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Stability of Equilibrial States and Limit Cycles in Sparsely Connected, Structurally Complex Boolean Nets

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Abstract. Through their natural dynamics, networks of Boolean functions can recompute, or return to, equilibrial states and limit cycles from which they have been displaced. How often such return occurs measures one kind of behavioral stability. For a class of functionally homogeneous, sparsely connected, disorderly structured networks, this paper examines how stability of equilibrial states and limit cycles is affected by the size of the net, size of the displacement, and the function used in the net. All functions in the class are examined, and displacement is varied over its full range. On the whole, for any given function, the effect of displacement size appears to be quite regular. However, from function to function, displacement effects vary widely. Though there are exceptions, at a given relative displacement, most functions show decreasing cyclic stability as nets become larger. On the other hand, the data suggest that for many functions, larger nets are more stable under small absolute displacements.

1. Introduction

Many-element models have long been used to understand complex real world systems. More recently, increased attention has been given to the study of many-element models in which element behavior can decisively influence overall system behavior (as an early example, see [1]). The present paper continues this very-complex-system theme using a class of dynamic Boolean nets to provide the objects of interest.

An important characteristic of any dynamic system is the extent to which its dynamics resist pertubations: its "behavioral stability." While behavioral stability is equally important in network systems, their high dimensionality can make it difficult to predict how a system will respond to disturbance. A number of workers have examined stability, under varying definitions, in nets of neuron analogs (e.g., [2–8]). Behavioral stability in Boolean nets has seen less work.

Although it was known earlier that, in nets, some binary functions exhibit behavioral neighborhoods around equilibrial states, in that they show trajectories whose Hamming-distances (the number of elements which differ) between net states progressively decrease to zero as the final state is approached [9,10], behavioral stability in nets of Boolean functions was apparently not studied systematically until 1969, by Stuart Kauffman.

Kauffman [11] found that it is highly likely ($P \approx 0.9$) for structurally random, functionally heterogeneous nets of two-input Boolean functions to return to the limit cycles and equilibrial states from which they are displaced by one-element displacements. (In what follows, both equilibrial states and limit cycles are referred to simply as "cycles.") A preliminary study [12] examined the stability of cycles in functionally homogeneous nets for all two-input Boolean functions. This study found that all two-input Boolean functions, except for "exclusive or" and "equivalence," are very stable for small displacements, and show regular effects for increasing size of displacement. It was also noted that the effect of net size on stability may depend on whether relative or absolute displacements are considered. The latter study was limited, however, in the range of displacement magnitudes examined, the size of the trajectory searched, and the sample sizes used.

Kaneko [13] has recently studied the stability of cycles in one-dimensional cellular automata. Such automata are simple-structure systems, strings or simple loops of elements, otherwise similar to the nets considered in this paper. In addition to their potential as models, simple-structure models are valuable for the qualitative insight into system dynamics they make accessible through direct observation. Nets with disorderly structure, while useful in their own modeling domains, are perhaps best studied by an indirect, statistical approach in the expectation that regularities in their behavior are more likely be found by comparing aggregated dynamics rather than particular behavioral trajectories. Such a statistical approach is used in the work reported here.

In the present paper, cycle stability is a straightforward measure of the average extent to which (behavior) space at a given distance from a cycle is behaviorally associated with that cycle. More exactly, it is the proportion of net states at a given distance from cyclic states which are in those cyclic states' basins of attraction (terms defined below). Depending on the modeling context, cycle stability can also be thought of as: 1) the probability of a net's natural dynamics recalculating a correct set of terminal values following an error of a given amount, 2) the likelihood of exact reproduction of a class exemplar (or exemplars) for a given degree of corruption in a stimulus probe, or 3) the average fraction of potential energy actually available at a given distance from a cycle.

Recent advances in understanding the capabilities of networks of simple elements have incresed interest in details of the behavior spaces of such systems. Additionally, accumulating knowledge in design and implementation of complex man-made systems makes work in sparsely connected, disorderly structured systems' behavior pertinent. Suitably complex architecture is required for systems of simple elements to be able to produce fully arbitrary behavior. The problems faced in massively connecting elements in large distributed systems such as neural networks makes knowledge of the effects of sparseness a significant issue in manufacturing technology. This paper provides information on behavior space and performance characteristics for a benchmark class of sparsely connected, structurally complex systems.

2. The nets examined

Any one network examined in this paper is autonomous, clocked, structurally rigid, and functionally homogeneous. That is, no inputs enter the network from outside the net. Time in the net, indexed by t, is discrete, and all update operations occur simultaneously throughout the net. For a particular net, the pattern of connections among elements is fixed and all elements are functionally identical.

Each element has exactly two binary inputs incident from network elements, and one "internal" binary input. The internal input carries the value of the element's *internal state*, which is also the element's *output*. The element computes a binary function T of the three inputs. The general form of T is given in figure 1.

Any one network is composed of exactly N elements. The binary N-tuple which gives the state of each element in the net at a given net time is the *net state* at that time. After being started at an arbitrary net state, a net proceeds stepwise from one net state to another, finally producing two sequences of net states. The first sequence, the *transient* or *run-in*, is followed by a second sequence of states, the *cycle*, in which the net is then permanently trapped, barring malfunction or disturbance to the net. The run-in may contain zero or more net states. The cycle may contain one or more distinct net states. The number of distinct net states in a sequence is the *length* of that sequence. A *disclosure length* is the length of a run-in plus the length of the cycle that follows it. A cycle's *basin of attraction* is the cycle and all run-ins to that cycle.

3. Procedure

Stability data were collected as follows. For displacement D, net size N, and a function T, a net connection table was chosen at random. Each element's "left" input connection was connected to an outputing element by equiprobable, independent, with-replacement (EIWR) sampling from the N net elements. The "right" connection table was similarly chosen. Note that, technically, net connection *tables* were chosen at random, not net *structures* as such. All sampling made use of computer generated pseudo-random numbers.

Following the choice of a net structure, a net state was chosen by EIWR sampling of element states. The net was then started at the chosen net state and allowed to proceed to a cycle. One of the states in the cycle was

L	R	0	1
0	0	a	e
0	1	b	f
1	0	c	g
1	1	d	h

Figure 1: The general form of T. L and R are the element's "left" and "right" inputs at time t. Under the internal state heading, 0 or 1 is the element's internal state value, at time t. The entries $a, b, \ldots, h$ are the element's computed internal state (and output state) at net time $t+1$. In a particular realization of T, a zero or a one replaces each entry. In this paperparticular Ts are referred to as $T(A, B)$, where A is the decimal equivalent of the binary integer $abcd$, the left column of T, and B is similarly derived from the right column. Thus, $T(0001, 0110)$ is $T(1, 6)$. $T(1, 6)$ is equivalent to Wolfram rule number 104 for binary, one-dimensional, nearest-neighbor cellular automata [14].

chosen EI. A state D percent Hamming distance units away from the cyclic state (that is, differing from the state on the cycle in exactly D percent of its element-states) was then chosen EI, and the net was allowed to resume operation.

If the displaced trajectory re-encountered the cycle from which the displacement was made, that trajectory, under that displacement, was a *stable* trajectory. An *unstable* trajectory was observed on those occasions when the displaced trajectory ran to a different cycle. The routine described above was repeated m times for each N, D, and T combination. In the data presented, m is at least 1000. Stability is given in terms of the percent of trajectories stable relative to m.

