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The Hydrodynamical Description for a Discrete Velocity Model of Gas

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Abstract. For a model of gas composed of identical particles with velocities restricted to a given finite set of vectors, the Boltzmann equation is replaced by a system of nonlinear coupled differential equations. The Chapman-Enskog method can be applied, and it gives the Navier-Stokes equations associated to the model. For the general model, we show that the dissipative terms in the Navier-Stokes equations do not depend on the mean number density nor on its gradient. For a gas near a homogeneous state, we give the transport coefficients explicitly.

1. Introduction

In discrete kinetic theory of gases, the main idea is to consider that the particle velocities belong to a given finite set of velocity vectors. J. E. Broadwell [1,2] has used some very simple models of gas to solve problems in which the Boltzmann equation must be introduced.

The presentation of a general model of gas with discrete velocities has been given in references 3 and 4, and the kinetic theory for such a gas has been built up. The Boltzmann equation is replaced by a system of partial differential equations. This system is more tractable than the Boltzmann equation, and the discrete models give some light about some fundamental problems such as the structure of the shock wave [1,5] or the Knudsen layer on a plate [2,6].

The system of kinetic equations is a semi-linear hyperbolic system, and it has a very interesting mathematical structure. Many papers concern this mathematical point of view; a review is given, for example, by H. Cabannes in reference 7. Also, for particular models, some exact solutions have been found [8,9]. Finally, we mention some generalizations for a mixture of gases [10–14].

We must emphasize that in discrete kinetic theory, only the velocity space is discretized, the space and time variables being continuous. For a lattice gas, as introduced for the first time in the paper of J. Hardy and Y. Pomeau [15], the space and time variables are discretized also. The main and very important consequence is to have one's way to study the hydrodynamical problems for such a lattice gas by simulation on a computer of cellular automaton type. This aspect is presented in the paper of U. Frisch, B. Hasslacher, and Y. Pomeau [16], and many classical problems of fluid dynamics have been studied with this point of view [17,18,19]. For the theoretical study of the hydrodynamics of a lattice gas, we must study a system of equations similar to the system of the discrete kinetic equations. (The difference comes from the exclusion principle used in the lattice gas theory.) The viscosity coefficient has been calculated for a lattice gas flowing out with a small Mach number [20,21].

In this paper, we briefly recall the discrete kinetic equations (section 2), describe the Maxwellian states (section 3), and apply the Chapman-Enskog method (section 4). So, we obtain the so-called Euler and Navier-Stokes equations associated with the model, and we prove that the mean number density n and its gradient $\vec{\nabla} n$ do not appear in the dissipative terms of the Navier-Stokes equations. In section 5, we investigate the hydrodynamical equations for a gas near a homogeneous state.

2. Description of the model

In earlier works we have described the general model of a gas with a discrete velocity distribution [3,4], and here we briefly recall the notations and the main results. The gas is composed of identical particles of mass m. The velocities of these particles are restricted to a *given* finite set of p vectors: $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p$. We denote by $N_i(\vec{r}, t)$ the number density of particles with the velocity $\vec{u}_i$ at point $\vec{r}$ and time t.

Only binary collisions are considered. Let $\vec{u}_i, \vec{u}_j$ and $\vec{u}_k, \vec{u}_l$ be the velocities of two molecules respectively before and after an encounter; these four velocities must belong to the original set, and they must satisfy the two relations expressing the conservation of momentum and energy. A "transition probability" A_{ij}^{kl} is associated with each collision, and we assume that the A_{ij}^{kl} coefficients satisfy the micro-reversibility principle

$$A_{ij}^{kl} = A_{kl}^{ij} \qquad \forall i, j, k, l. \tag{2.1}$$

Of course, the transition probabilities are positive or equal to zero and symmetrical with respect to the upper indices and to the lower ones. It is convenient to assign a zero value to the transition probability for an unrealizable collision.

The Boltzmann equation is replaced by a system of p nonlinear coupled differential equations [3,4]

$$\begin{aligned} \frac{\partial N_i}{\partial t} &+ \vec{u}_i \cdot \vec{\nabla} N_i = \tfrac{1}{2} \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} (A_{kl}^{ij} N_k N_l - A_{ij}^{kl} N_i N_j) \\ i &= 1, 2, \ldots, p. \end{aligned} \tag{2.2}$$

or

$$\frac{\partial}{\partial t}\, \mathrm{N} + \mathcal{A}\mathrm{N} = \mathcal{F}(\mathrm{N}, \mathrm{N}), \tag{2.3}$$

where $\mathbf{N} = (N_1, N_2, \ldots, N_p)$ is a p-component vector of the space $\mathbf{R}^p$, and $\mathcal{F}(\mathbf{U}, \mathbf{V})$ is a bilinear symmetric operator from $\mathbf{R}^p \times \mathbf{R}^p$ into $\mathbf{R}^p$:

$$\begin{aligned} \mathcal{F}_i(\mathbf{U}, \mathbf{V}) &= \tfrac{1}{4} \textstyle\sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} \Big\{ (A_{kl}^{ij}(U_k V_l + U_l V_k) \\ &- A_{ij}^{kl}(U_i V_j + U_j V_i) \Big\} \\ i &= 1, 2, \ldots, p. \end{aligned} \tag{2.4}$$

For a model with a given set of velocities, we define the summational invariants which are quantities ϕ associated with conservation laws through an encounter. In other words, $\mathbf{\Phi}$ is a p-component vector satisfying the following conditions:

$$A_{ij}^{kl}(\phi_i + \phi_j - \phi_k - \phi_l) = 0 \qquad \forall i, j, k, l. \tag{2.5}$$

In particular, $\mathbf{\Phi}$ is a summational invariant if ϕ_i is equal to m, $m\vec{u}_i$, or $\frac{1}{2}m\vec{u}_i^2$. In contrast to the classical kinetic theory for monoatomic gases, the geometric character of the set of the given velocities may allow other summational invariants. They generate a linear subspace $\mathbf{F}$ of $\mathbf{R}^p$ of dimension $q(1 \le q \le p)$. We denote by $\mathbf{F}^{\perp}$ the subspace of $\mathbf{R}^p$ orthogonal to $\mathbf{F}$.

We introduce orthonormal bases in $\mathbf{F}$ and in $\mathbf{R}^p$:

$$\begin{array}{ll} \mathbf{V}^1, \mathbf{V}^2, \ldots, \mathbf{V}^q & \text{in } \mathbf{F}, \\ \mathbf{V}^1, \mathbf{V}^2, \ldots, \mathbf{V}^q, \mathbf{W}^{q+1}, \ldots, \mathbf{W}^p & \text{in } \mathbf{R}^p. \end{array} \tag{2.6}$$

So we can write:

$$\begin{aligned} \mathbf{N} &= \textstyle\sum_{\alpha=1}^{q} a_\alpha \mathbf{V}^\alpha + \sum_{\beta=q+1}^{p} b_\beta \mathbf{W}^\beta, \\ N_i &= \textstyle\sum_{\alpha=1}^{q} a_\alpha V_i^\alpha + \sum_{\beta=q+1}^{p} b_\beta W_i^\beta, \end{aligned} \tag{2.7}$$

$$a_\alpha = \langle \mathbf{N}, \mathbf{V}^\alpha \rangle, \qquad b_\beta = \langle \mathbf{N}, \mathbf{W}^\beta \rangle; \tag{2.8}$$

the i-components of $\mathbf{V}^\alpha$ and $\mathbf{W}^\beta$ are denoted by V_i^α and W_i^β, and $\langle \mathbf{U}, \mathbf{V} \rangle = \sum_{i=1}^{p} U_i V_i$ denotes the scalar product in $\mathbf{R}^p$.

