# Transformations on Graphs and Convexity 

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#### Abstract

Graph transformations that may be reduced to the form $x(t+1)=x(t)+\nabla u(A x(t))$, with $x(t), x(t+1) \in R^{n}, u$ convex function and $A$ symmetric matrix, are studied. In particular, a reduction to the class is exhibited for some graph transformations recently introduced by Odlyzko and Randall. Further results on periods, quasi-periods, and pre-periods are presented. A class of multi-threshold transformations is introduced.


## 1. Introduction

We investigate transformations of the form

$$
\begin{equation*}
x(t+1)=x(t)+f(A x(t)), x(t) \in R^{n} \tag{1.1}
\end{equation*}
$$

where $A$ is a symmetric matrix of size $n \times n$, and $f$ is the gradient (or a subgradient) of a convex function $u$ on $R^{n}$, satisfying $u(-x)=u(x)$ for all $x \in R^{n}$.

Given an initial $x(0) \in R^{n}$, the sequence $\{x(t)\}_{t=0}^{\infty}$ received by successive iterations of (1.1) is called the trajectory of $x(0)$. We prove

Theorem 1.1. Let $A$ be a symmetric matrix and $f=\nabla u$ be the gradient of a convex and even function $u$. If a trajectory $\{x(t)\}_{t=0}^{\infty}$ is periodic, then the period is either one or two.

The paper is motivated by the recent paper [4] on the graph transformations of the following type. Let $G$ be a graph, and let an initial value be assigned to each vertex. The values of vertices are simultaneously updated such that the value of a vertex is moved by one in the direction of the average of the values of the neighboring vertices. A special rule is applied when equality holds.

We discuss possible applications of theorem 1.1 to these transformations in section 2. Theorem 1.1 provides us with a large range of possibilities to define other "reasonable" graph transformations by specifying the convex
function $u$ and its subgradient $f$. In particular, we introduce a class of multithreshold transformations.

An important tool for the study of transformations (1.1) is an "energy function"

$$
\begin{equation*}
E(t)=-x(t-1) A x(t), \tag{1.2}
\end{equation*}
$$

basically introduced in [1]. It has been shown that $\Delta E(t)=E(t+1)-$ $E(t) \leq 0$ for the transformations studied in [4]. In section 3, we show that $\Delta E(t)=0$ may happen for at most two consecutive iterations before the period is reached. This fact immediately bounds the number of steps before a transformation reaches its period. Transformations with $\Delta E(t) \leq 0$ for all $t$, the positive transformations, were studied in [2]. We show by an example that the multi-threshold transformations defined in section 2 are not positive. A different method is used to bound the number of iterations before they enter their periods.

The upper bound on pre-periods given in section 3 is of size $0\left(M^{2}\right)$ where $M$ is the maximum over $\left|x_{i}(0)\right|$, the initial values of vertices. Bounds of size $0(M)$ for the transformations from [4] were obtained in [3] by developing a new technique.

Clearly, a trajectory cannot be periodic unless it is bounded. On the other hand, if it is bounded and has integer values, it must have some period. In section 4, we show that a trajectory retains certain periodic properties even if the assumption about integrality is dropped and $x(t)$ may attain infinitely many distinct values. We prove in theorem 4.1 that a bounded trajectory always has a quasi-period of length one or two; i.e.,

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\|x(t+2)-x(t)\|=0 . \tag{1.3}
\end{equation*}
$$

Clearly, theorem 4.1 generalizes theorem 1.1. However, we will prove the latter separately since its proof is substantially simpler.

The paper uses the technique developed in [5-9] for the transformations of type

$$
\begin{equation*}
x(t+1)=f(A x(t)) . \tag{1.4}
\end{equation*}
$$

The main idea is that the behavior of some cellular automata may be best understood when its state at time $t$ is encoded by a vector $x(t) \in R^{n}$, the structure of the underlying network by a matrix $A$, and the way of forming a new state from the influences of neighbors by a gradient (or a subgradient) function $f$ of some convex function $u$.

