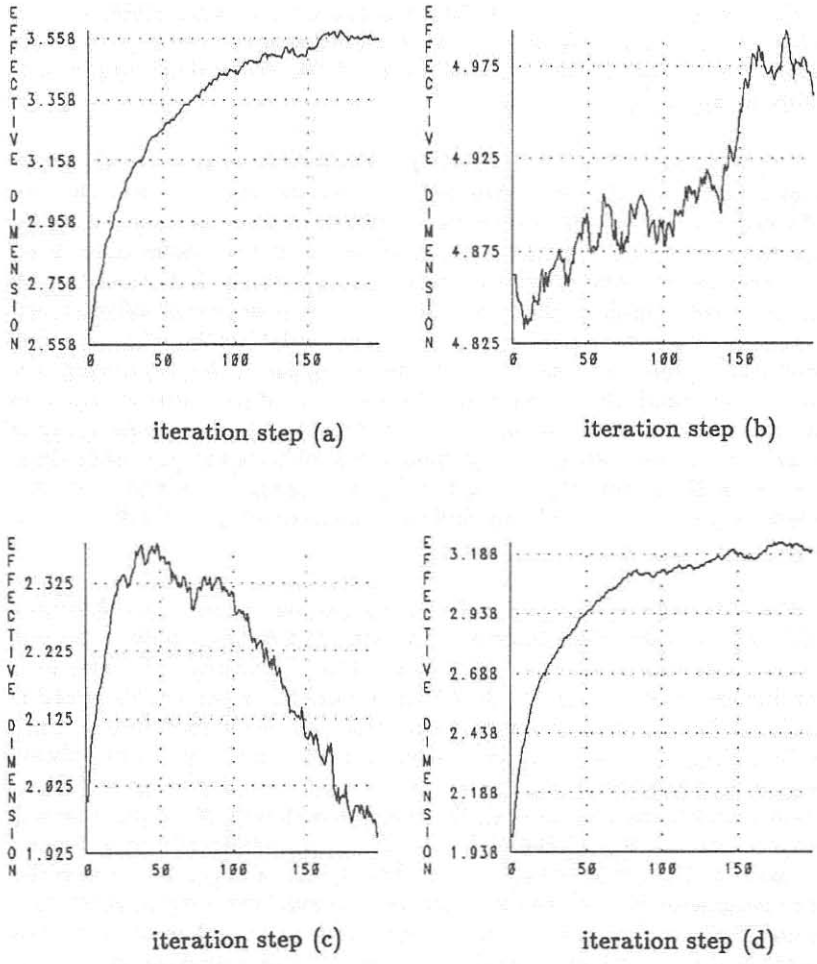

Figure 11: Development of D_{effec} for four different initial lattices: (a) 4-neighbor, (b) 8-neighbor, (c) $(10 \times 10 \times 10)$ three-dimensional, and (d) a random graph with $\langle d \rangle \sim 5$. In each case, the system evolves according to T-value and RT-link rules: $\vec{C} = (42, 69648, 32904)$ with constraints $[0, 15]$. *Qualitative* behavior, of course, depends critically on $D^{t=0}_{effec}$.

has been primarily on unraveling the structural evolution of these systems, the model can equally well be examined from either (a) the conventional cellular automaton view, where the focus remains on value configurations evolving on a dynamically responding spatial substructure, or (b) a global configurational view, where the behavior of the entire dynamically self-modifying state $|G\rangle$ is studied.

Although many important questions have not yet been considered—for example, it would be interesting to study the effects of link-changes and concomitant changes in effective information propagation speed on the universal computation abilities possessed by particular value rules Φ on fixed lattices—the qualitative picture of generic behavior is known. On the simplest level, one finds that the self-organizing property of value transition rules survives the coupling to the underlying lattice to induce evolving structural organization. In the most interesting cases, this organizing tendency is enhanced when combined with link rule induced correlations, producing geometries which are often of great complexity. Four broad types of behavior have been noted, though none are as strictly categorical as those found to well describe the qualitative behavioral classes of one- and two-dimensional CA; in general, one finds a rich variety of individually distinct developments.

