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The Effect of Galilean Non-Invariance in Lattice Gas Automaton One-Dimensional Flow

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Abstract. In the simple case of one-dimensional flow between plates, we show the effect of Galilean non-invariance of the usual hexagonal lattice gas mode. This effect leads to a distorted velocity profile when the velocity exceeds a value of 0.4. Higher-order corrections to the Navier-Stokes equations are considered in a discussion of the numerical importance of the distortion.

1. Introduction

It was argued by Frisch, Hasslacher, and Pomeau [1] that a (two-dimensional) hexagonal lattice gas model reproduces—upon coarse-graining—the fluid behavior described ordinarily by the Navier-Stokes equation. The significance of this proposal is that it could lead to new ways to simulate fluid flow, based upon simple binary arithmetic rather than high-precision floating-point calculations.

A characteristic feature of lattice gas automata is the appearance of higher-order corrections to the Navier-Stokes equation, once coarse-graining is performed [2]. These corrections are due to the discrete nature of both coordinate and velocity spaces. Since they are manifestations of the discrete lattice dynamics, they break Galilean invariance and show up as soon as the fluid velocity is no longer negligible compared to the microscopic velocity, or, equivalently, as soon as the Mach number (the ratio of fluid to sound velocity) approaches one. This transonic regime is easily obtained in lattice gases because here, the macroscopic fluid velocity is bounded by one, and the sound velocity itself takes the value $v_s = 17/\sqrt{2} \simeq 0.717$. The transonic regime can occur already at small Reynolds numbers, which is not normally the case in compressible fluids.

It is our aim here to study how large the Mach number can actually be (in the case studied) before effects due to the breaking of Galilean invariance set in. In particular, in as simple a fluid flow as one-dimensional flow

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between two plates, effects due to those higher-order lattice gas corrections to Navier-Stokes show up at high enough velocity. This was pointed out in the discussion of unsteady, one-dimensional flow in reference 3, and the demonstration and analysis of these effects is carried further in this work.

It is not that we believe it is very useful to study these corrections quantitatively. However, the case of one-dimensional fluid flow between plates is one of the simplest possible where Galilean invariance breaking will simply manifest itself as a distortion of the usual linear velocity profile given in the beginning of all textbooks on fluid mechanics. Moreover, while the linear profile is *independent* of fluid viscosity, the distorted one will depend on it. Also, while there is no pressure or density gradient in the "textbook" flow, a density gradient perpendicular to the direction of flow appears in the lattice gas. This one-dimensional flow thus provides a very clear case where some physical consequences of Galilean non-invariance and of higher-order corrections can be exhibited and discussed. Therefore, we consider the demonstration of the presence of these corrections and their consequences to be the main point of this work, not the crude analytical study which is only semi-quantitative.

In section 2, after briefly reviewing the physical system and the lattice gas model, we discuss the distortion of the velocity profile, the appearance of a density gradient, and the form of higher-order corrections to the Navier-Stokes equation for the flow considered. We summarize our results in section 3.

2. Numerical data and analysis

The lattice gas model is the usual one [1], with particles permanently moving on a two-dimensional hexagonal grid with coordination number equal to six. These particles moreover undergo two- and four-particle and so-called three-particle symmetric collisions, all of which conserve energy and momentum. The initial random distribution of particles corresponds to the same average number of particles in each of six possible directions at each site, and therefore leads to zero macroscopic velocity. The particles are enclosed [3] in a rectangle of dimensions $L = 84$ and $L' = 240$, where L is the width and L' the length. The direction of length corresponds to the y-axis, and the perpendicular one is taken to be the x-axis. Periodic boundary conditions are imposed at top ($y = L'$) and bottom ($y = 0$) of the system. The boundary condition at $x = L$ corresponds to no-slip, where particles bounce back along the incoming direction after hitting the wall. At the left wall ($x = 0$), boundary conditions are taken to be specular. As in reference 3, a tangential instability is introduced at $x = 0$ which creates a flow in the positive y direction by permitting some particles that normally would reflect on the wall going from top to bottom to bounce back into the direction of increasing y. We have checked that our results are insensitive to how the instability is created, another way being to start from no-slip boundary conditions at $x = 0$ and introduce a bias toward increasing y by allowing some particles to reflect specularly. In reference 3, the main object of study was how the instability propagates into the system with time, and it was shown that in the obtained steady state the expected linear velocity profile $u_y(x)$ is obtained, except for distortions at small x and high velocity. This steady state is the starting point of the present investigation.