We conclude the introduction with some definitions and notations that will be used in the paper. $R^{n}$ and $Z^{n}$ denote the $n$-dimensional Euclidean space and its lattice of integer points respectively. Let $\{x(t)\}_{t=0}^{\infty}$ be a sequence of points of $R^{n}$. We say that some $p>0$ is the period of the sequence, if

$$
\begin{equation*}
x(t+p)=x(t) \text { for all } t \text { sufficiently large, } \tag{1.5}
\end{equation*}
$$

and (1.5) does not hold for any smaller $p^{\prime}$. The sequence is integral if $x(t) \in$ $Z^{n}$ for all $t$, and it is bounded if there is some $M$ such that $\|x(t)\| \leq M$ for all $t$ where $\|$.$\| is some norm. We set \|A\|=\sum_{i, j}\left|a_{i j}\right|$ for a matrix $A=\left(a_{i j}\right)$. Mappings $u$ and $f$ are said to be even and odd, if $u(x)=u(-x)$ and $f(-x)=-f(x)$ for all $x$, respectively.

Let $u$ be a convex function on $R^{n}$. A vector $\xi$ is called a subgradient of $u$ at $x$ if

$$
\begin{equation*}
u(y)-u(x) \geq(y-x) \xi \text { for all } y \in R^{n} \tag{1.6}
\end{equation*}
$$

If the subgradient $\xi$ at $x$ is unique, then it is the gradient. If $u$ is convex and differentiable, then the gradient $\nabla u(x)$ is the vector of partial derivatives in $x$. We should mention that the gradient $\nabla u$ defines a mapping $R^{n} \rightarrow R^{n}$. For other notions and results of convex analysis, see reference [10].

We will often work with the "cyclic" sums like

$$
\sum_{t=a+1}^{b}(x(t)-x(t-1)) f(x(t))+(x(a)-x(b)) f(x(a))
$$

In order to abbreviate this lengthy notation, we will write

$$
\sum_{t=a}^{b}(x(t)-x[t-1]) f(x(t))
$$

instead of the above expression, where the square brackets indicate that the term inside should be replaced by its equivalent modulo $b-a+1$ within the interval $[a, b]$.

The notations $A x$ and $x y$ mean, respectively, the multiplication of a matrix $A$ by a vector $x$ and the inner product of $x$ and $y$.

## 2. Periods of graph transformations

In their recent paper [4], Odlyzko and Randall investigated the following transformations on the vertices of an undirected graph. Let $G$ be a graph with vertices $i=1, \ldots, n$. Some initial integer value $x_{i}(0)$ is assigned to each vertex $i$, and the values attached to the vertices are simultaneously updated by the following rule

$$
x_{i}(t+1)=\left\{\begin{array}{c}
x_{i}(t)-1  \tag{2.1}\\
x_{i}(t)+1
\end{array}\right\} \begin{aligned}
& \text { if the average value of } \\
& \text { the neighbors of } i \text { is }
\end{aligned}\left\{\begin{array}{c}
\text { less } \\
\text { greater }
\end{array}\right\} \text { than } x_{i}(t)
$$

In case $x_{i}(t)$ equals the average, one of the following rules (2.2), (2.3), or (2.4) is applied.

$$
\begin{align*}
& x_{i}(t+1)=x_{i}(t),  \tag{2.2}\\
& x_{i}(t+1)=x_{i}(t)+1 \tag{2.3}
\end{align*}
$$

$$
x_{i}(t+1)=\left\{\begin{array}{l}
x_{i}(t) \text { if } x_{i}(t)=x_{j}(t) \text { for each neighbor } j \text { of } i,  \tag{2.4}\\
x_{i}(t)+1 \text { otherwise. }
\end{array}\right.
$$

Odlyzko and Randall proved
Theorem 2.1. ([4]) For any graph $G$ and any assignment $x_{i}(0)$ of initial integer values to the vertices, the transformation defined by (2.1) and one of (2.2), (2.3), or (2.4) has a period of length at most two.

An argument from [4] shows that the trajectory $\{x(t)\}_{t=0}^{\infty}$ is bounded. It relies on the observation that $\max _{i} x_{i}(t)$ does not increase and $\min _{i} x_{i}(t)$ does not decrease as $t$ varies. Since the trajectory is bounded and integral, it must have some period. We show that the period is at most two by applying theorem 1.1.

Corollary 2.2. Theorem 2.1 holds for the transformation given by (2.1) and (2.2).

