Complex Systems **1** (1987) 821–829

Dimension Densities for Turbulent Systems with Spatially Decaying Correlation Functions

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Abstract. Lattices of coupled maps on the interval are used to test some ideas of Y. Pomeau concerning estimates of the number of degrees of freedom per unit length of a spatially incoherent system. Qualitative agreement is found between dimension densities obtained using two-point measurements at separated lattice points and dimension densities obtained using spatial decay of the correlation function.

1. Introduction

The introduction of nonlinear, deterministic, and low-dimensional dynamical systems with chaotic solutions led to many conjectures about how these chaotic systems might be related to fluid turbulence. It appears that the time series, produced by a chaotic solution can be, from the point of view of power spectra, as complex as experimentally observed signals from turbulent hydrodynamics (see [1–3] and references therein). Furthermore, certain transitions from laminar to turbulent flow have their analog in the transition from ordered to chaotic behavior of deterministic chaotic systems.

A basic problem in that context is how methods from nonlinear dynamical systems can be used to describe experimental turbulence. It was suggested to determine the fractal dimension [4–6] of a turbulent flow in order to estimate how many nonlinear equations would be needed for a model of turbulence. A frequent objection to the approach of using simple dynamical systems as models for turbulence is that these models might reproduce some temporal chaos but would not correspond to real turbulence, for which the spatial structure also is very irregular and chaotic. The dynamics in a turbulent flow especially is not expected to be spatially coherent and therefore cannot be described by a few global modes. Thus, a one-point measurement of a velocity component, say, should always contain information about the dynamics of the whole fluid and therefore yield

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a fractal dimension which is proportional to the size of the container. This has been confirmed, e.g., by Brandstätter [6] in an experiment on turbulent channel flow.

Thus, if we still want to use the framework of simple dynamical systems, then we have to consider lattices of coupled dynamical systems, such as coupled maps on the interval. It is known that these systems have very interesting properties with respect to spatio-temporal complexity [7,8]. In the following, we want to use the simplest of these maps in order to test the applicability of some ideas of Y. Pomeau [9,10] which should make it possible to estimate the the number of degrees of freedom per unit length of a system which is spatially incoherent.

This is done by computing the dimension density of the lattice system through a series of two-point measurements at separated lattice points. Then, we compare these results with the spatial decay of the correlation function and the mutual information content. We find a qualitative agreement with the expected dependence. For precise quantitative measurement, the general problem of accuracy and data limitations appear to become dominant.

There are several other approaches to this problem. The straightforward idea, which is unrealistic for basically all numerical simulations, is to compute the dimension of the full system and then divide the dimension by the volume of the system. The special cases, where the full dimension can be estimated through the Kaplan-Yorke conjecture, consist of mathematical systems for which the equations and their derivatives are known. Since derivatives reflect local dynamics and therefore in a sense correspond to infinite resolution, we would expect that dimension densities defined through Lyapunov spectra define an upper bound on the dimension densities from time series. It is also not quite clear to us if these methods can be compared at all. In our numerical simulations we could, however, confirm this inequality.

A different approach, which is completely based on the spatial structure of the systems and which does not explicitly take into account the temporal evolution, has been proposed by P. Grassberger [11]. We do not yet know how realistic this approach will turn out to be.

2. Dimension density

We want to discuss an intensive measure of complexity of the spatio-temporal dynamics of a system, i.e., an observable which does not depend on the size of the system. The natural approach would be to define the density of an (extensive) quantity, which grows proportionally with the size of the system. The quantity we choose is the dynamical dimension, or number of degrees of freedom of the system. Since for spatially coherent systems it is obvious that the dimension density must vanish in the limit of large extensions, we assume that the influence of the local dynamics at one position x decays with the distance from x. Thus, in the limit of infinite experimental resolution, we would expect that a dimension measurement at a given position of the system will yield a value which is extensive, since every point in the system contributes to a certain degree to the dynamics at x. Because of the decreasing influence of distant points, we expect that for finite resolution we are going to measure a finite dimension, since we cannot resolve the small amplitude dynamics. In a way, this is similar to the noise perturbation of a deterministic system, only in this case, deterministic dynamics and "stochastic" noise have the same dynamical origin and cannot be clearly distinguished.

The dimension value that we obtain for a finite resolution from a one-point measurement is composed of two different contributions from (i) the local dynamics, which can be of a dimensional complexity which can vary considerably with parameters and systems, or (ii) large amplitude influences from the dynamics of the neighborhood. The first situation we would expect for a small coupling between neighboring sites. The second case we would expect when the coupling is strong. With a one-point measurement we are not able to distinguish between these two cases.