While there are 256 distinct Ts, symmetries that exist among Ts and their behavior spaces have the effect of producing 88 identical populations of behavior spaces insofar as the sampling conditions used in this study are concerned [15]. The symmetries exist under 1) interchange in the functional roles of zero and one, 2) left-right switching of element connections, and 3) the join of the previous operations. Role interchange,"reflection," preserves the behavior space in that if, net structure remaining unchanged, $T(A, B)$ is replaced with $rT(A, B) = T(rB, rA) = T(h'g'f'e', d'c'b'a')$, where the prime indicates a complement (0-1 interchange) operation, the behavior spaces of T and rT are themselves net-state-by-net-state complements. That is, reflected Ts (T and rT) yield reflected (in the sense of complementary) behavior spaces.

A left-right switching of inputs throughout the net produces a behavior space identical to that produced by the operation s on $T : sT(A, B) = T(sA, sB) = T(acbd, egfh)$. Following the s operation with "reflection" yields rsT.

The end result is that the 256Ts are partitioned by a behavioral equivalence relation yielding 88 T, rT, sT, rsT equivalence classes. In the context of cellular automata, the r, s, and rs operations have been called called conjugation, reflection, and conjugation-reflection, respectively [16, page 492]. Considering the "number" of $T(A, B)$ to be the number indicated by the decimal-coded hexadecimal numeral AB, the one T used as a representative of its equivalence class is the lowest numbered T in the class. The equivalence classes are listed in [17], and, using a different coding of the functions, also in [16].

Net size

All nets were examined using net size 10. So as to get some indication of how net size affects cycle stability, all nets were examined with at least one other net size N'. My aim was to separate two net size stability curves enough so that some indication of the effect of net size would be elicited. That is, I attempted to get the ten empirical data points for N and for N' to show consistent differences, negative or positive.

Practical constraints on computation required that the smallest larger net size sufficient to establish the size effect be used. This was particularly true for Ts with long disclosure lengths [16]. Unless disclosure lengths made it impracticable, nets were also examined at $N' = 20$. Even larger net sizes were used in a few cases where disclosure lengths allowed and net size effect was small between $N = 10$ and $N' = 20$.

If I could find no N' that separated the curves (within effective limitations on computing), in some cases I increased m values on either the N or the N' curve. The cases in which m values were increased were those in which there appeared to be some possibility of separating the curves. This modification of procedure in response to inherently variable outcomes likely increases the error of claiming a size effect exists in the population, when it does not ("Type I error"). While the problem does exist, and net size data for some functions should probably be considered exploratory, on the other hand, 1) new N' curves were run in their entirety, not just for non-separating stability values at specific displacements, and 2) where additional observations were made, the new data were combined, when appropriate, with existing data.

A simple non-parametric test for population difference concludes that the probability of identical N and N' populations yielding ten independent observed differences of the same sign is $2(2^{-10})$, or about 0.2%. For nine consistent differences out of the ten, the probability in question is about 2%. Accordingly, those N and N' curves not tied at any stability values (except that for $D = 0$) which show nine or ten consistent differences, are taken to be evidence of a clear net size effect. Similar calculations for nine, and for eight consistent differences, that is, where there are one or two ties respectively, indicate that one can operate with a decision rule of at most one inconsistent difference, and not exceed a probability of approximately 6% that the observations, while favoring a conclusion that a net size effect exists, do not in fact arise from a true net size effect. Similarly, with as many as five tied observations, if there are no inconsistent differences, the (Type

I) error rate does not exceed approximately 6%.

Displacement

Where H is Hamming distance in absolute units, namely the actual number of element state differences between two net states, $D = 100H/N$ is the percent Hamming distance. Early results suggested that displacement in terms of D, rather than H, might give stability curves similar at different net sizes. Displacement size, for all Ts, was varied from $D = 10, \ldots (10) \ldots 100$. In the case of a few Ts, additional displacement values were used. Some N' values made 10 percentage point increments impossible. Note that $D = 100$ is a displacement that yields the complement of a cyclic net state.

4. Results

Stability results are displayed graphically in Appendix A. In all graphs, the horizontal axis is percent displacement D, and the vertical axis is percent stability. The results are plotted from $D = 0$. Obviously, stability at $D = 0$ is 100% by definition. Net function is indicated using $T(A, B)$ notation and by Wolfram rule number. For example, $T(1,6)$ is also noted as $W(104)$. Ts are ordered by their number AB. N values appearing in a given graph are indicated thereon. If no m value appears with N, m is 1000 for each stability value in that function line. K indicates 1000 in connection with m, that is, $m = 5K$ means $m = 5000$.

5. Analysis and discussion

Effects of net function and displacement on stability. The strong dependence of stability on net function can easily be seen in Appendix A. Stability curves vary widely. The trivial Ts, $T(0,0)$, $T(0,15)$, and $T(15,0)$ give some idea of the variability: high stability throughout the range of displacement values, low stability throughout the range of (non-zero) displacement values, and high stability at extremes of displacement, respectively. As to non-trivial functions exhibiting similar stability response, we have as examples,

1. showing high stability throughout: $T(1,0)$, $T(1,2)$, $T(2,0)$, $T(7,2)$, $T(8,0)$ and $T(8,8)$;
2. showing low stability throughout: $T(0,9)$, $T(0,10)$, $T(0,11)$, $T(0,14)$, $T(1,9)$, $T(1,14)$, $T(2,9)$, $T(2,11)$, $T(6,9)$, and $T(11,4)$;
3. showing high stability at displacement extremes: $T(1,8)$, $T(2,4)$, $T(3,0)$, $T(3,12)$, $T(6,6)$, $T(7,0)$, $T(7,14)$, $T(10,2)$, $T(10,8)$, $T(11,0)$, $T(11,2)$, $T(11,8)$, $T(14,0)$ and $T(14,8)$.

It is not surprising that at least some functions have high stability at high displacements. Where T is of the form $T(abcd, dcba)$, the complement of a cyclic state is in the cycle's basin of attraction [18], therefore T must show 100% stability at $D = 100$. All Ts which are of the latter form, namely, $T(0,0)$, $T(1,8)$, $T(2,4)$, $T(3,12)$, $T(6,6)$, and $T(7,14)$, do show the predicted response.

Not all Ts can be placed in the categories sketched above. A few Ts show generally high stability with a sharp drop-off at displacements approaching 100%, namely $T(0,1)$ and $T(1,1)$. Other Ts show stability that gradually declines over the displacement range: $T(0,3)$, $T(1,3)$, $T(2,3)$, and $T(3,3)$. Still others can perhaps best be described as moderately stable over most of the displacement range: $T(3,10)$, $T(6,1)$, and $T(6,14)$.

In any case, the categorization of functions as to the general effect of displacement is made more difficult by not knowing for certain what the large-net stability contour is. For example, while $T(8,10)$ is possibly classifiable as a moderately stable function at $N = 10$, its apparent change of stability with net size suggests that this function might well show highly unstable cycles in large nets.

Effect of net size on stability. At least for the net sizes used, the clear generalization for net size effects on cyclic stability is that for most Ts, if net size is increased, cycles become less stable. Looking at percent displacement over its range of possible values, 68 Ts appear to have cycles that are less stable in larger nets. Only a very few Ts appear to have cycles that are more stable in larger nets: $T(0,1)$, $T(1,0)$, $T(1,1)$, $T(8,0)$, and $T(8,8)$.