We have shown that equations (2.2) possess the essential properties of the Boltzmann equation [3,4]. There are two ways of describing the gas: first, a *microscopic* description corresponding to the knowledge of the densities N_i or equivalently to the knowledge of the quantities a_α and b_β; and second, a *macroscopic* description corresponding to the knowledge of the q quantities a_α alone. The quantities a_α are called macroscopic state variables of the gas. Among them there are the number density n, the mean velocity $\vec{v}$, and the temperature T. We give hereafter the macroscopic conservation laws for the quantities a_α:

$$\frac{\partial a_\alpha}{\partial t} + \langle \mathcal{A}\mathbf{N}, \mathbf{V}^\alpha \rangle = 0 \qquad \alpha = 1, 2, \ldots, q. \tag{2.9}$$

This system of q equations for the q quantities a_α is not a closed system: the b_β variables are present in them. To obtain a closed system, we are going to use the well-known Chapman-Enskog method, which is applicable when the Knudsen number of the gas is small. It is possible to use this method for a discrete model gas [3,4] and so obtain the constitutive laws for our macroscopic medium.

3. Maxwellian state, Euler equations

For a discrete model of gas, an H-theorem is valid [3,4]. The H-function here is defined by $H = \sum_{i=1}^{p} N_i \operatorname{Log} N_i = \langle \mathbf{N}, \operatorname{Log} \mathbf{N} \rangle$, and the Maxwellian state is a state in which Log $\mathbf{N}$ is a summational invariant. That is,

$$\begin{aligned} \operatorname{Log} \mathbf{N} &= \textstyle\sum_{\alpha=1}^{q} c_\alpha \mathbf{V}^\alpha \\ \operatorname{Log} N_i &= \textstyle\sum_{\alpha=1}^{q} c_\alpha \mathbf{V}_i^\alpha . \end{aligned} \tag{3.1}$$

(Here, Log $\mathbf{N}$ denotes the sequence (Log N_1, Log $N_2, \ldots,$Log N_p).) From the definition (2.8) of the macroscopic state variables a_α, we have

$$a_\alpha = \langle \mathbf{N}, \mathbf{V}^\alpha \rangle = \sum_{i=1}^{p} V_i^\alpha \exp\Big(\sum_{\gamma=1}^{q} c_\gamma V_i^\gamma\Big). \tag{3.2}$$

The quantities a_α are functions of the q variables c_γ. The correspondence between the a_α and the c_γ is one to one [4]. Thus, in a Maxwellian state, the densities N_i are well-defined functions of the macroscopic state variables a_α. The macroscopic conservation laws (2.9) are the so-called Euler equations. Notice that the exact form of these Euler equations depends on the choice of the model of gas.

Later on, we want to study in what way the mean number density appears in the Euler and Navier-Stokes equations. To this end, we introduce $n = \sum_{i=1}^{p} N_i$, and we put

$$V_i^1 = \frac{1}{\sqrt{p}}, \quad i = 1, 2, \ldots, p; \quad a_1 = \langle \mathbf{N}, \mathbf{V}^1 \rangle = \frac{1}{\sqrt{p}}\, n; \tag{3.3}$$

$$a_\alpha = n X_\alpha, \quad \alpha = 1, 2, \ldots, q; \quad b_\beta = n Y_\beta, \quad \beta = q+1, \ldots, p; \tag{3.4}$$

$$\mathbf{N} = n\tilde{\mathbf{N}} = n\Big(\frac{1}{\sqrt{p}}\mathbf{V}^1 + \sum_{\alpha=2}^{q} X_\alpha \mathbf{V}^\alpha + \sum_{\beta=q+1}^{p} Y_\beta \mathbf{W}^\beta\Big). \tag{3.5}$$

In a Maxwellian state, from equation (3.2) we deduce

$$n = \exp\Big(\frac{1}{\sqrt{p}}\, c_1\Big) \sum_{i=1}^{p} \exp\Big(\sum_{\gamma=2}^{q} c_\gamma V_i^\gamma\Big) \tag{3.6}$$

$$X_\alpha = \frac{\sum_{i=1}^{p} V_i^\alpha \exp(\sum_{\gamma=2}^{q} c_\gamma V_i^\gamma)}{\sum_{i=1}^{p} \exp(\sum_{\gamma=2}^{q} c_\gamma V_i^\gamma)}, \quad \alpha = 2, 3, \ldots, q, \tag{3.7}$$

$$Y_\beta = \frac{\sum_{i=1}^{p} W_i^\beta \exp\,(\sum_{\gamma=2}^{q} c_\gamma V_i^\gamma)}{\sum_{i=1}^{p} \exp\,(\sum_{\gamma=2}^{q} c_\gamma V_i^\gamma)}, \quad \beta = q+1, \ldots, p. \tag{3.8}$$

The relations (3.7) define an application from $\mathbf{R}^{q-1}$ into $\mathbf{R}^{q-1}$ which associates $X_2, X_3, \ldots, X_q$ to $(c_2, c_3, \ldots, c_q)$. Let D_x be its image. This application is a bijection between $\mathbf{R}^{q-1}$ and D_x as it is easy to prove by using the bijective properties of the application defined by the relations (3.2) [4].

The relations (3.7) may be written

$$X_\alpha = X_\alpha(c_2, \dots, c_q), \qquad \alpha = 2, \dots, q, \tag{3.9}$$

with the Jacobian matrix

$$\begin{array}{rcl} \frac{\partial X_\alpha}{\partial c_\gamma} & = & (\sum_{i=1}^p \tilde{N}_i V_i^\alpha V_i^\gamma) - (\sum_{i=1}^p \tilde{N}_i\, V_i^\alpha)(\sum_{i=1}^p \tilde{N}_i V_i^\gamma) \\ \alpha & = & 2, \dots, q; \;\; \gamma = 2, \dots, q. \end{array} \tag{3.10}$$

In relations (3.10), we have introduced the reduced densities $\tilde{N}_i$ by:

$$N_i = n\tilde{N}_i \quad i = 1, 2, \dots, p. \tag{3.11}$$

In the same way, we write

$$Y_\beta = Y_\beta \;\; (c_2, \dots, c_q), \qquad \beta = q+1, \dots, p, \tag{3.12}$$

$$\begin{array}{rcl} \frac{\partial Y_\beta}{\partial c_\gamma} & = & (\sum_{i=1}^p \tilde{N}_i W_i^\beta V_i^\gamma) - (\sum_{i=1}^p \tilde{N}_i W_i^\beta)(\sum_{i=1}^p \tilde{N}_i V_i^\gamma) \\ \beta & = & q+1, \dots, p; \;\; \gamma = 2, \dots, q. \end{array} \tag{3.13}$$