Thus, let us assume we are measuring a signal $S(x,t)$ at a position x at a time t. It can be decomposed into local and coupling terms:

$$S(x,t) \approx S_0(x,t) + \xi_{U_x}(t), \tag{2.1}$$

where $S_0(x,t)$ denotes the contribution from the local dynamics and $\xi_{U_x}(t)$ stands for the influence from a neighborhood U_x of x. We would like to mention that we do not consider here the finite propagation speed of perturbations, but are only interested in the stationary dynamics at different points. For the sake of simplicity, let us assume that we have an exponential decay of the influence of spatially separated points. By this, we mean that the dynamics at a point $y \in U_x$, which is separated by a distance $||x-y|| = L$ from x, will generate a perturbation ξ_y of size:

$$\xi_y = \underbrace{\hat{S}(y)}_{\sim 1} \cdot e^{\frac{-L}{\lambda}} \tag{2.2}$$

where $\hat{S}(y)$ represents the local dynamics at position y and we have assumed a spatial homogeneity, i.e., the mean amplitude of the dynamics should not depend on the position. Of course, this excludes complex spatial patterns, which would severely complicate the arguments. The exponential factor contains a characteristic length λ, but again, the main argument does not depend on the exact form in which the spatial perturbation decays. From equation (2.2), we can see how for infinite resolution ($\xi_y \to 0$) we asymptotically measure the dimension of the whole system.

In the next part, we introduce a finite experimental resolution $\varepsilon > 0$, which means that we do not compute dimension values for signals of amplitude $r < \varepsilon$. From equation (2.2), we see that this means that we only pick up perturbations from a neighborhood of size L_ε for which: $e^{-\frac{L_\varepsilon}{\lambda}} = \varepsilon$.

From this, we obtain the effective range $L_\varepsilon = \lambda \ell n \frac{1}{\varepsilon}$. Again, we note that this quantity cannot be obtained from a one-point measurement.

From the measured dimension D_ε at a single point with resolution ε and the range L_ε, we can now define the dimension density ρ of the system by:

$$N_\varepsilon = \rho \cdot L_\varepsilon^d, \tag{2.3}$$

where d is the geometrical dimension of the system: in our case, of a string of coupled maps we have $d = 1$.

The next assumption which we make is that of an additivity of the dimensions for *combined* signals, i.e. if we superimpose the signals from two oscillators $S_1(t)$,$S_2(t)$ of the same amplitude ($||S_1(t)|| \sim ||S_2(t)||$), then we expect to observe a signal which has a dimension which equals the sum of the dimensions of the two separate signals. There are several ways in which this superposition can be realized. For instance, the combined signal could be the sum $S(t) = S_1(t) + S_2(t)$ of the separate signals or it could consist of an interleaved time series, which could be interpreted as coming from two "orthogonal" sources.

3. Two-point measurements

We now intend to specify the two signals $S_i(t)$ an originating from the same system but from different locations:

$$\begin{aligned} S_1(t) &= S(x,t) \\ S_2(t) &= S(y,t) \qquad |x-y| = \ell \end{aligned} \tag{3.1}$$

We can now roughly distinguish between three different cases depending on the separation ℓ. The characteristic distances we have in our model are λ, which determines a distance over which a perturbation has decreased significantly. The second characteristic length L_ε is given by the experimental resolution (or by the smallest structures we wish to resolve). Thus, we have for the two-point dimension $D_2^{(2)}(\ell)$ obtained from points separated by a distance ℓ:

$$D_2^{(2)}(\ell) = \begin{cases} D_\varepsilon & \text{for } \ell \le \lambda \\ 2D_\varepsilon & \text{for } \ell \ge L_\varepsilon \end{cases} \tag{3.2}$$

For intermediate distances $\lambda \le \ell \le L_\varepsilon$, we have :

$$D_2^{(2)}(\ell) = \rho \cdot V(B_x(L_\varepsilon) \cup B_y(L_\varepsilon)) \tag{3.3}$$

where $B_x(L_\varepsilon)$ denotes a ball of radius L_ε centered at point x and V is the corresponding volume in the appropriate embedding space. In one dimension, we have: $V(B_x(L_\varepsilon)) = 2 \cdot L_\varepsilon$. From this, we get for the two-point dimension of a lattice string:

$$D_2^{(2)}(\ell) = \begin{cases} \rho(2L_\varepsilon + \ell) & \ell \leq 2L_\varepsilon \\ \rho \cdot 4L_\varepsilon & \ell \geq 2L_\varepsilon \end{cases} \tag{3.4}$$