Proof. Let $G$ be a graph on $n$ vertices and $x(t)=\left(x_{1}(t), \ldots, x_{n}(t)\right)$ be the vector of the values of vertices at time $t$. Let $A=\left(a_{i j}\right)$ be the $n \times n$ matrix with the entries

$$
a_{i j}=\left\{\begin{array}{cl}
1 & \text { if } i \neq j \text { and } i j \text { is an edge of } G,  \tag{2.5}\\
-d_{i} & \text { if } i=j \text { where } d_{i} \text { is the degree of vertex } i, \\
0 & \text { otherwise. }
\end{array}\right.
$$

Let $g$ be the real function of one variable defined by

$$
g(x)= \begin{cases}\left(1+x^{2}\right) / 2 & \text { for }|x| \leq 1  \tag{2.6}\\ |x| & \text { otherwise }\end{cases}
$$

Clearly, $g$ is convex, differentiable, and even. The derivative $g^{\prime}$ of $g$ satisfies $g^{\prime}(x)=-1$ for $x \leq-1, g^{\prime}(x)=1$ for $x \geq 1$, and $g^{\prime}(0)=0$. Let us define the function $u(x)$, for $x \in R^{n}$, by $u\left(x_{1}, \ldots, x_{n}\right)=\sum g\left(x_{i}\right)$. It follows from the properties of $g$ that $u$ satisfies the assumptions of theorem 1.1. Now, it is easy to see that the considered transformation satisfies $x(t+1)=x(t)+\nabla u(A x(t))$. Hence, $\{x(t)\}$ has a period of at most two by theorem 1.1.

Corollary 2.3. Theorem 2.1 holds for the transformation given by (2.1) and (2.3).

Proof. We may assume that all the values $x_{i}(0)$, and hence also all $x_{i}(t)$, are nonnegative, since augmenting all of them by some positive constant does not change the behavior of the transformation. Let $A$ be the matrix (2.5). We augment each entry $a_{i i}$ from $-d_{i}$ to $-d_{i}+\varepsilon$ where $\varepsilon>0$ is small enough $\left(0<\varepsilon<\left(n \max _{i} x_{i}(0)\right)^{-1}\right.$ is sufficient). Then $\sum\left\{x_{j}(t) \mid j\right.$ is neighbor of $i\}<d_{i} x_{i}(t)$ implies that the left-hand side is also smaller than $\left(d_{i}-\varepsilon\right) x_{i}(t)$. We also redefine $g(x)$ in the previous proof so that $g(x)=|x|$ for all $x$ with
$|x|>\delta>0$ where $\delta$ is sufficiently small. Then the gradient $\nabla u$ has only $\pm 1$ components at all points $A x(t)$ that may occur during the iterations.

The bound of two for the length of the period of the transformation given by (2.1) and (2.4) cannot be obtained directly from theorem 1.1. Odlyzko and Randall [4] proved that if the period is not one, then $x_{i}(t) \neq x_{i}(t+1)$ for all $i$ and for all large $t$ 's. In this case, the rule (2.4) coincides with (2.3) and the result follows.

Let $u$ be a convex function. Let $y(1), \ldots, y(k) \in R^{n}, k \geq 2$, be arbitrary points, and $\xi(1), \ldots, \xi(k)$ be subgradients of $u$ in these points. It easily follows from (1.6) that

$$
\begin{equation*}
\sum_{i=1}^{k}(\xi(i)-\xi[i-1]) y(i) \geq 0 \tag{2.7}
\end{equation*}
$$

This property of subgradients is called cyclical monotonicity (see [10]). We need a stronger property of gradients that was formulated in [8].

Lemma 2.4. ([8]) If $\xi(1), \ldots, \xi(k)$ are gradients and (2.7) holds with equality, then $\xi(1)=\ldots=\xi(k)$.

Lemma 2.5. Let $f: R^{n} \rightarrow R^{n}$ be an odd mapping. Then the expression

$$
\begin{equation*}
\sum_{t=1}^{k}(f(y(t))+f(y[t-1])) y(t) \tag{2.8}
\end{equation*}
$$

may be written as

$$
\begin{equation*}
\sum_{t=1}^{k}(f(z(t))-f(z[t-1])) z(t) \tag{2.9}
\end{equation*}
$$

for $k$ even where $z(t)=(-1)^{t-1} y(t)$, or

$$
\begin{equation*}
\frac{1}{2} \sum_{t=1}^{2 k}(f(z(t))-f(z[t-1])) z(t) \tag{2.10}
\end{equation*}
$$

for $k$ odd where $z(t), t=1, \ldots, k$, is above and $z(t+k)=-z(t)$.
Proof. We use $f(-y)(-y)=f(y) y$. For $k$ even, we write (2.8) as $(f(y(1))-$ $f(-y(k))) y(1)+(f(-y(2))-f(y(1)))(-y(2))+\ldots+f(-y(k))-f(y(k-$ 1))) $(-y(k))$.