The three trivial Ts, of course, show no effect of size at all. Twelve Ts have stability curves for N and N' that can not be deemed to be separated by the criteria used here. Of the latter group, $T(1,3)$, $T(2,1)$, $T(3,1)$, $T(6,2)$, $T(6,8)$, $T(6,10)$, and $T(7,6)$, all show curves which strongly suggest a differential response of stability to displacement as net size is increased. $T(0,3)$, $T(2,3)$, $T(7,14)$, $T(9,8)$, and $T(9,10)$ have curves that appear difficult to disentangle at the net size and displacement values used, especially given the lack of useful theory.

Absolute displacement

The displacements considered in this paper are all given in percentage units. One of the findings of the work reported here is that this relative scaling of displacement is a useful procedure. Nevertheless, how stable cycles are under small absolute displacements, say, of one or two units Hamming distance, is an interesting question. Infrequent error in net computation would likely be modeled best in terms of small absolute displacements. While the present study was not designed to speak directly to that question (however, see $T(3,1)$, $T(6,2)$, $T(6,8)$, $T(7,6)$, and $T(7,14)$ for relevant data), the available relative displacement data can be rescaled to provide indirect evidence. The rescaled data suggest that often, even for Ts that clearly decline in stabilty as net size increases when stability is characterized by relative displacement, when absolute one-unit displacements are considered these same Ts may show stability increases as nets become larger. Additional work to clarify this point may be warranted.

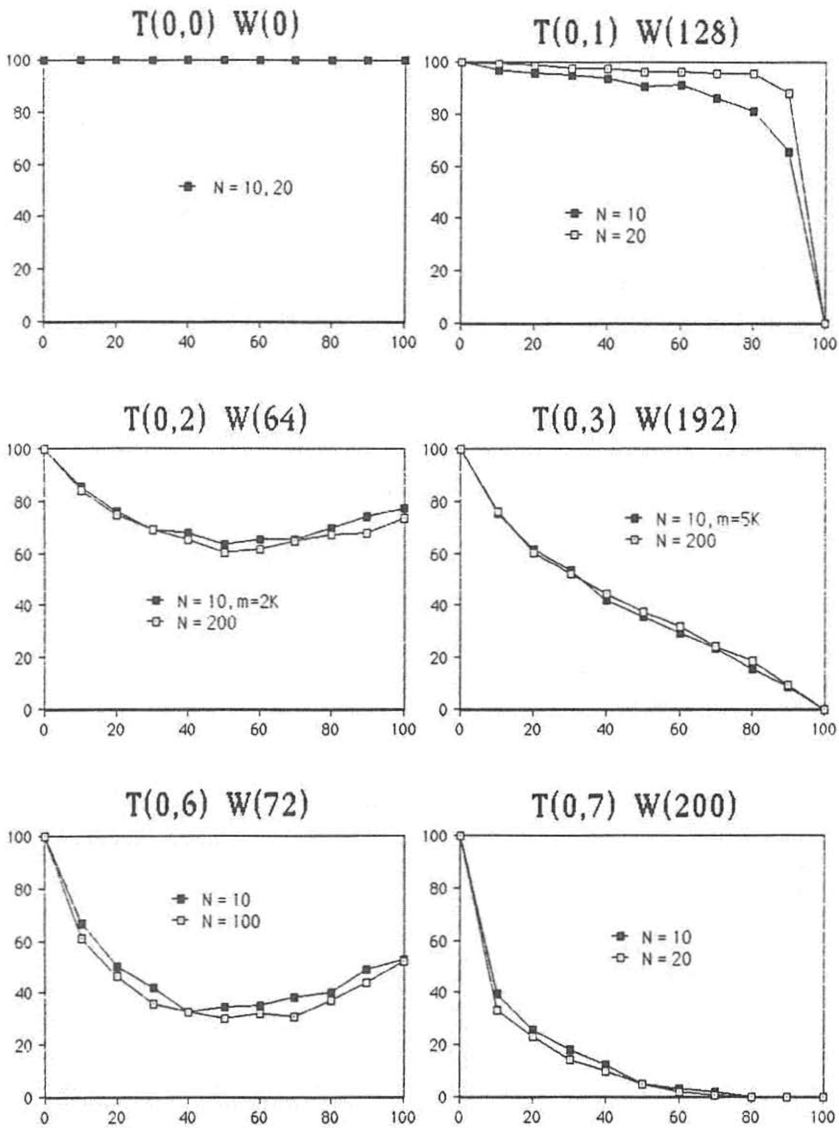
Acknowledgement

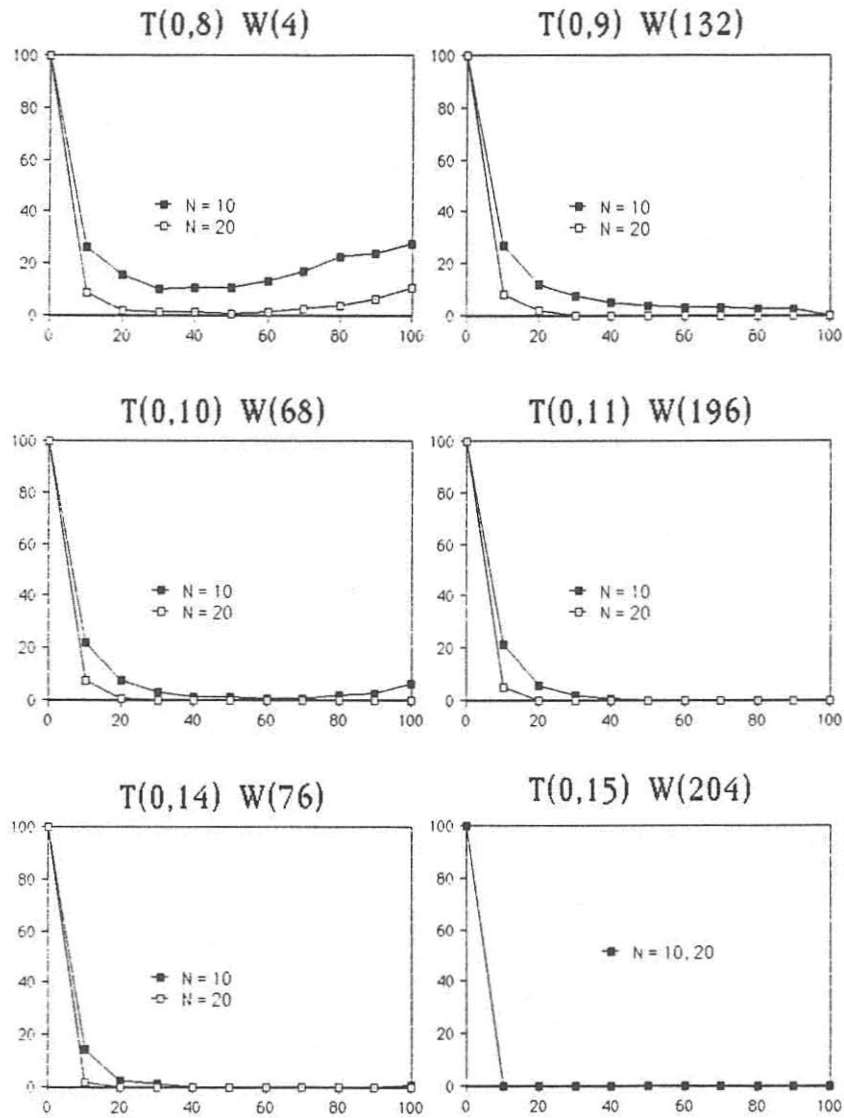
I thank John M. Walker for valuable help during data preparation and analysis.