We now eliminate the characteristic length L_ε from equation (3.4) by insertion: $D_2^{(2)}(0) = \rho \cdot 2L_\varepsilon$. Then we obtain from equation (3.4):

$$D_2^{(2)}(\ell) = D_2^{(2)}(0) + \ell \cdot \rho. \tag{3.5}$$

Finally, for the dimension density ρ:

$$\rho = \frac{D_2^{(2)}(\ell) - D_2^{(2)}(0)}{\ell}. \tag{3.6}$$

In the case that we do not have a discrete spatial lattice, but a continuous system, we get:

$$\rho = \lim_{\ell \to 0} \frac{D_2^{(2)}(\ell) - D_2^{(2)}(0)}{\ell}. \tag{3.7}$$

4. Numerical realizations

To investigate the introduced notions of correlation length and density of degrees of freedom, we choose as a model a one-dimensional lattice of coupled "tent"-maps

$$x_{n+1}^{(i)} = (1 - 2\overline{c})h(x_n^{(i)}) + \sum_{j=-n_\kappa}^{n_\kappa} c_j\left(h(x_n^{(j-i)}) + h(x_n^{(j+i)})\right) + \eta_n^{(i)} \tag{4.1}$$

with

$$h(x) = 1 - 2|x - \frac{1}{2}|$$

$$\overline{c} = \sum_{j=1}^{n_\kappa} c_j$$

$$c_j = c_1 e^{-\Gamma(j-1)}$$

n_κ is the number of neighboring points which one point is coupled to on either side, Γ denotes the decay rate of coupling strength, and $\eta_n^{(i)}$ is a small Gaussian distributed noise term of the order of 10^{-6}.

This system may serve as a crude model of fully developed turbulence by yielding essentially random data with certain spatiotemporal correlation properties. Being restricted to discrete space and time, it nevertheless exhibits features also found in real physical situations.

We intend to describe the interdependence of signals $u(x,t)$ derived from two separated points in the system's domain. Specifically, we consider the time series measured at a reference point x_0, $u(x_0,t)$ and the time series measured at a point $x_0 + \Delta x$ that may also be shifted in time by a certain amount Δt, i.e. $u(x+\Delta x, t+\Delta t)$. In the following, we refer to the sequence $(u(x_0,t), u(x_0+\Delta x, t+\Delta t))$ as the *combined time series* belonging to the displacement $(\Delta x, \Delta t)$.

The aim of our investigation is to find out how "low order approximations" describing spatiotemporal correlations compare. To that end, we compute three different quantities: the correlation function $\Phi(\Delta x, \Delta t)$ and the mutual information $\mu(\Delta x, \Delta t)$ as a function of spatial and temporal displacement, and the two-point correlation dimension $D_2^{(2)}(\Delta x)$ as a function of the spatial displacement. The first two quantities describe features of the joint probability distribution $p(u, u')$ generated by the scalar series $u = u(x_0, t)$ and $u' = u(x_0 + \Delta x, t + \Delta t)$, and are defined in the standard way:

$$\Phi(\Delta x, \Delta t) = \frac{\langle uu' \rangle - \langle u \rangle \langle u' \rangle}{\sigma_u \sigma_{u'}} \tag{4.2}$$

σ_u denotes the standard deviation of the distribution $p(u)$ generated by u,

$$\begin{aligned} \mu(\Delta x, \Delta t) &= H(u) + H(u') - H(u, u') \\ &= \int_{-\infty}^{+\infty} p(u, u') \log(\frac{p(u, u')}{p(u)p(u')}) du du' \end{aligned} \tag{4.3}$$

where $H(u)$ is the entropy of the distribution $p(u)$.

The two-point dimension [9,10] is conceptually different because it also takes into account the dynamics of the lattice. It is defined as the correlation dimension of the combined series (u, u'). It is determined by the well-known method of reconstruction in a space of embedding dimension D. In the case of a combined signal, we embed pairs of points in an effectively $2D$-dimensional phase space. The state of the system is thus represented by a pair of vectors $\binom{\mathbf{u}_D(x,t)}{\mathbf{u}_D(x+\Delta x, t)}$. The norm in this space is defined via $\|\binom{\mathbf{a}}{\mathbf{b}}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$.

In our numerical simulations of system (4.1) we use 100 maps, nearest-neighbor coupling ($n_\kappa = 1$), and open boundary conditions (i.e., one-sided coupling at the boundary). In all runs, the noise level is 10^{-6}. As reference point x_0, we pick point number 47 (near the middle of the lattice) and determine the above defined quantities with the second point ("probe") lying in the range $47 \ldots 54$, corresponding to $\Delta x = 0 \ldots 7$. For the determination of the mutual information content of the combined signal, we use an algorithm proposed by Frazer [12]. The number of data points considered for both the correlation Φ and mutual information μ is about 16400. The two-point correlation exponent $D_2^{(2)}$ is determined by computing the well-known correlation graphs for embedding dimensions D up to 20, using 10000 to 25000 data points. As discussed above, these curves cannot be assigned a unique asymptotic slope. Thus, we fix a certain level of resolution $\log(\epsilon_0)$ and