For $k$ odd, we may write it as one half of $(f(y(1))-f(-y(k))) y(1)$ $+\ldots+(f(y(k))-f(-y(k-1))) y(k)+(f(-y(1))-f(y(k)))(-y(1))+\ldots$ $+(f(-y(k))-f(y(k-1)))(-y(k))$.

Proof of theorem 1.1. Assume the trajectory is periodic with some period $k \geq 2$, and let the vectors $x(1), \ldots, x(k)$ form the period. Set

$$
\begin{equation*}
E=\sum_{t=1}^{k}(x[t-1]-x[t+1]) A x(t) . \tag{2.11}
\end{equation*}
$$

We have

$$
\begin{equation*}
E=0 \tag{2.12}
\end{equation*}
$$

as $A$ is symmetric. For notational convenience, we denote the gradient $\nabla u$ by $f$, and set

$$
\begin{equation*}
y(t)=A x(t) \quad t=1, \ldots, k . \tag{2.13}
\end{equation*}
$$

By (1.1) we have

$$
\begin{equation*}
x(t+1)-x(t-1)=f(y(t))+f(y(t-1)), \tag{2.14}
\end{equation*}
$$

and (2.11) is equivalent to

$$
\begin{equation*}
-E=\sum(f(y(t)+f(y[t-1])) y(t) . \tag{2.15}
\end{equation*}
$$

As $u$ is even, its gradient $f$ is odd, and lemma 2.5 can be applied. Assume $k$ is odd. (The case with $k$ even is similar.) Then (2.15) turns into $\sum_{t=1}^{k}(f(z(t))-$ $f(z[t-1])) z(t)=0$ by (2.12). Using lemma 2.4, we get $f(z(t))=f(z[t-1])$, and hence $f(y(t))=-f(y[t-1])$ for all $t=1, \ldots, k$. Thus, $x(t+1)=x(t-1)$ for all $t$ in the period.

We will reformulate theorem 1.1 in a seemingly stronger (but equivalent) form that is more convenient for applications.

Let $u$ be a convex function not necessarily differentiable. We say that a mapping $f: R^{n} \rightarrow R^{n}$ is an acyclic subgradient if $f(x)$ is a subgradient of $u$ at $x$ for every $x \in R^{n}$, and if $\sum_{t=1}^{k}(f(x(t))-f(x[t-1])) x(t)=0$ for some $x(1), \ldots, x(k), k \geq 2$, then $f(x(1))=f(x(2))=\ldots=f(x(k))$.

Theorem 2.6. Let $A$ be a symmetric matrix, $f$ be an acyclic subgradient, and $f$ be odd. Then the trajectory $\{x(t)\}_{t=0}^{\infty}$ given by (1.1) has bound of two for the length of period of arbitrary $x(0)$ provided that the trajectory is bounded.

The proof of theorem 2.6 is quite analogous to the proof of theorem 1.1.
Theorem 2.6 is more convenient for applications as the convex function $u$, whose acyclic subgradient the mapping $f$ is, may be given by a simpler formula. For example, we can take $u\left(x_{1}, \ldots, x_{n}\right)=\sum\left|x_{i}\right|$ instead of (2.6) in the proof of corollary 2.2. It might seem that what we gained by a simpler definition of $u$ we would lose by proving that $f$ is an acyclic subgradient. However, the property of "begin acyclic" is not much restrictive. For example, we may use

Lemma 2.7. ([5]) If $f(x) \in$ rel int $\partial(x)$ for each $x$, where $\partial(x)$ is the set of all subgradients of $u$ in $x$, then $f$ is acyclic.

The condition of lemma 2.7 is not necessary, and also many extremal choices of subgradients are acyclic (see [5]). In fact, theorems 1.1 and 2.6 are equivalent, as for every acyclic subgradient $f$ defined on a finite subset $S \subset R^{n}$ there is a differentiable convex function $u$ such that $f(x)$ is the gradient of $u(x)$ for each $x \in S$ (see [6]).

Based on theorem 2.6, one can derive a lot of other concrete transformations when taking into account various convex functions and their subgradients. Some constructions of "reasonable" subgradient have been presented in [6]. Here we will formulate only one such example that generalizes the transformation given by (2.1) and (2.2), in the sense that the increment of $x_{i}(t)$ need not be only $0, \pm 1$ but depends on the gap between $x_{i}(t)$ and the average value of the neighbors of $i$.