In figure 1, the velocity profile $u_y(x)$ is shown as a function of the fractional distance in the direction perpendicular to the flow. The macroscopic average of density and velocity is done over cells of width 6 (in the lattice units) and length 200, after checking that the flow is y independent and the velocity component in the x direction is negligible. The density is 1/3. The profile is linear for large x, the linear part extrapolating at $x = 0$ to a maximum allowed velocity of $\sqrt{3/2} = 0.866$. For small x, the velocity profile is rounded off with an effective intercept of 0.53 at $x = 0$. This distortion, moreover, extends over half of the width.

Further study shows on one hand that the round-off disappears at small velocity, and on the other that it extends over larger absolute distances when the width of the system is increased, scaling approximately with the system width (cf. figure 2). It cannot therefore be a result of gas slip velocity at the left boundary ($x = 0$) of the rectangular box in which the fluid flows.

In figure 2, the data are shown for two different widths $L = 84$ and $L = 156$, everything else being equal. As mentioned, the rounding-off of the velocity profile occurs in both cases, from $x = 0$ to $x/L \simeq 0.5$, the linear part extrapolating to $u_y(0) = 0.866$. (The data for the system $L = 156$ do not fall exactly on top of those for $L = 84$; however, one must not forget that the larger system is less one-dimensional than the smaller one, the length of the system ($L' = 240$) being the same in both cases). The fact that the velocity profile depends on the ratio x/L is a characteristic of the linear profile (see below). The data of figure 2 show that this remains true through the region of rounding off.

In figure 3, data are shown for the system of width $L = 84$, which correspond to different maximum velocities at $x = 0$. The data demonstrate that when this velocity drops below 0.4, the rounding-off mostly disappears and the full linear profile is recovered. A small effect, presumably due to gas slip velocity, remains close to $x = 0$.

The distortion cannot be due to gas slip velocity, and we conclude that what is seen goes beyond the usual fluid description of the Navier-Stokes equation. The distortion shows up as soon as the velocity reaches a value of 0.4, which corresponds to a Mach number of $0.4\sqrt{2} = 0.56$. However, it disappears for small enough velocities. Clearly, Galilean invariance can only be restored if the fluid velocity is sufficiently small compared to the unit microscopic velocity. This was already pointed out for the discrete velocity model in reference 4.

In the textbook case where one of the plates, say the left plate, moves with velocity U, the linear profile (with respect to the choice of axes of

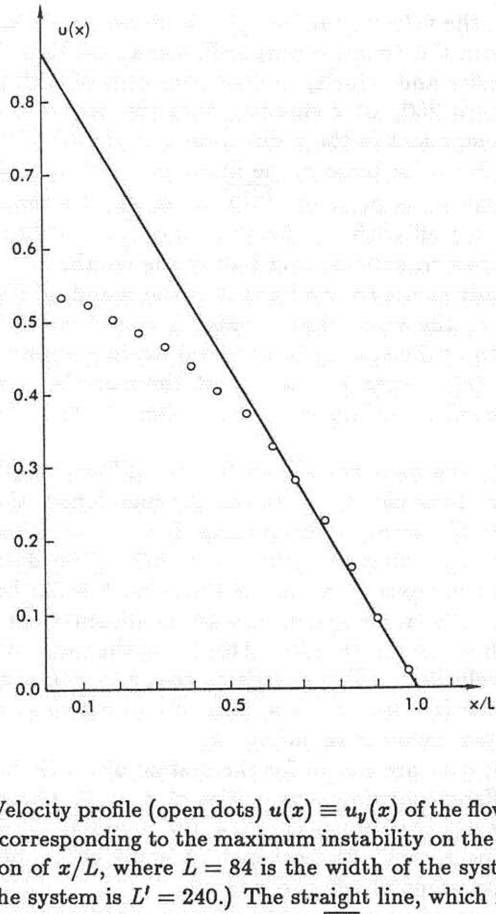

Figure 1: Velocity profile (open dots) $u(x) \equiv u_y(x)$ of the flow between two plates corresponding to the maximum instability on the left plate, as a function of x/L, where $L = 84$ is the width of the system. (The length of the system is $L' = 240$.) The straight line, which is a fit for $x/L > 0.5$, intersects the velocity axis at $\sqrt{3/2} = 0.866$, the maximum velocity in the y direction.