The Jacobian matrix $(\partial X_\alpha / \partial c_\gamma)$, $\alpha = 2, \dots, q$, $\gamma = 2, \dots, q$, is a symmetric definite positive one. Indeed,

$$\begin{aligned} \sum_{\alpha=2}^{q} \sum_{\gamma=2}^{q} \frac{\partial X_\alpha}{\partial c_\gamma} \lambda_\alpha \lambda_\gamma &= \sum_{i=1}^{p} \tilde{N}_i \Big(\sum_{\alpha=2}^{q} \lambda_\alpha V_i^\alpha \Big) \Big(\sum_{\gamma=2}^{q} \lambda_\gamma V_i^\gamma \Big) \\ &- \Big[\sum_{i=1}^{p} \tilde{N}_i \, \Big(\sum_{\alpha=2}^{q} \lambda_\alpha V_i^\alpha \Big) \Big] \Big[\sum_{i=1}^{p} \tilde{N}_i \, \Big(\sum_{\gamma=2}^{q} \lambda_\gamma V_i^\gamma \Big) \Big] \\ &= \Big(\sum_{i=1}^{p} \tilde{N}_i \; \Lambda_i \Lambda_i \Big) - \Big(\sum_{i=1}^{p} \tilde{N}_i \Lambda_i \Big)^2 \\ &= \Big(\sum_{i=1}^{p} \Big(\tilde{N}_i^{1/2} \Big)^2 \Big) \Big(\sum_{i=1}^{p} \Big(\tilde{N}_i^{1/2} \Lambda_i \Big)^2 \Big) \\ &- \Big(\sum_{i=1}^{p} \Big(\tilde{N}_i^{1/2} \Big) \Big(\tilde{N}_i^{1/2} \Lambda_i \Big) \Big)^2 \end{aligned}$$

that is strictly positive for all $\Lambda_i = \sum_{\alpha=2}^{q} \lambda_\alpha V_i^\alpha$, except for $\Lambda_i = 0$, $i = 1, 2, \dots, p$, or, equivalently, $\lambda_\alpha = 0$, $\alpha = 2, \dots, q$, (we recall that $\sum_{i=1}^{p} \tilde{N}_i = 1$).

Now we emphasize the Maxwellian state near homogeneous state. In other words, we suppose that all the number densities N_i are close to n/p and that the quantities X_α, $\alpha = 2, \dots, q$ are very small of order ε, say. We remark from equations (3.2) and (3.3) that the two sequences $(c_1, 0, \dots, 0)$ and $(n, 0, \dots, 0)$ correspond each other when $c_1 = \sqrt{p}$ Log (n/p). Then, the Jacobian matrix $\partial X_\alpha / \partial c_\gamma)(0, \dots, 0)$ is the unit matrix.

By some algebra on the expressions (3.7), (3.8) and (3.5), (3.6) and by recalling that the base $\mathbf{V}^1, \dots, \mathbf{V}^q, \mathbf{W}^{q+1}, \dots, \mathbf{W}^p$ is an orthonormal base, we obtain the following results:

$$c_\alpha = pX_\alpha + O(\varepsilon^2), \qquad \alpha = 2, 3, \dots, q, \tag{3.14}$$

$$Y_\beta = O(\varepsilon^2), \quad \beta = q+1,\ldots,p, \tag{3.15}$$

$$N_i = n\Big(\frac{1}{p} + \sum_{\alpha=2}^{q} X_\alpha\, V_i^\alpha + O(\varepsilon^2)\Big), \quad i = 1,2,\ldots,p, \tag{3.16}$$

$$\exp\Big(\frac{c_1}{\sqrt{p}}\Big) = \frac{n}{p}\Big(1 + O(\varepsilon^2)\Big) \tag{3.17}$$

It is important to notice that the microscopic variables Y_β are small quantities of order two in ε at least. Then, we pay attention to the mean velocity $\vec{v}$ and to the pressure tensor $\mathbf{P}$ of the gas. From the hypotheses, $\vec{v}$ is small and of order εc, where c denotes the order of magnitude of the velocities $\vec{u}_i$, $i = 1,2,\ldots,p$. Indeed, $\vec{v}/c$ must be a macroscopic variable, that is to say a linear expression of the variables X_α, $\alpha = 2,\ldots,q$. To this end, we must take the initial set of velocities $\vec{u}_i$ such that $\sum_{i=1}^{p} \vec{u}_i = 0$, and so in the homogeneous state the gas is at rest. Then, for the pressure tensor, as defined by [3,4]:

$$\mathbf{P} = m \sum_{i=1}^{p} N_i\Big(\vec{u}_i - \vec{v}\Big)\Big(\vec{u}_i - \vec{v}\Big),$$

we obtain the approximate expression

$$\begin{aligned}\mathbf{P} &= mn\Big\{\Big(\sum_{i=1}^{p}\frac{1}{p}\,\vec{u}_i\vec{u}_i\Big) + \sum_{\alpha=2}^{q} X_\alpha\Big(\sum_{i=1}^{p} V_i^\alpha \vec{u}_i\vec{u}_i\Big)\\ &+ c^2 O(\varepsilon^2)\Big\}.\end{aligned} \tag{3.18}$$

In the homogeneous state, $N_i = n/p$, $\vec{v}$ is zero and the tensor $\mathbf{P}$ reduces to $\mathbf{P}_o = mn(\sum_{i=1}^{p}(1/p)\vec{u}_i\vec{u}_i)$. For the regular coplanar models with four or six velocities of magnitude c we have [5,6]

$$\mathbf{P}_o = mn\,\frac{c^2}{2}\,\mathbf{1}$$

and for the regular coplanar models with $2r$ velocities [4], we have also

$$\mathbf{P}_o = mn\,\frac{c^2}{2}\,\mathbf{1}.$$

Let us remark that if the tensor $\mathbf{P}_o = mn(\sum_{i=1}^{p}(1/p)\vec{u}_i\vec{u}_i)$ is spherical, then we have, necessarily, in the plane $\mathbf{P}_o = mn(c^2/2)\mathbf{1}$ and in the space $\mathbf{P}_o = mn(c^2/3)\mathbf{1}$ where $c^2 = \sum_{i=1}(1/p)\mid \vec{u}_i \mid^2$ [20].

At last, we give the Euler equations for a Maxwellian gas near the homogeneous state. These equations are the equations (2.9) with the densities N_i replaced by the expressions (3.16). We have

$$\begin{aligned}\tfrac{\partial}{\partial t}(nX_\alpha) &+ \Big(\textstyle\sum_{i=1}^{p} \frac{1}{p}\, V_i^\alpha \vec{u}_i\Big)\cdot\vec{\nabla}n\\ &+ \textstyle\sum_{\beta=2}^{q}\Big(\sum_{i=1}^{p} V_i^\alpha V_i^\beta \vec{u}_i\Big)\cdot\vec{\nabla}(nX_\beta) = 0\\ \alpha &= 1,2,\ldots,q.\end{aligned}$$

By recalling that $\sum_{i=1}^{p} \vec{u}_i = 0$ and by distinguishing the equation relative to the density n from the others, we obtain:

$$\begin{array}{rcl}\frac{\partial n}{\partial t} & + & \sum_{\gamma=2}^{q}\left(\sum_{i=1}^{p} \vec{u}_i V_i^{\gamma}\right)\cdot\vec{\nabla}\left(nX_{\gamma}\right)=0\\ \frac{\partial}{\partial t}(nX_{\alpha}) & + & \frac{1}{p}\left(\sum_{i=1}^{p} \vec{u}_i \vec{V}_i^{\alpha}\right)\cdot\vec{\nabla}n\\ & + & \sum_{\gamma=2}^{q}\left(\sum_{i=1}^{p} \vec{u}_i V_i^{\alpha} V_i^{\gamma}\right)\cdot\vec{\nabla}\left(nX_{\gamma}\right)=0\\ \alpha & = & 2,3,\ldots,q.\end{array} \tag{3.19}$$