Corollary 2.8. Let $G$ be a graph with $n$ vertices. Assume that each vertex $i$ is equipped with an odd nondecreasing integer valued function $f_{i}: R \rightarrow Z$ satisfying $\left|f_{i}(x)\right| \leq\left\lceil|x| / d_{i}\right\rceil$ where $d_{i}$ is the degree of $i$. Let $x_{i}(0)$ be some initial values of the vertices, and let the values be simultaneously updated by

$$
x_{i}(t+1)=x_{i}(t)+f_{i}\left(\sum_{j=1}^{n} a_{i j} x_{j}(t)\right)
$$

where $A=\left(a_{i j}\right)$ is the matrix (2.5). Then the period of the transformation is either one or two.

Proof. The trajectory $\{x(t)\}$ is integral and bounded. Each function $f_{i}$ is the subgradient function of the convex function $u_{i}=\int f_{i}$. Then $u\left(y_{1}, \ldots, y_{n}\right)=$ $\sum u_{i}\left(y_{i}\right)$ is convex and $f\left(y_{1}, \ldots, y_{n}\right)=\left(f_{1}\left(y_{1}\right), \ldots, f_{n}\left(y_{n}\right)\right)$ is its subgradient function.

We will call the transformations defined in corollary 2.8 multi-threshold, as a counterpart to the multi-threshold transformations of type (1.4) which were studied by Goles and Olivos.

## 3. Pre-periods

In this section, we study the behavior of the transformations from section 2 before they reach the period. We present two results. We give an upper bound on the number $q$ of iterations before a multi-threshold transformation reaches its period. We call $q$ the pre-period of the transformation.

Theorem 3.1. The pre-period of a multi-threshold transformation on a graph $G$ is at most

$$
\begin{equation*}
24 M^{2} e+2 \tag{3.1}
\end{equation*}
$$

where $e$ is the number of edges of $G$ and $M=\max _{i}\left|x_{i}(0)\right|$.

We also study the change $\triangle E(t)=E(t+1)-E(t)$ in the energy function during iterations. While $\Delta E(t)$ may be arbitrary for a multi-threshold transformation, it is $\Delta E(t) \leq 0$ for the transformation given by (2.1) and (2.2). The latter has been shown in [4] together with an example that $\triangle E(t)$ may be zero for $t$ not in the period. (It is easy for $t$ in the period.) We prove that it cannot happen in more than two subsequent iterations.

Theorem 3.2. Let $\{x(t)\}$ be the trajectory of the transformation given by (2.1) and (2.2). If $\Delta E(t)=\Delta E(t+1)=\Delta E(t+2)=0$ then $x(t)=x(t+2)$.

We need the following lemmas 3.3 and 3.4 for the proof of theorem 3.1.
Lemma 3.3. ([9]) Let $g: Z \rightarrow Z$ be nondecreasing. Then

$$
\sum_{t=1}^{k}(g(z(t))-g(z(t-1))) z(t) \geq \frac{1}{2}|\{t \mid g(z(t)) \neq g(z(t-1)), t=1, \ldots, k\}|
$$

for every $k \geq 2$ and $z(1), \ldots, z(k)=z(0) \in Z$.
Combining lemmas 3.3 and 2.5 , we get
Lemma 3.4. Let $g: Z \rightarrow Z$ be an odd nondecreasing mapping. Then

$$
\begin{aligned}
& \sum_{t=1}^{k}(g(y(t))+g(y(t-1))) y(t) \geq \\
& \frac{1}{2}|\{t \mid g(y(t)) \neq-g(y(t-1)), t=1, \ldots, k\}|
\end{aligned}
$$

for every $k \geq 2$ and $y(1), \ldots, y(k)=y(0) \in Z$.
Proof of theorem 3.1. Consider the initial segment $\{x(t)\}_{t=1}^{q}$ of the trajectory before the period. Let $y(t)=A x(t)$ and set $c_{i}=\mid\left\{t \mid f_{i}\left(y_{i}(t)\right) \neq\right.$ $\left.-f_{i}\left(y_{i}(t-1)\right), t=2, \ldots, q\right\} \mid$. Since for every $t=1, \ldots, q$ there is some $i$ such that $f_{i}\left(y_{i}(t)\right) \neq-f_{i}\left(y_{i}(t-1)\right)$, we have

$$
\begin{equation*}
\sum_{i=1}^{n} c_{i} \geq q-1 \tag{3.2}
\end{equation*}
$$

We estimate

$$
\begin{align*}
& V= \sum_{t=1}^{q}(f(y(t))+f(y[t-1])) y(t)=  \tag{3.3}\\
& \sum_{i=1}^{n} \sum_{i=1}^{q}\left(f_{i}\left(y_{i}(t)\right)+f_{i}\left(y_{i}[t-1]\right)\right) y_{i}(t) \\
& \frac{1}{2} \sum_{i=1}^{n} c_{i} \geq \frac{1}{2}(q-1)
\end{align*}
$$