Appendix A. Stability graphs

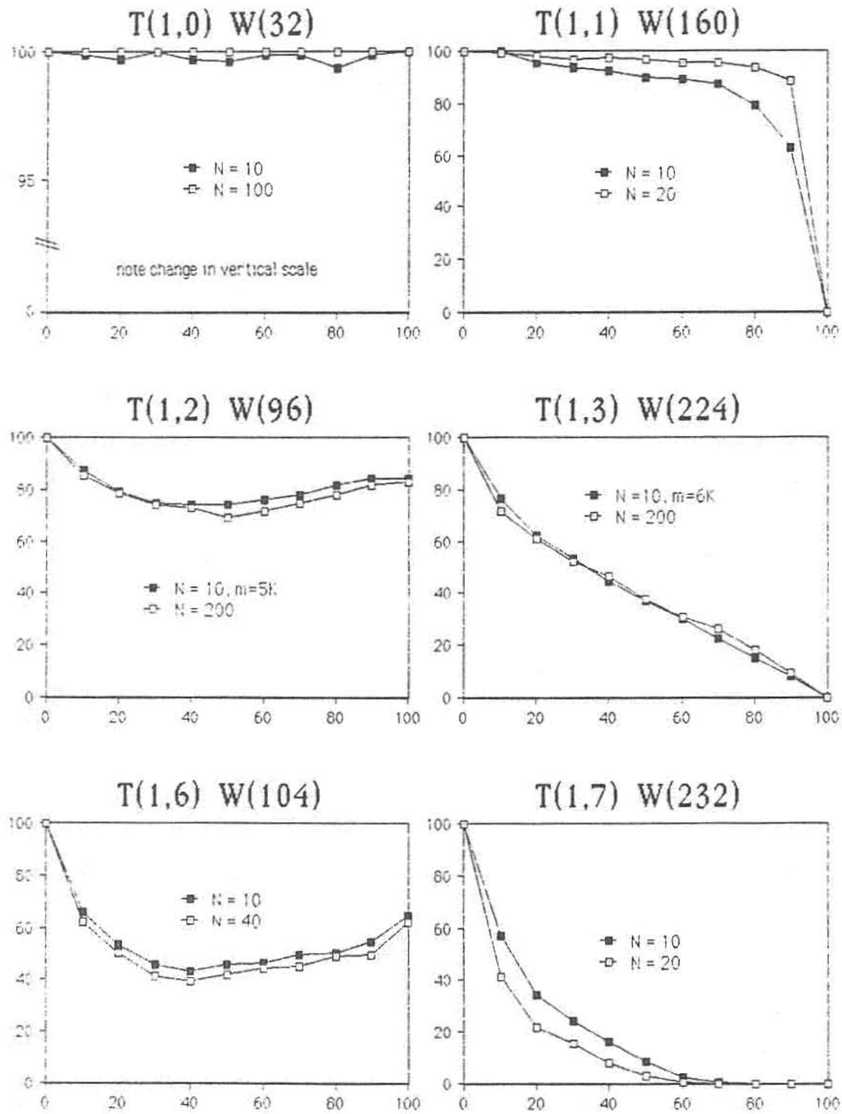
In all graphs, the horizontal axis is displacement from the cycle in percent Hamming distance. Percent Hamming distance is the percent of element-states which differ, between the cyclic state from which the displacement is measured, and the displaced net state which begins the displaced trajectory, relative to the number of elements in the net. The vertical axis is percent stability, that is, the percent of trajectories which returned to the cycle, relative to the total number of trajectories initiated (m). Where not indicated on the graph, $m = 1000$. N is the net size (the number of elements in the net).

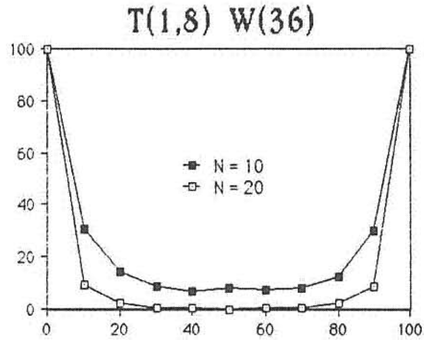
The function used by net elements is indicated both in $T(A, B)$ notation, and by the use of Wolfram rule numbers, the latter indicated by $W(C)$. Further details on functions are given in section 2, and section 3 above.

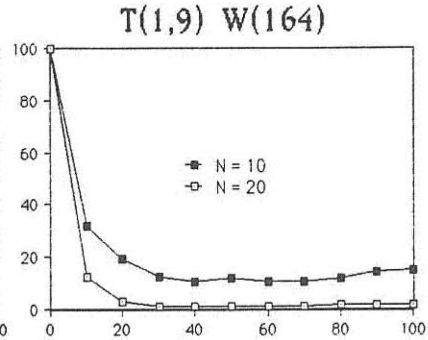

T(0,0) W(0)
N = 10, 20
T(0,1) W(128)
N = 10
N = 20
T(0,2) W(64)
N = 10, m=2K
N = 200
T(0,3) W(192)
N = 10, m=5K
N = 200
T(0,6) W(72)
N = 10
N = 100
T(0,7) W(200)
N = 10
N = 20

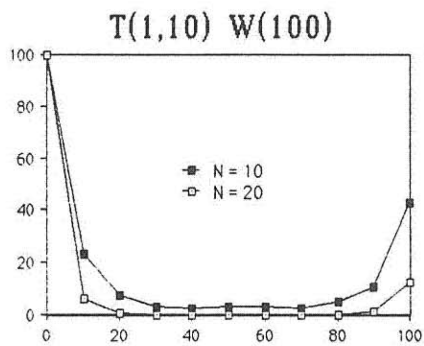


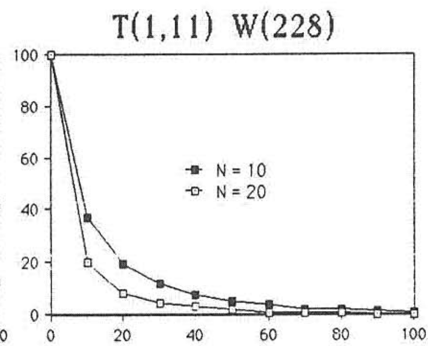
T(0,8) W(4)
N = 10
N = 20
T(0,9) W(132)
N = 10
N = 20
T(0,10) W(68)
N = 10
N = 20
T(0,11) W(196)
N = 10
N = 20
T(0,14) W(76)
N = 10
N = 20
T(0,15) W(204)
N = 10, 20