Type-D systems—those capable of sustaining a prolonged dynamics within a relatively small interval of an effective dimensionality—deserve further study. As pointed out in the text, the simulations run so far have been for too short a time ($t < 1200$) and on graphs which are too small in size ($N < 2500$) to permit a conclusive statement to be made about limiting behavior. Moreover, restrictions on computer memory did not always allow observations of pure unconstrained dynamics to be observed. There is evidence, however, supporting the conjecture that as $N \to \infty$, there will always exist Γs maintaining a desired D_{effec} for arbitrarily long periods of time. If this proves true, we can then speak of D_{effec} as an effective order parameter in lattice systems capable of inducing a dynamically self-generated dimensionality. In larger systems, one can speak of characteristic length: scales over which more rigorously defined discrete dimensions D_{disc} [11] attain stable values.

In a more detailed study of these systems, it will be important to ask about global characterizations of behavior. If we consider a completely disordered initial state $|G\rangle$ where all sites take on values and all site-pairs are linked with equal independent probabilities, the irreversibility of most Γs must induce increasing deviations from this initial statistical randomness. In particular, one expects that the same sort of contraction which takes place in the space of all value configurations under a majority of Φs in CA models occurs in the much larger space of all possible labeled graphs G when these simpler systems are coupled with the set $\{\Psi, \Omega\}$. Measures of

deviation at each time t is provided by a global state-entropy

$$S(t) \equiv \binom{N+1}{2}^{-1} \sum_{\{|G\rangle\}} p_t(G)\ \log_2\ p_t(G), \tag{4.1}$$

where the sum is taken over all possible states $|G\rangle$, with probability $p_t(G)$ of occurring at time t. Exactly how small a fraction of the set of all possible structures is actually accessible through a typical Γ presently remains an unanswered question.

We mention three extensions to the basic scheme which may also be of some interest:

1. Purely structural analogs of CA models, in which random initial geometries evolve according to transition rules which are functions only of local topology. Sets of sites connected in a triangular fashion, for example, may be made to contract to a single point, or two sites may be linked if the numbers of sites in the locally partitioned neighborhoods A and B (see figure 1) match a set of prescribed transition values. Quiescent states would be explicit sets of local topologies remaining unchanged by the dynamics, say that of d-dim hypercubic lattices. The dynamics of many real random cellular structures may be modeled in this way.

2. Probabilistic transition rules Γ. Instead of either coupling or decoupling sites with $prob = 1$ for specified neighborhood sums, relax the definitions to span a larger set of possibilities. In the notation introduced in section 2.1, we would go from $prob\big[\psi(\beta_1,\beta_2,\beta_3)=0\big]=1$ for a particular 3-tuple defining an RT link deletion, to

$$prob\big[\psi(\beta_1,\beta_2,\beta_3)=0\big]=p, \quad 0<p<1. \tag{4.2}$$

   It is suspected that a critical-like behavior will emerge about specific sets $\{p_i\}$ such that $p_c < p_i^c$ will induce decaying behavior and $p_i > p_i^c$ will induce asymptotically growing configurations.

3. More physically realistic models incorporating link lengths as additional dynamic parameters. The original structures may be discretizations of selected continuous manifolds. Various structural deformations may then be directly realized by these generalized structural automata. Link lengths may also be introduced as dynamic elements inducing an asynchronization of local site and link transitions, whereby a unique discrete signal transmission speed would become a fundamental parameter.

The most ambitious use of SDCA models resides in their possible application to microphysical field theories. There have already been a few attempts made to construct a cellular automaton-based microphysics [11,12,13], in which emerging physical complexity is due to some basic discrete

local cooperative interactivity. Such attempts invariably start from the assumption that in any finite volume of space nature can only process a finite amount of information. T. D. Lee, among others, has recently advocated that a random discrete space-time be seriously considered as fundamental [14]. It has also been proposed that the topology and dimension of space-time be treated as dynamical constructs [7,15]. The present framework allows for just such a dynamical self-structuring to occur, particularly if Type-D systems turn out *not* to be artifacts of dynamics taking place within a limited spatio-temporal arena. The microphysical view of space-time which would emerge from such a picture would be one in which a fundamentally discrete pregeometry continually evolves as a completely amorphous structure but with a globally well-defined dimensionality. *Particles* would be introduced as locally persistent substructures (i.e. *topological solitons*) with dimensionalities differing from the globally stable value. *String-like* and other varieties of topological structures are very naturally described in this framework. Although it is easy to imagine a topologically deformed trail left behind gliders and other propagating value states in Conways "Life" model starting on 8-neighbor lattices, for example, it remains to be seen whether locally correlated propagating geometric substructures actually emerge under more general conditions.

These and other intriguing possibilities, as well as the rich variety of interesting behaviors already observed in the basic model, clearly recommend its continued study.

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