4. Chapman-Enskog method

Let us take again the kinetic equations (2.3) and the conservation laws (2.9). We write these equations in an adimensional form, and we introduce the *Knudsen* number K_n [4]:

$$\frac{\partial}{\partial t}\,\mathrm{N} + \mathcal{A}\mathrm{N} = \frac{1}{K_n}\,\mathcal{F}(\mathrm{N},\mathrm{N}) \tag{4.1}$$

$$\frac{\partial}{\partial t}\,a_{\alpha} + \langle\,\mathcal{A}\mathrm{N},\mathrm{V}^{\alpha}\rangle = 0, \qquad \alpha = 1,2,\ldots,q. \tag{4.2}$$

The Chapman-Enskog method for the classical kinetic theory is explained, for example, in the paper of Grad [22]. This method is applied to discrete models of gas in reference 3. It gives the Euler and Navier-Stokes equations associated to the model. Here, we briefly recall this method: for the densities N and for the time derivatives of the macroscopic variables, we assume the following expansions:

$$\mathrm{N} = \mathrm{N}^{(0)}(a,Da) + K_n\mathrm{N}^{(1)}(a,Da) + \ldots \tag{4.3}$$

$$\begin{array}{rcl}\frac{\partial}{\partial t}\,a_{\alpha} & = & F_{\alpha}^{(0)}(a,Da) + K_nF_{\alpha}^{(1)}(a,Da) + \ldots\\ \alpha & = & 1,2,\ldots,q.\end{array} \tag{4.4}$$

Here, a and Da represent the macroscopic variables and their spatial derivatives. By substituting the expansions (4.3) and (4.4) in the equations (4.1) and (4.2), we obtain:

$$\mathcal{F}\left(\mathrm{N}^{(0)},\mathrm{N}^{(0)}\right) = 0, \tag{4.5}$$

$$2\mathcal{F}\left(\mathrm{N}^{(0)},\mathrm{N}^{(1)}\right) = \frac{\partial \mathrm{N}^{(0)}}{\partial t} + \mathcal{A}\mathrm{N}^{(0)}, \tag{4.6}$$

$$F_{\alpha}^{(0)} = -\langle\,\mathcal{A}\mathrm{N}^{(0)},\mathrm{V}^{\alpha}\rangle, \qquad \alpha = 1,2,\ldots,q, \tag{4.7}$$

$$F_{\alpha}^{(1)} = -\langle\,\mathcal{A}\mathrm{N}^{(1)},\mathrm{V}^{\alpha}\rangle, \qquad \alpha = 1,2,\ldots,q. \tag{4.8}$$

We must also impose the Chapman-Enskog conditions:

$$\begin{array}{rcl}\langle \mathrm{N}^{(0)},\mathrm{V}^{\alpha}\rangle & = & a_{\alpha}, \quad \langle \mathrm{N}^{(1)},\mathrm{V}^{\alpha}\rangle = 0,\\ \alpha & = & 1,2,\ldots,q.\end{array} \tag{4.9}$$

4.1 The first approximation $\mathbf{N}^{(0)}$

From equation (4.5), the densities $\mathbf{N}^{(0)}$ are Maxwellian. In the previous section, we studied such densities and we choose for them the expressions (3.5) with (3.6), (3.7), and (3.8). In particular, $\mathbf{N}^{(0)}$ depends on the macroscopic variables a_α alone.

4.2 The second approximation $\mathbf{N}^{(1)}$

To obtain $\mathbf{N}^{(1)}$ we must solve equation (4.6). In the left member, we put $\mathbf{N}^{(1)} = \mathbf{A}^{(0)}\mathbf{X}^{(1)}$, that is, $\mathrm{N}_i^{(1)} = \mathrm{N}_i^{(0)}\, \mathrm{X}_i^{(1)}$, $\mathrm{i} = 1, 2, \ldots, \mathrm{p}$, and we obtain

$$2\mathcal{F}(\mathbf{N}^{(0)}, \mathbf{A}^{(0)}\mathbf{X}^{(1)}) = \mathbf{I}^{(0)}\mathbf{X}^{(1)}. \tag{4.10}$$

The operator $\mathbf{I}^{(0)}$ is the linearized operator of collision about a Maxwellian state. This operator is symmetric and negative; it possesses the eigenvalue 0, and the eigenspace associated with it is the subspace $\mathbf{F}$ of the summational invariants. We give below the elements of the matrix $\mathbf{I}^{(0)}$ [4]:

$$\begin{aligned} I_{ij}^{(o)} &= \sum_{k=1}^{p}\sum_{l=1}^{p}\Big\{A_{jl}^{ik} N_j^{(o)} N_l^{(o)} - \frac{1}{2} A_{ij}^{kl} N_i^{(o)} N_j^{(o)} \\ &\quad - \frac{1}{2}\sum_{m=1}^{p} A_{im}^{kl} N_i^{(o)} N_m^{(o)} \delta_{ij}\Big\}. \end{aligned} \tag{4.11}$$

Let us return to the equations (4.6) and let us use the relations (4.4) at the order $0(1)$. We have

$$\mathbf{I}^{(0)}\mathbf{X}^{(1)} = \sum_{\alpha=1}^{q} \frac{\partial \mathbf{N}^{(o)}}{\partial a_\alpha} F_\alpha^{(o)} + \mathcal{A}\mathbf{N}^{(0)}. \tag{4.12}$$

The quantities $F_\alpha^{(0)}$ are given in equation (4.7) in such a manner that the compatibility conditions for the system (4.12) are satisfied. The solution of (4.12) is defined save on the addition of any summational invariant. With the conditions (4.9), the solution of (4.12) is then unique. We can take as unknown quantities the variables $b_\beta^{(1)}$ such that

$$\mathbf{N}^{(1)} = \sum_{\beta=q+1}^{p} b_\beta^{(1)} \mathbf{W}^\beta \tag{4.13}$$

and we have

$$\mathbf{X}^{(1)} = \mathbf{A}^{(0)-1}\mathbf{N}^{(1)} = \sum_{\beta=q+1}^{p} b_\beta^{(1)} \mathbf{A}^{(0)-1}\mathbf{W}^\beta.$$

The system (4.12) becomes

$$\mathbf{I}^{(0)}\mathbf{A}^{(0)-1}\Big(\sum_{\beta=q+1}^{p} b_\beta^{(1)}\mathbf{W}^\beta\Big) = \sum_{\alpha=1}^{q} \frac{\partial \mathbf{N}^{(0)}}{\partial a_\alpha} F_\alpha^{(0)} + \mathcal{A}\mathbf{N}^{(0)}. \tag{4.14}$$

Taking the aforesaid properties for the operator $\mathbf{I}^{(0)}$ into account, we see that the system (4.14) is an equality between two vectors of $\mathbf{F}^{\perp}$ and so is equivalent to

$$\begin{aligned} \textstyle\sum_{\beta=q+1}^{p} & \langle \mathbf{I}^{(0)}\mathbf{A}^{(0)-1}\mathbf{W}^{\beta}, \mathbf{W}^{\gamma}\rangle b_{\beta}^{(1)} \\ &= \langle \textstyle\sum_{\alpha=1}^{q} \frac{\partial \mathbf{N}^{(0)}}{\partial a_\alpha} F_{\alpha}^{(0)} + \mathcal{C}\mathbf{N}^{(0)}, \mathbf{W}^{\gamma}\rangle \\ \gamma &= q+1,\ldots,p, \end{aligned} \tag{4.15}$$

or, in condensed notations:

$$\sum_{\beta=q+1}^{p} \mathcal{B}_{\gamma\beta}\, b_{\beta}^{(1)} = \mathcal{B}_{\gamma}, \quad \gamma = q+1,\ldots,p. \tag{4.16}$$