by lemma 3.4 and (3.2).
On the other hand, we have

$$
\begin{aligned}
V & =\sum_{t=1}^{q}(f(y(t))+f(y(t-1))) y(t)+(f(y(q))-f(y(0))) y(1) \\
& =x(q+1) A x(q)-x(0) A x(1)+(f(y(q))-f(y(0))) y(1) \\
& =x(q+1) A x(q)-x(1) A x(1)+f(A x(q)) A x(1) \leq 3 M^{2}\|A\| .
\end{aligned}
$$

Comparing it with the lower bound (3.3) on $V$, and using $\|A\|=4 e$, we get (3.1).

Proof of theorem 3.2. For every $t$, define the set $Z(t)=\left\{i \mid A_{i} x(t)=0\right\}$ where $A_{i}$ is the $i^{\text {th }}$ row of the matrix $A$. We claim that

$$
\begin{equation*}
Z(t-1) \neq Z(t) \tag{3.4}
\end{equation*}
$$

provided $\Delta E(t)=0$ and $t-1$ is not in the period. Set $\Delta x(t)=x(t+1)-x(t)$ and $\Delta x_{i}(t)=x_{i}(t+1)-x_{i}(t)$. We have $0=\Delta E(t)=(\Delta x(t)+\Delta x(t-1)) y(t)$, and hence

$$
\begin{equation*}
\left(\Delta x_{i}(t)+\triangle x_{i}(t-1)\right) y_{i}(t)=0 \text { for each } i \tag{3.5}
\end{equation*}
$$

For a contradiction, assume there is some $i \in Z(t-1) \backslash Z(t)$. Then $\Delta x_{i}(t-$ $1)=0\left(\right.$ since $i \in Z(t-1)$ ), and $\Delta x_{i}(t) \neq 0 \neq y_{i}(t)$ (since $\left.i \notin Z(t)\right)$, which contradicts (3.5). From (3.5) we also have

$$
\begin{equation*}
\Delta x_{i}(t)=-\Delta x_{i}(t-1) \text { for each } y_{i}(t) \neq 0 \tag{3.6}
\end{equation*}
$$

Since $t-1$ is not in the period, there is some $i$ for which $\Delta x_{i}(t) \neq \triangle x_{i}(t-1)$, and hence $y_{i}(t)=0$ by (3.5). It follows that $\Delta x_{i}(t)=0$ while $\Delta x_{i}(t-1) \neq 0$. Hence, $i \in Z(t) \backslash Z(t-1)$, which proves the claim.

Define sets $S^{+}(t)$ and $S^{-}(t)$ by

$$
\begin{aligned}
S^{+}(t) & =\left\{i \mid A_{i} x(t)=0 \text { and } A_{i} x(t-1)>0\right\} \\
S^{-}(t) & =\left\{i \mid A_{i} x(t)=0 \text { and } A_{i} x(t-1)<0\right\}
\end{aligned}
$$

Clearly,

$$
\begin{equation*}
Z(t) \backslash Z(t-1)=S^{+}(t) \cup S^{-}(t) \tag{3.7}
\end{equation*}
$$

Clearly, it is sufficient to prove the theorem only for $t=1$. For a contradiction, assume that $\triangle E(1)=\Delta E(2)=\triangle E(3)=0$ and $x(1) \neq x(3)$.

If some $i \in S^{+}(2)$, then $A_{i} x(2)=0, A_{i} x(1)>0$ by the definition of $S^{+}(2)$, and $A_{i} x(0)<0$ by (3.6). Hence

$$
\begin{equation*}
0<A_{i}(x(2)-x(0))=\sum\left\{a_{i j} \mid j \in S^{+}(1)\right\}-\sum\left\{a_{i j} \mid j \in S^{-}(1)\right\} \tag{3.8}
\end{equation*}
$$

for $i \in S^{+}(2)$. Analogously, for $i \in S^{-}(2)$, we have

$$
\begin{equation*}
0<\sum\left\{a_{i j} \mid j \in S^{-}(1)\right\}-\sum\left\{a_{i j} \mid j \in S^{+}(1)\right\} \tag{3.9}
\end{equation*}
$$