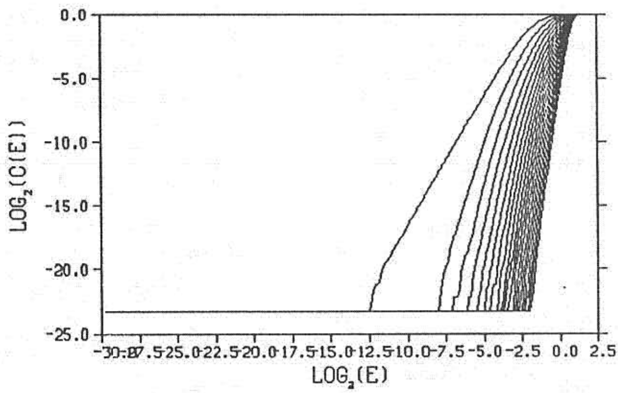

Figure 1: Correlation graph of the coupled tent-map lattice (equation (3.1)) for $n_\kappa = 10$, $\kappa = 0.3$. The two-point dimension is computed for a distance $\Delta = 1$. The curves for the first 15 embedding dimensions are plotted.

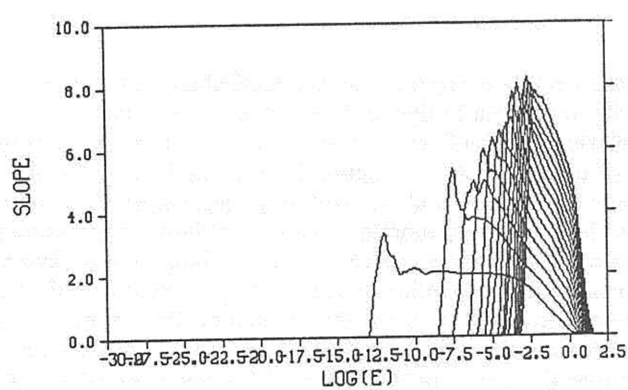

Figure 2: Local slope (dimension) for system of figure 3.1.

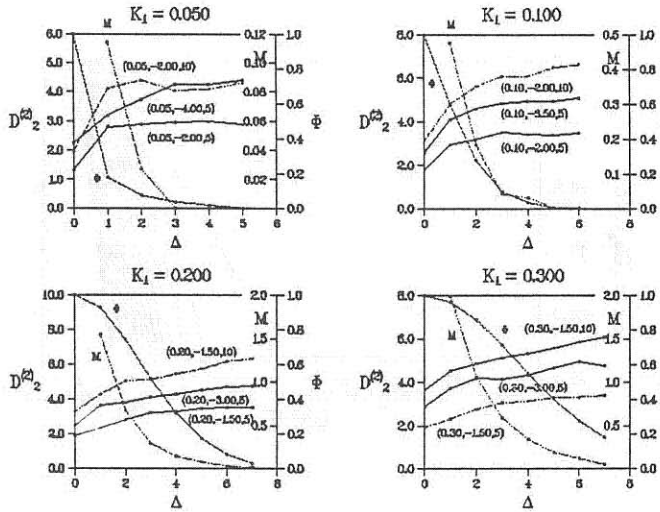

Figure 3: Correlation function Φ, mutual information M and two-point dimension $D_2^{(2)}$ as a function of the point separation Δ. The two-point dimension $D_2^{(2)}$ is plotted for several values of resolution ϵ and embedding dimension D. The four subplots correspond to different coupling strengths $K_1 = 0.05, \ldots, 0.3$. Here, the coupling neighborhood is $n_\kappa = 5$.

define the correlation exponent as the local slope of the correlation graph. The results are shown in figures 1 and 2, where we give an example of how the correlation graph of this model (figure 1) behaves. We clearly see the increase of dimension (slope, figure 2) with the increase of the resolution. For various combinations of embedding dimension and resolution, we have displayed the correlation function, the mutual information content, and the two-point dimension for a lattice with a coupling between five neighboring points for increasing coupling strength. We see qualitatively the behavior which we expect. We compare the estimated dimension density, e.g., for $\kappa = .3$ with the value which we obtain from the Lyapunov spectrum. For the Lyapunov dimension of the system of 100 maps, we observe a dimension of about 67, which yields a density of 0.67. From the two-point dimension, we obtain a value which appears to be below 0.5 for the resolutions we could realize. This is also in agreement with the theoretical expectation.

We think it became clear in this paper that quantitative results are very preliminary in this field and the complexity exhibited by spatially extended systems is much richer than what is known from low dimensional chaos.

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