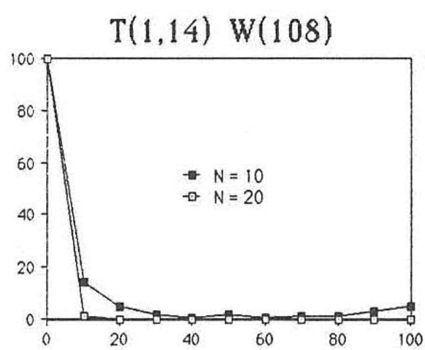

T(1,0) W(32)
N = 10
N = 100
note change in vertical scale
T(1,1) W(160)
N = 10
N = 20
T(1,2) W(96)
N = 10, m=5K
N = 200
T(1,3) W(224)
N =10, m=6K
N = 200
T(1,6) W(104)
N = 10
N = 40
T(1,7) W(232)
N = 10
N = 20

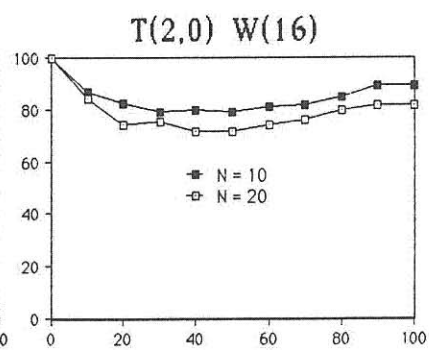

T(1,8) W(36)
N = 10
N = 20
0
20
40
60
80
100


T(1,9) W(164)
N = 10
N = 20
0
20
40
60
80
100

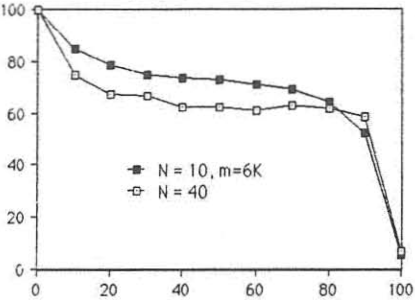

T(1,10) W(100)
N = 10
N = 20
0
20
40
60
80
100


T(1,11) W(228)
N = 10
N = 20
0
20
40
60
80
100

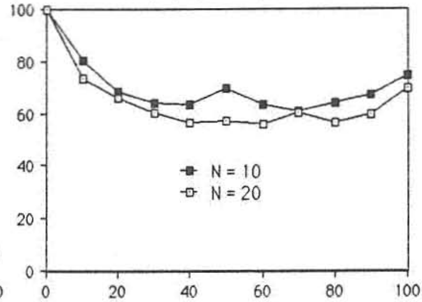

T(1,14) W(108)
N = 10
N = 20
0
20
40
60
80
100


T(2,0) W(16)
N = 10
N = 20
0
20
40
60
80
100

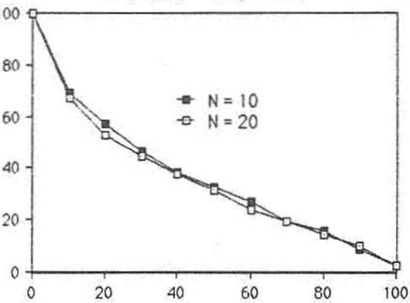
T(2,1) W(144)



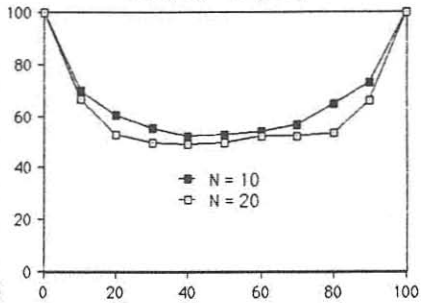
T(2,2) W(80)



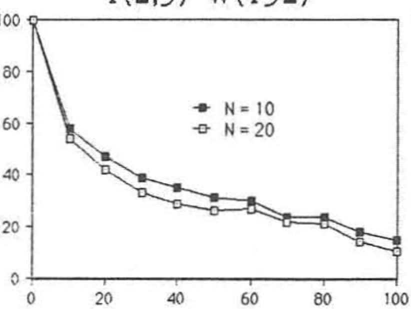
T(2,3) W(208)



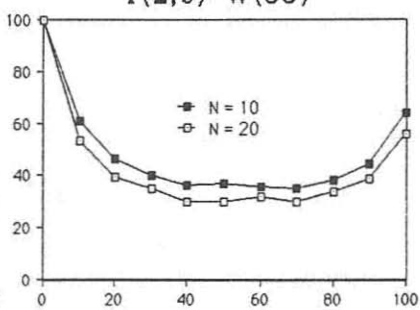
T(2,4) W(24)