The union $S^{+}(2) \cup S^{-}(2)$ is nonempty due to (3.4) and (3.7). Summing up (3.8) and (3.9) over all $i \in S^{+}(2) \cup S^{-}(2)$, we get

$$
\begin{align*}
& \sum\left\{a_{i j} \mid i \in S^{+}(2), j \in S^{+}(1)\right\}+\sum\left\{a_{i j} \mid i \in S^{-}(2), j \in S^{-}(1)\right\} \\
& >\sum\left\{a_{i j} \mid i \in S^{+}(2), j \in S^{-}(1)\right\}+\sum\left\{a_{i j} \mid i \in S^{-}(2), j \in S^{+}(1)\right\} . \tag{3.10}
\end{align*}
$$

Now, consider an $i \in S^{+}(1)$. As $S^{+}(1) \subset Z(3)$ by (3.4), we have

$$
\begin{equation*}
A_{i} x(1)=A_{i} x(3)=0 \text { for } i \in S^{+}(1) . \tag{3.11}
\end{equation*}
$$

By (3.5) and (3.6), the components of $x(3)-x(1)$ are

$$
x_{j}(3)-x_{j}(1)=\left\{\begin{align*}
0 & \text { if } j \notin Z(2) \text { or } j \in Z(1)  \tag{3.12}\\
1 & \text { if } j \in S^{+}(2) \\
-1 & \text { if } j \in S^{-}(2)
\end{align*}\right.
$$

Hence (3.11) and (3.12) yield

$$
A_{i}(x(3)-x(1))=\sum\left\{a_{i j} \mid j \in S^{+}(2)\right\}-\sum\left\{a_{i j} \mid j \in S^{-}(2)\right\}=0 .(3.13)
$$

Summing up (3.13) over all $i \in S^{+}(1)$, we get

$$
\sum\left\{a_{i j} \mid j \in S^{+}(2), i \in S^{+}(1)\right\}-\sum\left\{a_{i j} \mid j \in S^{-}(2), i \in S^{+}(1)\right\}=0 . \text { (3.14) }
$$

For $S^{-}(1)$ we have analogously

$$
\sum\left\{a_{i j} \mid j \in S^{-}(2), i \in S^{-}(1)\right\}-\sum\left\{a_{i j} \mid j \in S^{+}(2), i \in S^{-}(1)\right\}=0 .(3.15)
$$

But (3.14) and (3.15) contradict (3.10), which concludes the proof.
Corollary 3.5. Let $G$ be a graph with at least two edges. Then the preperiod of the transformation given by (2.1) and (2.2) is at most 60 Me where $e$ is the number of edges of $G$ and $M=\max _{i}\left|x_{i}(0)\right|$.

Proof. Goles and Odlyzko have shown in the proof of theorem 2 of [3] that $|E(t)| \leq 8 M e$ for $t \geq 2 M+1$. (Their result is for the transformation given by (2.4) but holds also for (2.2).) Theorem 3.2 and $E(t+1) \leq E(t)$ gives the bound.

Unfortunately, theorem 3.2 does not hold when (2.2) is replaced by either (2.3) or (2.4). Consider the path of length $n$ with the initial values $(2,1,0,1,0,1, \ldots)$. Then $\Delta E(t)=0 n-2$ times in sequence in the preperiod.

We close the section with an example showing that $\Delta E(t) \leq 0$ need not hold for a multi-threshold transformation.

Example. Consider the transformation given by

$$
x_{i}(t+1)=x_{i}(t)+g\left(-x_{i}(t)+\frac{1}{d_{i}} \sum a_{i j} x_{j}(t)\right)
$$

where

$$
g(x)= \begin{cases}2 & \text { for } x>1 \\ 1 & \text { for } 0<x \leq 1 \\ 0 & \text { for } x=0\end{cases}
$$

$$
\text { and } g(x)=-g(-x) \text { for } x<0
$$

Assume $G=K_{3}$ is the complete graph on 3 vertices, and the initial values are $x(0)=(2,3,5)$. Then $x(1)=(4,4,3)$ and $x(2)=(3,3,4)$. We have $\Delta E(1)=(x(0)-x(2)) A x(1)=3$.

## 4. Quasi-periods

In this section, we study the transformation (1.1) when the mapping $f$ may attain infinitely many distinct values. The trajectory $\{x(t)\}_{t=0}^{\infty}$ need not be periodic even if it is bounded. However, the trajectory still retains certain periodic properties.