4.3 Some remarks on the matrix $\mathcal{B}_{\gamma\beta}$.

It is easy to verify that

$$\mathcal{B}_{\gamma\beta} = \sum_{i=1}^{p}\sum_{j=1}^{p} I_{ij}^{(0)} \frac{1}{N_j^{(0)}} W_j^{\beta} W_i^{\gamma}. \tag{4.17}$$

Let us write $\Phi = \sum_{\beta=q+1}^{p} \lambda_\beta \mathbf{W}^{\beta}$. Using the value (4.11) for $I_{ij}^{(0)}$ and the Maxwellian properties of the densities $N_i^{(0)}$, we have

$$\begin{aligned} \sum_{\beta=q+1}^{p}\sum_{\gamma=q+1}^{p} \mathcal{B}_{\gamma\beta}\lambda_\gamma\lambda_\beta &= \sum_{i=1}^{p}\sum_{j=1}^{p} I_{ij}^{(0)} \frac{1}{N_j^{(0)}}\phi_j\phi_i \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} \frac{1}{2} A_{kl}^{ij}\Big\{ \frac{N_k^{(0)}N_l^{(0)}}{N_i^{(0)}}\phi_i(\phi_k+\phi_l) \\ &- \frac{N_i^{(0)}N_j^{(0)}}{N_i^{(0)}}\,\phi_i(\phi_k+\phi_l)\Big\} \\ &= -\frac{1}{8}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} A_{ij}^{kl}\Big(N_k^{(0)}\phi_l + N_l^{(0)}\phi_k - N_j^{(0)}\phi_i - N_i^{(0)}\phi_j\Big) \\ &(\phi_k+\phi_l-\phi_i-\phi_j) \ . \end{aligned} \tag{4.18}$$

Consequently, if the Maxwellian state $N^{(0)}$ is homogeneous ($N_i^{(0)} = \mathrm{n/p}$), then the last right-hand side of equation (4.18) is negative or zero, and the matrix $\mathcal{B}_{\gamma\beta}$ is then symmetric and negative definite.

The system (4.16) is a Cramer system because $\sum_{\beta=q+1}^{p} \mathcal{B}_{\gamma\beta}\lambda_\beta = 0$, $\gamma = q+1,\ldots,p$, has the unique solution $\lambda_\beta = 0$, $\beta = q+1,\ldots,p$. Indeed, $\mathbf{Y} = \sum_{\beta=q+1}^{p} \lambda_\beta \mathbf{A}^{(0)-1}\mathbf{W}_\beta$ belongs to $\mathbf{F}$ and $\mathbf{A}^{(0)}\ \mathbf{Y}$ to $\mathbf{F}$. Then, from the properties of $\mathbf{I}^{(0)}$, we conclude that $\mathbf{Y} = 0$, ([4], page 63).

Now, we pay attention to the right member of equation (4.15) or (4.16). We recall that the $F_{\alpha}^{(0)}$ are given by the relations (4.7), and that the densities $\mathbf{N}^{(0)}$ are Maxwellian densities. As previously, we introduce the variables $(n, X_2,\ldots,X_q)$ defined in (3.4) and we put according to (3.5)

$$\mathbf{N}^{(0)}(a_1,a_2,\ldots,a_q)=\hat{\mathbf{N}}^{(0)}(n,X_2,\ldots,X_q)=n\tilde{\mathbf{N}}^{(0)}(X_2,\ldots,X_q). \tag{4.19}$$

We have:

$$\begin{aligned}\frac{\partial \mathbf{N}^{(0)}}{\partial a_1} &= \sqrt{p}\Big(\frac{\partial \hat{\mathbf{N}}^{(0)}}{\partial n}-\textstyle\sum_{\alpha=2}^{q}\frac{X_\alpha}{n}\frac{\partial \hat{\mathbf{N}}^{(0)}}{\partial X_\alpha}\Big)\\ \frac{\partial \mathbf{N}^{(0)}}{\partial a_\alpha} &= \frac{1}{n}\frac{\partial \hat{\mathbf{N}}^{(0)}}{\partial X_\alpha}, \qquad \alpha=2,3,\ldots,q.\end{aligned} \tag{4.20}$$

By introducing the last expressions (4.20) and the expressions (4.7) for $F_\alpha^{(0)}$, in the definition of B_γ, we obtain:

$$\begin{aligned}B_\gamma=\sum_{j=1}^{p}\Big\{ &-\sqrt{p}\Big(\frac{\partial \hat{N}_j^{(0)}}{\partial n}-\sum_{\alpha=2}^{q}\frac{X_\alpha}{n}\frac{\partial \hat{N}_j^{(0)}}{\partial X_\alpha}\Big)\\ &\cdot\Big(\sum_{i=1}^{p}(\vec{u}_i\cdot\vec{\nabla}\hat{N}_i^{(0)})V_i^1\Big)\\ &-\sum_{\alpha=2}^{q}\frac{1}{n}\frac{\partial \hat{N}_j^{(0)}}{\partial X_\alpha}\Big(\sum_{i=1}^{p}(\vec{u}_i\cdot\vec{\nabla}\hat{N}_i^{(0)})V_i^\alpha\Big)\\ &+\vec{u}_j\cdot\vec{\nabla}\hat{N}_j^{(0)}\Big\}W_j^\gamma.\end{aligned} \tag{4.21}$$

We have $V_i^1=(1/\sqrt{p})$, $i=1,2,\ldots,p$, $\hat{N}_i^{(0)}=n\tilde{N}_i^{(0)}$, and $\vec{\nabla}\hat{N}_i^{(0)}=\tilde{N}_i^0\vec{\nabla}n+\sum_{\alpha=2}^{q}n(\partial\tilde{N}_i^{(0)}/\partial X_\alpha)\vec{\nabla}X_\alpha$. Therefore,

$$\begin{aligned}B_\gamma &= -\sum_{j=1}^{p}\Big\{\Big(\tilde{N}_j^{(0)}-\sum_{\alpha=2}^{q}X_\alpha\frac{\partial\tilde{N}_j^{(0)}}{\partial X_\alpha}\Big)\Big(\sum_{i=1}^{p}\Big(\tilde{N}_i^{(0)}\vec{u}_i\cdot\vec{\nabla}n\\ &+ n\sum_{\beta=2}^{q}\frac{\partial\tilde{N}_i^{(0)}}{\partial X_\beta}\,\vec{u}_i\cdot\vec{\nabla}X_\beta\Big)\Big)\\ &+ \sum_{\alpha=2}^{q}\frac{\partial\tilde{N}_j^{(0)}}{\partial X_\alpha}\Big(\sum_{i=1}^{p}\Big(\tilde{N}_i^{(0)}\vec{u}_i\cdot\vec{\nabla}n+n\sum_{\beta=2}^{q}\frac{\partial\tilde{N}_i^{(0)}}{\partial X_\beta}\vec{u}_i\cdot\vec{\nabla}X_\beta\Big)V_i^\alpha\Big)\\ &- \Big(\tilde{N}_j^{(0)}\vec{u}_j\cdot\vec{\nabla}n+n\sum_{\beta=2}^{q}\frac{\partial\tilde{N}_j^{(0)}}{\partial X_\beta}\vec{u}_j\cdot\vec{\nabla}X_\beta\Big)\Big\}W_j^\gamma.\end{aligned} \tag{4.22}$$