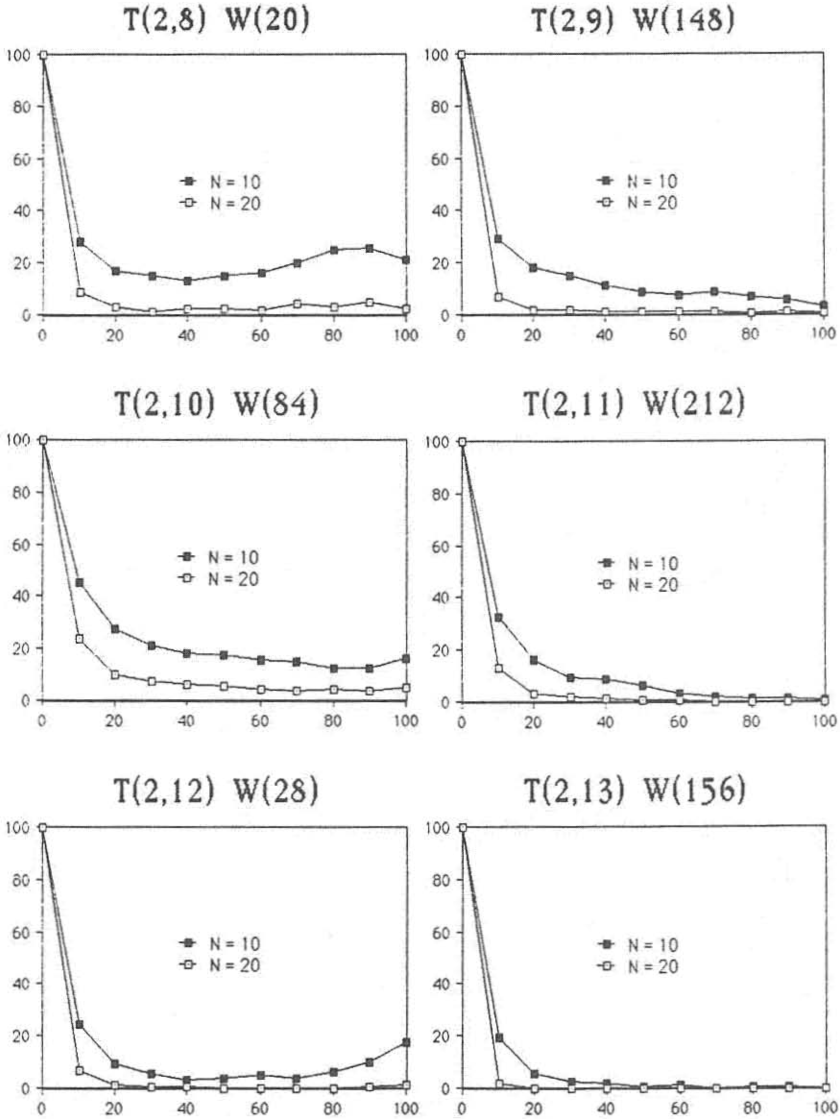


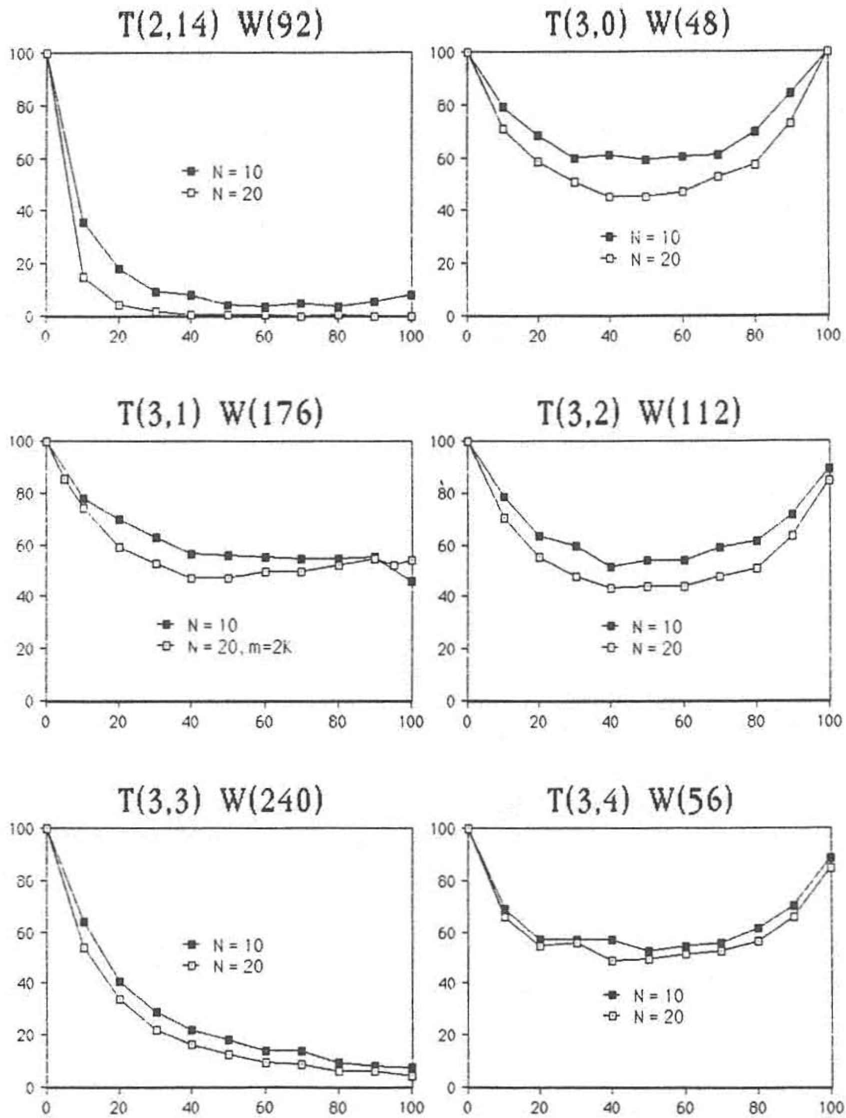
T(2,5) W(152)

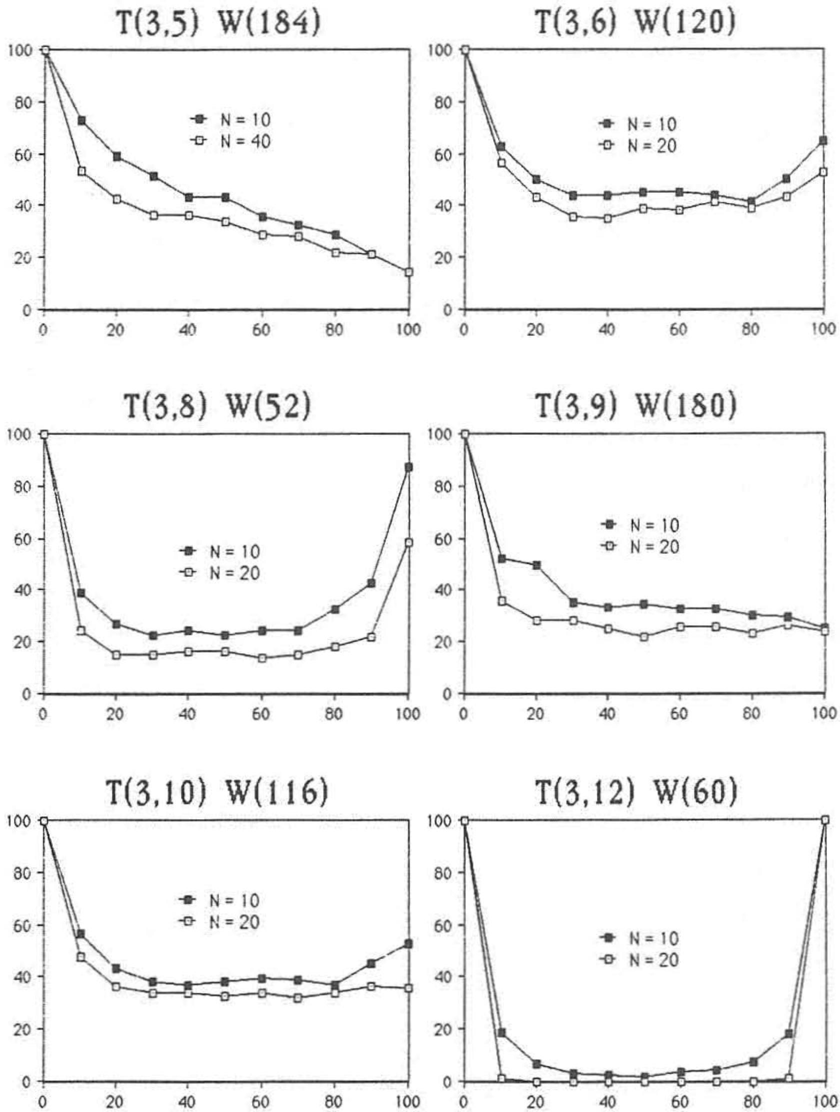


T(2,6) W(88)

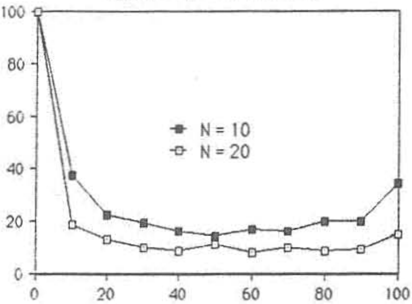



T(2,8) W(20)
N = 10
N = 20
T(2,9) W(148)
N = 10
N = 20
T(2,10) W(84)
N = 10
N = 20
T(2,11) W(212)
N = 10
N = 20
T(2,12) W(28)
N = 10
N = 20
T(2,13) W(156)
N = 10
N = 20
0 20 40 60 80 100

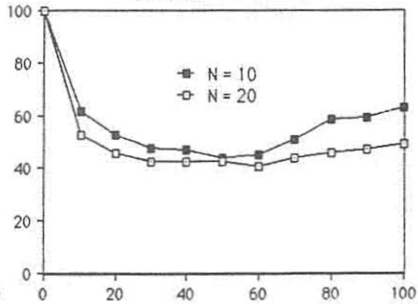

T(2,14) W(92)
N = 10
N = 20
T(3,0) W(48)
N = 10
N = 20
T(3,1) W(176)
N = 10
N = 20, m=2K
T(3,2) W(112)
N = 10
N = 20
T(3,3) W(240)
N = 10
N = 20
T(3,4) W(56)
N = 10
N = 20