We say that a sequence $\{x(t)\}_{t=0}^{\infty}$ of vectors from $R^{n}$ has the quasi-period $p>0$ if

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\|x(t+p)-x(t)\|=0 \tag{4.1}
\end{equation*}
$$

and (4.1) does not hold for any $0<p^{\prime}<p$.
Theorem 4.1. Let $A$ be a symmetric matrix of size $n \times n, u$ be a convex differentiable function that is even, and $x(0) \in R^{n}$. If the trajectory $\{x(t)\}_{t=0}^{\infty}$ is bounded, then it has a quasi-period of either one or two.

We need the following two lemmas from [6].
Lemma 4.2. ([6]) Let $\{z(t)\}_{t=0}^{\infty}$ be a bounded sequence in $R^{n}$, and $f: R^{n} \rightarrow$ $R^{n}$ be the gradient of a convex function $u$. Assume that for every $\varepsilon>0$ there exists an infinite set $S$ of integers such that

$$
\begin{equation*}
0 \leq \sum_{t=r}^{s-1} f(z(t))(z(t)-z[t-1])<\varepsilon \tag{4.2}
\end{equation*}
$$

for every $r, s \in S, r<s-1$. Then $\lim _{t \rightarrow \infty}\|f(z(t))-f(z(t-1))\|=0$.
Lemma 4.3. ([6]) Let $\{x(t)\}_{t=1}^{\infty}$ be a bounded sequence in $R^{n}$. Then for every $\delta>0$ there is an infinite set $S$ of integers satisfying

$$
\begin{equation*}
\|x(r-j)-x(s-j)\| \leq \delta \text { for every } r, s \in S, j=0,1 \tag{4.3}
\end{equation*}
$$

and

$$
\begin{equation*}
r-s=0 \bmod 2 \text { for every } r, s \in S \tag{4.4}
\end{equation*}
$$

Proof of theorem 4.1. Let $\{x(t)\}_{t=0}^{\infty}$ be a trajectory of the transformation, and let $M$ be such that $\|x(t)\| \leq M$ for all $t$. The function $f=\nabla u$ is continuous as $u$ is differentiable [10]. Hence

$$
\begin{equation*}
\forall \varepsilon>0 \exists \delta_{0}>0 \forall\|x\|,\left\|x^{\prime}\right\| \leq M,\left\|x-x^{\prime}\right\|<\delta_{0}:\left\|f(A x)-f\left(A x^{\prime}\right)\right\|<\varepsilon \tag{4.5}
\end{equation*}
$$

Set $\delta=\delta_{0}(2 M\|A\|)^{-1}$. By lemma 4.3, there is an infinite set $S$ of integers such that (4.3) and (4.4) hold. We have

$$
\begin{aligned}
& 0=\sum_{t=r}^{s-1}(x[t+1]-x[t-1]) A x(t)= \\
& \sum_{t=r}^{s-1}(f(y(t))+f(y(t-1))) y(t)+(x(r-1)-x(s-1)) y(r)+ \\
& (x(r)-x(s)) y(s-1) .
\end{aligned}
$$

Using (4.3), we get

$$
\begin{equation*}
\left|\sum_{t=r}^{s-1}(f(y(t))+f(y(t-1))) y(t)\right|<\delta_{0} . \tag{4.6}
\end{equation*}
$$

Using $f(y)=-f(-y)$ and (4.4), we may write

$$
\begin{align*}
& \sum_{t=r}^{s-1}(f(y(t))+f(y(t-1))) y(t)= \\
& \sum_{t=r}^{s-1}(f(z(t))-f(z(t-1))) z(t)= \\
& \sum_{t=r}^{s-1} f(z(t))(z(t)-z[t+1])+(f(z(s-1))-f(z(r-1))) z(r) \tag{4.7}
\end{align*}
$$

where

$$
z(t)=\left\{\begin{aligned}
y(t) & \text { if } t \text { is even } \\
-y(t) & \text { if } t \text { is odd }
\end{aligned}\right.
$$

We have $\|f(z(s-1))-f(z(r-1))\|=\|-f(y(s-1))+f(y(r-1))\|<\varepsilon$ by (4.5) and hence

$$
\left|\sum_{t=r}^{s-1} f(z(t))(z(t)-z[t+1])\right|<\delta_{0}+\varepsilon M\|A\|
$$

by (4.6) and (4.7). Applying lemma 4.2 to $\{z(t)\}$, we get $\| f(z(t))-f(z(t-$ 1)) $\| \rightarrow 0$; hence, $\|f(y(t))+f(y(t-1))\| \rightarrow 0$, and finally $\|x(t+1)-x(t-1)\| \rightarrow$ 0 as $t \rightarrow \infty$.

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