We are interested by the coefficient of $\vec{\nabla}n$ in the expression of B_γ. Let us denote it by $\vec{K}_\gamma$.

$$\begin{aligned}\vec{K}_\gamma &= \sum_{j=1}^{p}W_j^\gamma\Big\{\tilde{N}_j^{(0)}\vec{u}_j-\tilde{N}_j^{(0)}\Big(\sum_{i=1}^{p}\tilde{N}_i^{(0)}\vec{u}_i\Big)\\ &- \sum_{\beta=2}^{q}\frac{\partial\tilde{N}_j^{(0)}}{\partial X_\beta}\Big(\sum_{i=1}^{p}\tilde{N}_i^{(0)}\vec{u}_iV_i^\beta\Big)\\ &+ \sum_{\beta=2}^{q}X_\beta\frac{\partial\tilde{N}_j^{(0)}}{\partial X_\beta}\Big(\sum_{i=1}^{p}\tilde{N}_i^{(0)}\vec{u}_i\Big)\Big\}.\end{aligned} \tag{4.23}$$

But

$$\begin{array}{rcl} nX_\alpha & = & n\sum_{i=1}^p \tilde{N}_i^{(0)} V_i^\alpha \\ nY_\beta^{(0)} & = & b_\beta^{(0)} = n\sum_{i=1}^p \tilde{N}_i^{(0)} W_i^\beta \end{array}$$

and we have

$$\begin{array}{rcl} \vec{K}_\gamma & = & \left(\sum_{i=1}^p \tilde{N}_i^{(0)} W_i^\gamma \vec{u}_i\right) - \left(\sum_{i=1}^p \tilde{N}_i^{(0)} W_i^\gamma\right)\left(\sum_{i=1}^p \tilde{N}_i^{(0)} \vec{u}_i\right) \\ & - & \sum_{\beta=2}^q \frac{\partial Y_\gamma^{(0)}}{\partial X_\beta}\left\{\left(\sum_{i=1}^p \tilde{N}_i^{(0)} V_i^\beta \vec{u}_i\right)\right. \\ & - & \left.\left(\sum_{i=1}^p \tilde{N}_i^{(0)} V_i^\beta\right)\left(\sum_{i=1}^p \tilde{N}_i^{(0)} \vec{u}_i\right)\right\}. \end{array} \tag{4.24}$$

The sequence $\vec{u}_i, i = 1, 2, \ldots, p$ is a summational invariant; then we can write

$$\vec{u}_i = \sum_{\delta=1}^q \vec{k}_\delta V_i^\delta,$$

where $\vec{k}_\delta$, $\delta = 1, 2, \ldots, q$, is a sequence of vectors of $\mathbf{R}^3$. We substitute this expression for $\vec{u}_i$ in (4.24) and we obtain

$$\begin{array}{l} \vec{K}_\gamma = \sum_{\delta=1}^q \vec{k}_\delta\left\{\left(\sum_{i=1}^p \tilde{N}_i^{(0)} W_i^\gamma V_i^\delta\right) - \left(\sum_{i=1}^p \tilde{N}_i^{(0)} W_i^\gamma\right)\left(\sum_{i=1}^p \tilde{N}_i^{(0)} V_i^\delta\right)\right. \\ \left. - \sum_{\beta=2}^q \frac{\partial Y_\gamma^{(0)}}{\partial X_\beta}\left[\left(\sum_{i=2}^p \tilde{N}_i^{(0)} V_i^\beta V_i^\delta\right) - \left(\sum_{i=1}^p \tilde{N}_i^{(0)} V_i^\beta\right)\left(\sum_{i=1}^p \tilde{N}_i^{(0)} V_i^\delta\right)\right]\right\} \end{array}$$

The coefficient of $\vec{k}_\delta$ is zero: if $\delta = 1$ it is clear because $V_i^1 = (1/\sqrt{p})$ and $\sum_{i=1}^p \tilde{N}_i^{(0)} = 1$; if $\delta = 2, \ldots, q$ this result is a consequence of the relations (3.10) and (3.13) because

$$\vec{K}_\gamma = \sum_{\delta=2}^q \vec{k}_\delta\left\{\frac{\partial Y_\gamma^{(0)}}{\partial c_\delta} - \sum_{\beta=2}^q \frac{\partial Y_\gamma^{(0)}}{\partial X_\beta}\frac{\partial X_\beta}{\partial c_\delta}\right\} = 0.$$

In conclusion, $\vec{K}_\gamma = 0$ and the coefficient of $\vec{\nabla} n$ in B_γ is 0. *The right member B_γ in equation (4.16) does not depend on $\vec{\nabla} n$.* The expression for B_γ is a linear combination of $\vec{\nabla} X_\alpha$, $\alpha = 2, \ldots, q$ alone.

Let us put

$$\begin{array}{rcl} \tilde{N}_i^{(0)} \vec{u}_i & = & \sum_{\alpha=1}^q \vec{Z}_\alpha^{(0)} V_i^\alpha + \sum_{\beta=q+1}^p \vec{Z}_\beta^{(0)} W_i^\beta \\ i & = & 1, 2, \ldots, p. \end{array} \tag{4.25}$$

Naturally, $\vec{Z}_1^{(0)}$ is equal to $(1/\sqrt{p})\vec{v}$, and with $\tilde{N}_i^{(0)}$, the quantities $\vec{Z}_\alpha^{(0)}$, $\alpha = 1, 2, \ldots, p$, depend on the variables $X_2, \ldots, X_q$ only. Then, we can write B_γ in the form:

$$\begin{aligned} B_\gamma &= n\Big(-Y_\gamma^{(0)} + \sum_{\alpha=2}^{q} X_\alpha \frac{\partial Y_\gamma^{(0)}}{\partial X_\alpha}\Big)\Big(\sum_{\beta=2}^{q} \frac{\partial \vec{v}}{\partial X_\beta} \cdot \vec{\nabla} X_\beta\Big) \\ &- n \sum_{\alpha=2}^{q} \frac{\partial Y_\gamma^{(0)}}{\partial X_\alpha}\Big(\sum_{\beta=2}^{q} \frac{\partial \vec{Z}_\alpha^{(0)}}{\partial X_\beta} \cdot \vec{\nabla} X_\beta\Big) \\ &+ n \sum_{\beta=2}^{q} \frac{\partial \vec{Z}_\gamma^{(0)}}{\partial X_\beta} \cdot \vec{\nabla} X_\beta \end{aligned}$$

$$\left\{ \begin{aligned} B_\gamma &= n \textstyle\sum_{\beta=2}^{q} \Big\{ \Big(-Y_\gamma^{(0)} + \sum_{\alpha=2}^{q} X_\alpha \frac{\partial Y_\gamma^{(0)}}{\partial X_\alpha} \Big) \frac{\partial \vec{v}}{\partial X_\beta} \\ &- \Big(\textstyle\sum_{\alpha=2}^{q} \frac{\partial Y_\gamma^{(0)}}{\partial X_\alpha} \frac{\partial \vec{Z}_\alpha^{(0)}}{\partial X_\beta} \Big) + \frac{\partial \vec{Z}_\gamma^{(0)}}{\partial X_\beta} \Big\} \cdot \vec{\nabla} X_\beta \\ \gamma &= q+1, \ldots, p. \end{aligned} \right. \tag{4.26}$$