T(3,5) W(184)
N = 10
N = 40
T(3,6) W(120)
N = 10
N = 20
T(3,8) W(52)
N = 10
N = 20
T(3,9) W(180)
N = 10
N = 20
T(3,10) W(116)
N = 10
N = 20
T(3,12) W(60)
N = 10
N = 20

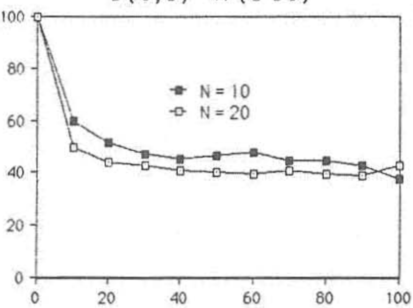
T(3,14) W(124)



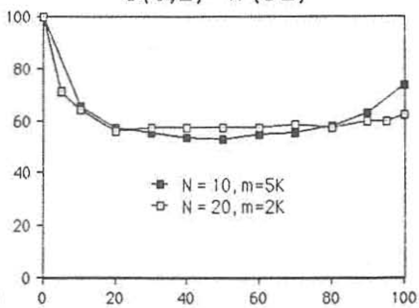
T(6,0) W(18)



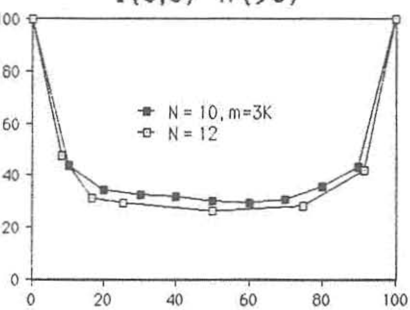
T(6,1) W(146)



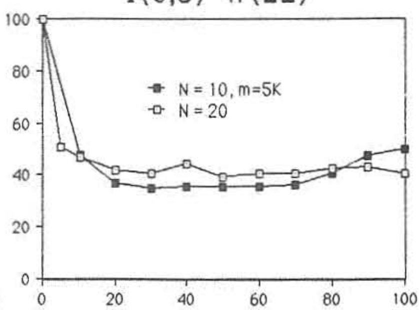
T(6,2) W(82)

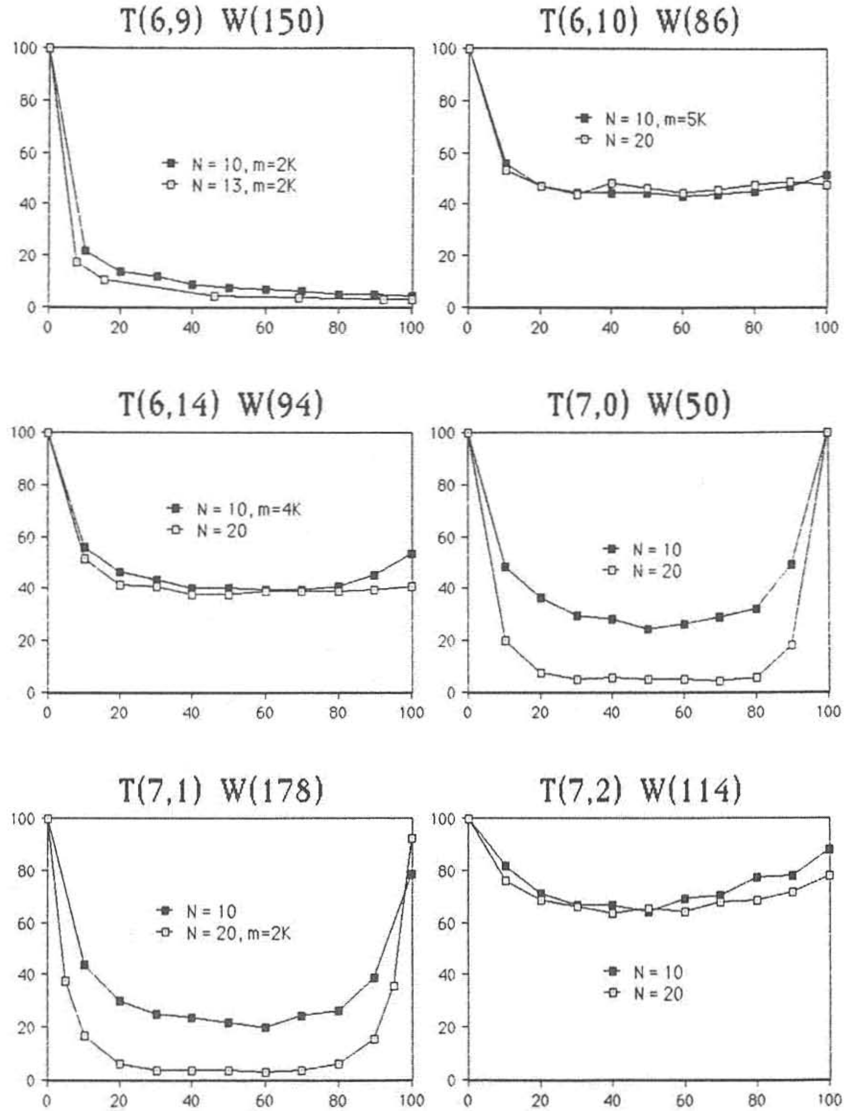


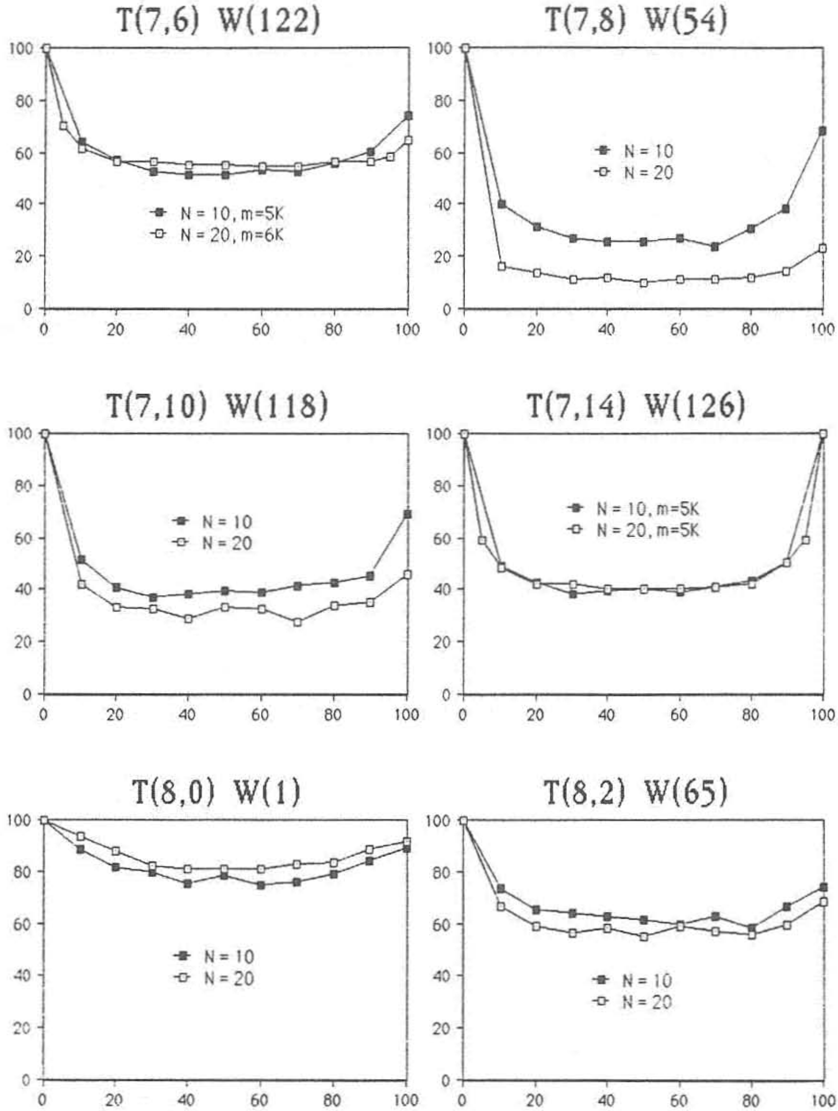
T(6,6) W(90)