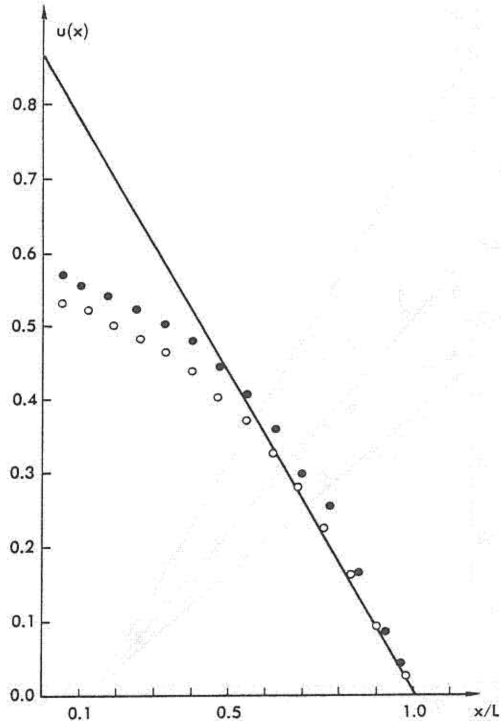

Figure 2: Velocity profile $u(x) \equiv u_y(x)$ as a function of x/L, where the open dots correspond to a system of width $L = 84$ (the same as in figure 1), and the full dots to a system of width $L = 156$. The length $L' = 240$ is the same in both cases and the straight line is the same as in figure 1.

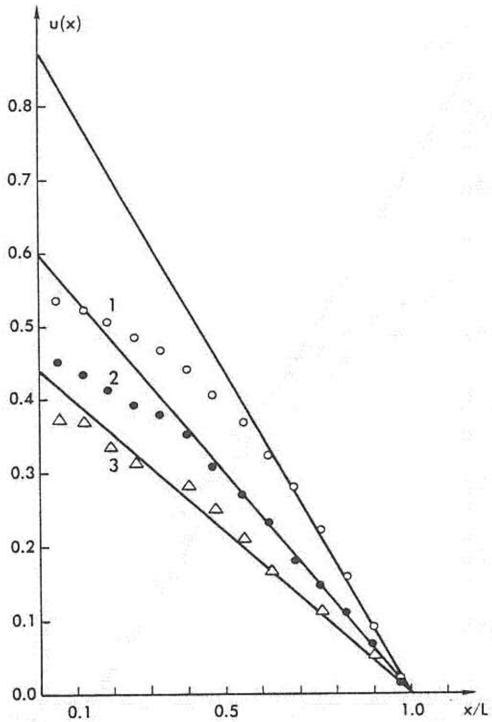

Figure 3: The velocity profiles $u(x) = u_y(x)$ for a system of width $L = 84$ and length $L' = 240$, of decreasing maximum velocity. Curve 1 is the same as in figure 1, with its straight line of intercept 0.866; curve 2 has lower maximum velocity, with a corresponding straight line of intercept 0.6. For curve 3, where the maximum velocity is about 0.4, the straight line has approximately the same intercept. The respective straight lines fit the profile for x/L closer and closer to zero as the maximum velocity is lowered.

figure 1) is given by

$$u(x) = U\left(1 - \frac{x}{L}\right), \tag{2.1}$$

where u is the component of velocity in the y direction. This is a solution of the Navier-Stokes equation:

$$\nu \frac{d^2u}{dx^2} = 0 \tag{2.2}$$

with the appropriate boundary conditions (ν denotes the kinematic viscosity). Notice that the velocity profile, expression (2.1), is independent of ν. The linear shape of the profile is the same as long as the relative velocity of the two plates is U. However, from the distorted profile of figure 2, one can tell at which plate tangential flow occurs; this is a sign of Galilean invariance breaking.