In conclusion, we have explicited the right member of the system (4.16). Let us denote by $\mathcal{B}_{\beta\gamma}^{-1}$ the elements of the inverse matrix of the matrix $\mathcal{B}_{\beta\gamma}$. The solution of equation (4.16) is

$$b_\beta^{(1)} = \sum_{\gamma=q+1}^{p} \mathcal{B}_{\beta\gamma}^{-1} B_\gamma. \tag{4.27}$$

In expression (4.26), we see that B_γ is a linear combination of the gradients $\vec{\nabla} X_\beta$, $\beta = 2, \ldots, q$ and does not depend on the gradient $\vec{\nabla} n$. The matrix element $\mathcal{B}_{\gamma\beta}$ is a homogeneous function of order + 1 of the densities $N_i^{(0)}$; therefore, $\mathcal{B}_{\beta\gamma}^{-1}$ is a homogeneous function of order + 1 of the densities $N_i^{(0)}$ and is written as a product of the inverse density $1/n$ by a homogeneous function of order - 1 of the $\tilde{N}_i^{(0)}$. As a consequence, $b_\beta^{(1)}$ *does depend neither on* n, *nor on* $\vec{\nabla} n$. The Navier-Stokes equations associated to the model are the conservation (4.2) with $\mathbf{N} = \mathbf{N}^{(0)} + K_n \mathbf{N}^{(1)}$, where $\mathbf{N}^{(0)}$ is the local Maxwellian state associated to the macroscopic variables a_α, $\alpha = 1, 2, \ldots, q$, and where $\mathbf{N}^{(1)}$ is equal to $\sum_{\beta=q+1}^{p} b_\beta^{(1)} W^\beta$ with $b_\beta^{(1)}$ given by equation (4.27).

5. The Navier-Stokes equations near the homogeneous state

Now we investigate the Navier-Stokes equations in the case where the gas is near a homogeneous state, that is, where each density is close to the value (n/p). The Maxwellian densities $N_i^{(0)}$ with value close to (n/p) have been studied in section 3. From equations (3.15), (3.16), and (4.19), we have

$$\begin{aligned} N_i^{(0)} &= n\Big(\tfrac{1}{p} + \textstyle\sum_{\alpha=2}^{q} X_\alpha V_i^\alpha + 0(\varepsilon^2)\Big) = n\tilde{N}_i^{(0)} \\ i &= 1, 2, \ldots, p, \end{aligned} \tag{5.1}$$

$$Y_\gamma^{(0)} = 0(\varepsilon^2), \qquad \gamma = q+1,\ldots,p, \tag{5.2}$$

and from equation (4.25) and the hypothesis $\sum_{i=1}^p \vec{u}_i = 0$, we have

$$\begin{aligned} \vec{v} = \sqrt{p}\ \vec{Z}_1^{(0)} &= \sum_{i=1}^{p} \tilde{N}_i^{(0)} \vec{u}_i \\ &= \sum_{\alpha=2}^{q} X_\alpha \Big(\sum_{i=1}^{p} \vec{u}_i V_i^\alpha \Big) + O(\varepsilon^2) \end{aligned} \tag{5.3}$$

$$\begin{aligned} \vec{Z}_\beta^{(0)} &= \textstyle\sum_{i=1}^p \tilde{N}_i^{(0)} \vec{u}_i V_i^\beta \\ &= \tfrac{1}{p}\Big(\textstyle\sum_{i=1}^p \vec{u}_i V_i^\beta \Big) + \textstyle\sum_{\alpha=2}^q X_\alpha \Big(\textstyle\sum_{i=1}^p \vec{u}_i V_i^\beta V_i^\alpha \Big) + O(\varepsilon^2) \\ \beta &= 1,2,\ldots,q; \end{aligned} \tag{5.4}$$

$$\begin{aligned} \vec{Z}_\gamma^{(0)} &= \textstyle\sum_{i=1}^p \tilde{N}_i^{(0)} \vec{u}_i W_i^\gamma \\ &= \textstyle\sum_{\alpha=2}^q X_\alpha \Big(\textstyle\sum_{i=1}^p \vec{u}_i W_i^\gamma V_i^\alpha \Big) + O(\varepsilon^2) \\ \gamma &= q+1\ldots,p. \end{aligned} \tag{5.5}$$

In equation (5.5), we have used the property for the sequence $\vec{u}_i$ to be a summational invariant and consequently to be orthogonal to W_i^γ.

Let us return to the expression (4.26) for B_γ. With equations (5.2), (5.3), (5.4), and (5.5) we have

$$\begin{aligned} \frac{\partial \vec{v}}{\partial X_\alpha} &= O(1), \frac{\partial Y_\gamma^{(0)}}{\partial X_\alpha} = O(\varepsilon), \frac{\partial \vec{Z}_\beta^{(0)}}{\partial X_\alpha} = O(1), \\ \frac{\partial \vec{Z}_\gamma^{(0)}}{\partial X_\alpha} &= \textstyle\sum_{i=1}^p \vec{u}_i W_i^\gamma V_i^\alpha + O(\varepsilon), \\ \alpha &= 2,3,\ldots,q; \ \ \beta = 1,2,\ldots,q; \ \ \gamma = q+1,\ldots,p. \end{aligned} \tag{5.6}$$

By assuming that the quantities X_α and $\vec{\nabla} X_\alpha$ are of the same order, we have

$$B_\gamma = n\Big\{ \sum_{\alpha=2}^{q} \Big(\sum_{i=1}^{p} \vec{u}_i W_i^\gamma V_i^\alpha \Big) \cdot \vec{\nabla} X_\alpha + O(\varepsilon) \Big\}. \tag{5.7}$$

With the densities given in equation (5.1), it is easy to see that the expression (4.17) for the $\mathcal{B}_{\gamma\beta}$ elements has the following form

$$\mathcal{B}_{\gamma\beta} = n(\mathring{\mathcal{B}}_{\beta\delta}^{-1}$$

with

$$\mathring{\mathcal{B}}_{\beta\delta}^{-1} \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} \frac{1}{p} \Big\{ A_{jl}^{ik} - \frac{1}{2} A_{ij}^{kl} - \frac{1}{2} \sum_{m=1}^{p} A_{im}^{kl} \delta_{ij} \Big\} W_j^\beta W_i^\gamma.$$

Then

$$\sum_{\beta=q+1}^{p} \mathring{\mathcal{B}}_{\gamma\beta}\, b_\beta^{(1)} = \sum_{\alpha=2}^{q} \Big(\sum_{i=1}^{p} \vec{u}_i W_i^\gamma V_i^\alpha \Big) \cdot \vec{\nabla} X_\alpha + O(\varepsilon).$$

We write the solution in the form:

$$\begin{aligned} b_\beta^{(1)} &= \textstyle\sum_{\alpha=2}^q \vec{K}_{\beta\alpha} \cdot \vec{\nabla} X_\alpha + O(\varepsilon) \\ \beta &= q+1\ldots,p. \end{aligned} \tag{5.8}$$