Let us now assume that the distortion of the velocity profile can be represented by corrections to the usual Navier-Stokes equations. The form of higher-order corrections, both in the convective and viscous part of the Navier-Stokes equations, is given in reference 2. Because of the symmetry of one-dimensional flow, only corrections to the viscous term appear in our case. To lowest order in the velocity, they are of the form

$$u^2 \frac{d^2u}{dx^2} \quad \text{and} \quad u\left(\frac{du}{dx}\right)^2.$$

Adding these to the right-hand side of equation (2.2) preserves the $u \to -u$ symmetry of the equation, which remains a property of the flow in figure 2. The term in $u^2(d^2u/dx^2)$ leads to a velocity-dependent kinematic viscosity. Replace this by an effective ν_{eff} averaged over velocity and consider the influence of term $u(du/dx)^2$, which plays a more important role. The extended Navier-Stokes equation reads:

$$\frac{d^2u}{dx^2} = \frac{\alpha}{\nu_{\mathit{eff}}}\, u \left(\frac{du}{dx}\right)^2, \tag{2.3}$$

where α measures the strength of the extra term. (The profile now depends on viscosity.) This can be written as (in obvious notation)

$$\frac{u''}{u'} = \frac{\alpha}{\nu_{\mathit{eff}}}\, u\, u' = \frac{\alpha}{2\nu_{\mathit{eff}}}(u^2)'.$$

The solution of this equation is related for $\alpha < 0$ to the error function $w(u)$ [5]. An expansion to the lowest orders in velocity gives the solution

$$u + \frac{|\alpha|}{6\nu_{\mathit{eff}}} u^3 = U\left(1 - \frac{x}{L}\right), \tag{2.4}$$

where we have written $\alpha = -\,|\,\alpha\,|$. Obviously, to even approach a description of the round-off in figure 3, α must be taken negative, the u^3 term compensating for the difference between the measured u and the extrapolated velocity profile. An estimate of $|\alpha|/\delta\nu_{eff}$ from the curves in figure 3 leads to a value between 1 and 2 for curve 1 (intercept $U = 0.866$) and between 1 and 1.5 for curve 2 (intercept $U = 0.6$). This is not really satisfactory. However, the error on the coefficient is large since it is related to that of $1/u^3$. Moreover, if the effective viscosity increases with u, the spread in values is reduced. (The same result is achieved if fifth-order terms in the velocity add to the third-order ones.) The point is, however, not a detailed numerical study of these higher-order terms, but the realization that for high velocities (Mach number greater than 0.56) they are very important, because $|\alpha|$ turns out to be of order $6\nu_{eff}$.

Our last remark concerns density. To the same order (order u^2) in which there are corrections to the Navier-Stokes equation, there appears a correction to the pressure [2]. The pressure becomes

$$p = 3\rho\left(1 - \frac{c^{(2)}}{4}u^2\right),$$

where ρ is the density and $c^{(2)}$ is a density dependent coefficient, equal to 1 when the average density is $1/3^{(2)}$. Thus, although there is no contribution from pressure to equation (2.3), one now obtains a transverse density gradient from $dp/dx = 0$, namely (with $c^{(2)} = 1$)

$$\frac{d\rho}{dx}\left(1 - \frac{u^2}{4}\right) = \frac{\rho}{4}\frac{d}{dx}u^2, \tag{2.5}$$

where $u = u_y(x)$. As long as u_x is small compared to u_y, and it is, the equation of continuity continues to be satisfied. This density gradient is seen in the data. Excluding the two points closest to the plates where the density is always smaller, the density drops continuously from small x, where it is 0.3541, to high x, where it is 0.3243. A fit to the data with formula (2.5) works well until one gets close to $x = 0$, where the data rises faster than the numbers given by the fit, indicating as in the previous discussion of the velocity profile the need for even higher-order corrections in velocity.

3. Summary

We have shown how at large fluid velocity (Mach number above 0.56), the lattice gas automaton fails to reproduce the linear velocity profile of one-dimensional flow between two plates. We interpret this as a sign of Galilean invariance breaking due to the discrete nature in coordinate and velocity space of the hexagonal lattice gas model of fluid mechanics. Higher-order corrections to the Navier-Stokes equations are introduced and their numerical importance estimated by comparison with the observed effects. The appearance of a density gradient in the direction transverse to the flow is discussed.

Acknowledgment

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