In conclusion, for the densities we have

$$\begin{aligned} N_i &= n\Big\{\frac{1}{p} + \sum_{\alpha=2}^q X_\alpha V_i^\alpha \\ &+ K_n \frac{1}{n} \sum_{\beta=q+1}^p \Big(\sum_{\alpha=2}^q \vec{K}_{\beta\alpha} \cdot \vec{\nabla} X_\alpha\Big) W_i^\beta + O(\varepsilon)\Big\} \end{aligned} \tag{5.9}$$

Last, we give the Navier-Stokes equations for the general model and for a gas near the homogeneous state. We write these equations in the dimensional form (formally, we make $K_n = 1$):

$$\begin{aligned} \tfrac{\partial}{\partial t}(nX_\alpha) &+ \textstyle\sum_{i=1}^p V_i^\alpha \vec{u}_i \cdot \vec{\nabla}\Big(\frac{n}{p} + \sum_{\gamma=2}^q nX_\gamma\Big) \\ &+ \textstyle\sum_{i=1}^p V_i^\alpha \vec{u}_i \cdot \vec{\nabla}\Big(\sum_{\beta=q+1}^p \Big(\sum_{\gamma=2}^q \vec{K}_{\beta\gamma} \cdot \vec{\nabla} X_\gamma\Big) W_i^\beta\Big) = 0 \\ \alpha &= 1,2,\ldots,q. \end{aligned}$$

The equation with $\alpha = 1$ is the equation for the density n; this equation does not contain the second derivatives of X_γ, $\gamma = 2,\ldots,q$. Indeed,

$$\sum_{i=1}^p \frac{1}{\sqrt{p}}\, \vec{u}_i \cdot \vec{\nabla}\Big(\sum_{\beta=q+1}^p \Big(\sum_{\gamma=2}^q \vec{K}_{\beta\gamma} \cdot \vec{\nabla} X_\gamma\Big) W_i^\beta\Big) = 0$$

because

$$\sum_{i=1}^p \vec{u}_i W_i^\beta = 0.$$

At last, by distinguishing the equation for the density n from the others, we have:

$$\begin{aligned} \tfrac{\partial n}{\partial t} &+ \textstyle\sum_{\gamma=2}^q \Big(\sum_{i=1}^p \vec{u}_i V_i^\gamma\Big) \cdot \vec{\nabla}(nX_\gamma) = 0 \\ \tfrac{\partial(nX_\alpha)}{\partial t} &+ \tfrac{1}{p}\Big(\textstyle\sum_{i=1}^p \vec{u}_i V_i^\alpha\Big) \cdot \vec{\nabla} n \\ &+ \textstyle\sum_{\gamma=2}^q \Big(\sum_{i=1}^p \vec{u}_i V_i^\alpha V_i^\gamma\Big) \cdot \vec{\nabla}\Big(nX_\gamma\Big) \\ &- \textstyle\sum_{\gamma=2}^q \mathbf{\Lambda}_{\alpha\gamma} : \vec{\nabla}\vec{\nabla}(X_\gamma) = 0 \end{aligned} \tag{5.10}$$

with

$$\begin{aligned} \mathbf{\Lambda}_{\alpha\gamma} &= -\sum_{i=1}^p \sum_{\beta=q+1}^p V_i^\alpha W_i^\beta \vec{u}_i \vec{K}_{\beta\gamma} \\ &= -\sum_{\beta=q+1}^p \Big(\sum_{i=1}^p V_i^\alpha W_i^\beta \vec{u}_i\Big)\Big(\sum_{\delta=q+1}^p \overset{\circ}{B}{}_{\beta\delta}^{-1}\Big(\sum_{j=1}^p V_j^\gamma W_j^\delta \vec{u}_j\Big)\Big) \\ &= -\sum_{\delta=q+1}^p \sum_{\beta=q+1}^p \overset{\circ}{B}{}_{\beta\delta}^{-1}\Big(\sum_{i=1}^p V_i^\alpha W_i^\beta \vec{u}_i\Big)\Big(\sum_{j=1}^p V_j^\gamma W_j^\delta \vec{u}_j\Big). \end{aligned} \tag{5.11}$$

Thus, for the general discrete velocity model we have defined and calculated the transport coefficients. We remark that the tensor of order two $\mathbf{\Lambda}_{\alpha\gamma}$ $(\alpha, \gamma = 2, \ldots, q)$ is a symmetric tensor in the indexes α and γ. Moreover, if α and γ are fixed, $\mathbf{\Lambda}_{\alpha\gamma}$ is a symmetric tensor of order two in the physical space. We call the tensor $\mathbf{\Lambda}_{\alpha\gamma}$ *the viscosity tensor.*

The matrix $\overset{\circ}{\mathcal{B}}{}^{-1}_{\beta\delta}$ is a definite negative matrix. By using this property, it is easy to show the inequality:

$$\sum_{\alpha=2}^{q} \sum_{\gamma=2}^{q} (\mathbf{\Lambda}_{\alpha\gamma} \lambda_\alpha \lambda_\gamma) : \vec{A}\vec{A} \geq 0$$
$$\forall\, \lambda_\alpha \in R \ , \ \alpha = 2, \ldots, q \ ; \ \forall \vec{A} \in \mathbf{R}^3.$$

Consequently, the viscosity tensor corresponds to dissipative phenomena.

In references 5 and 6, we have given the tensor pressure in the Navier-Stokes approximation for two particular models, first for a coplanar model with four velocities, and second for a regular coplanar model with six velocities. In the two cases, we have verified that the pressure tensor $\mathbf{P} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)}$ is such that $\mathbf{P}^{(1)}$ does not depend on n and $\vec{\nabla} n$. Using the notations of reference 4 with the four velocity model and with the hypothesis $\mid \vec{U} \mid \ll c$, we have:

$$\mathbf{P} \simeq \rho \, \frac{c^2}{2} \, \mathbf{1} - \frac{mc}{2\sigma} \, \mathbf{E}_o : \vec{\nabla} \vec{U},$$

where $\mathbf{E}_o$ is a tensor of order two related with the geometry of the model. With the six-velocity model and with the hypothesis $\mid \vec{u} \mid \ll c$ and $\mid \delta \mid \ll 1$, we have:

$$\mathbf{P} \simeq \rho \, \frac{c^2}{2} \, \mathbf{1} \ - \ \frac{3mc}{8\sigma} \Big\{ \Big(c\vec{e}_o \cdot \vec{\nabla}\delta + \mathbf{E}_o : \vec{\nabla}\vec{U} \Big) \mathbf{E}_o - \Big(c\vec{e}_1 \cdot \vec{\nabla}\delta + \mathbf{E}_1 : \vec{\nabla}\vec{U} \Big) \mathbf{E}_1 \Big\},$$

where $\mathbf{E}_o$, $\mathbf{E}_1$ and $\vec{e}_0$, $\vec{e}_1$ are respectively two tensors of order two and two vectors related with the geometry of the model. The theory presented here is a generalization of these results. Let us notice that these results are to compare to those of J. P. Rivet and U. Frisch [20] and to those of M. Hénon [21] for a hexagonal lattice gas.

6. Conclusion

In conclusion, we emphasize that the discrete kinetic equations have a good structure. The Chapman-Enskog method has been applied and has exhibited the Euler and Navier-Stokes equations associated with the model. We recall that the Euler equation system is a hyperbolic system of conservation laws [4]. Here, we have given the Navier-Stokes equations. Near the homogeneous state, we have explicitly given the transport coefficients and shown that these coefficients really correspond to dissipative phenomena.

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