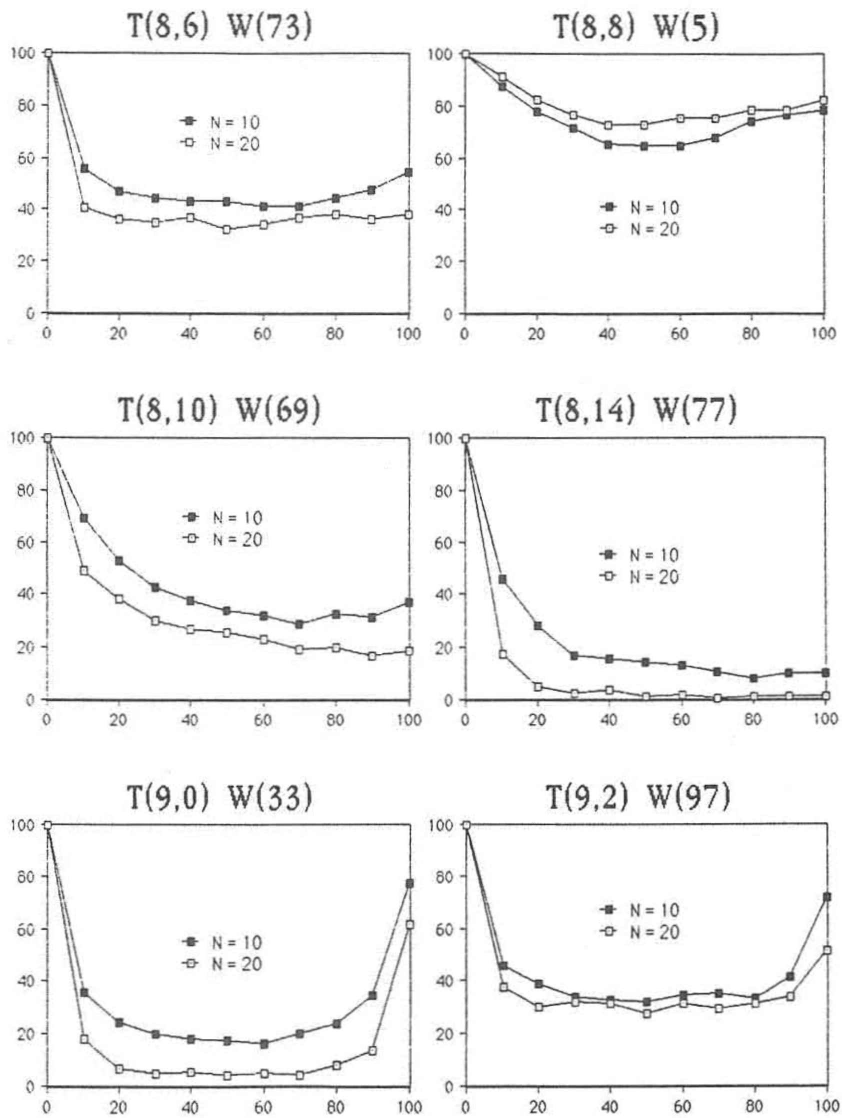


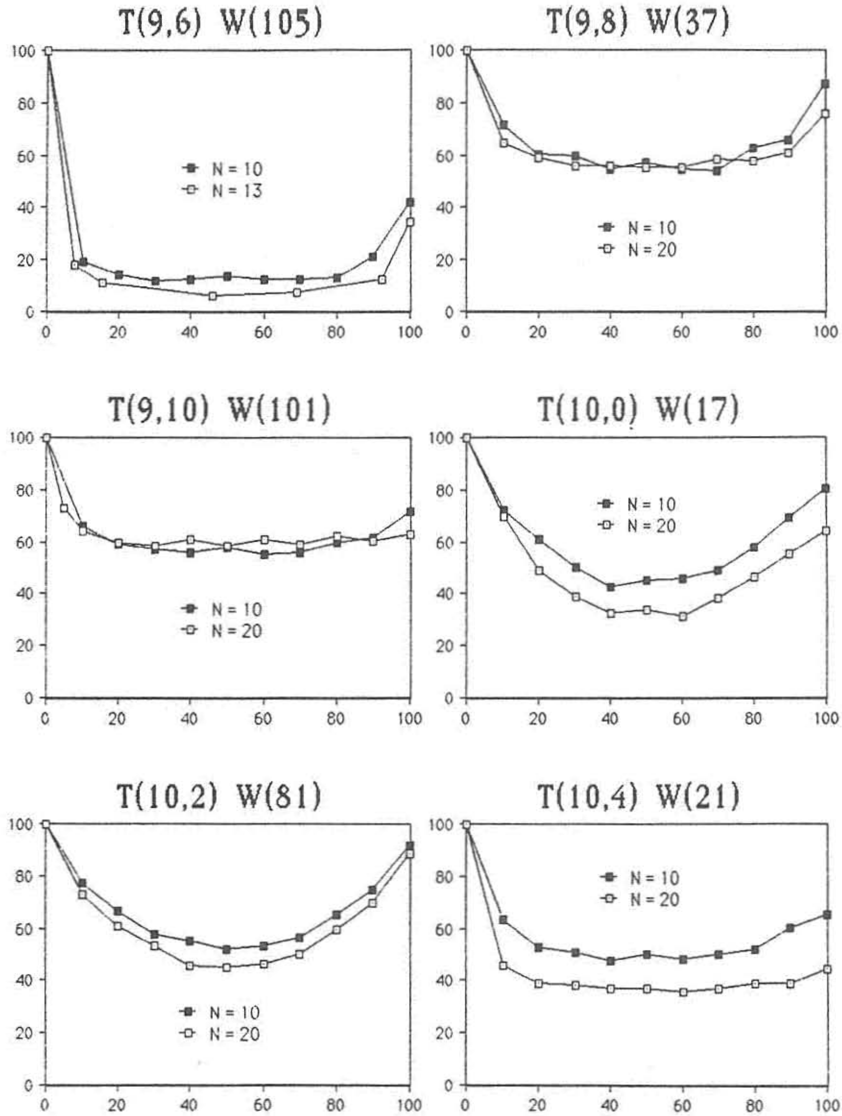
T(6,8) W(22)

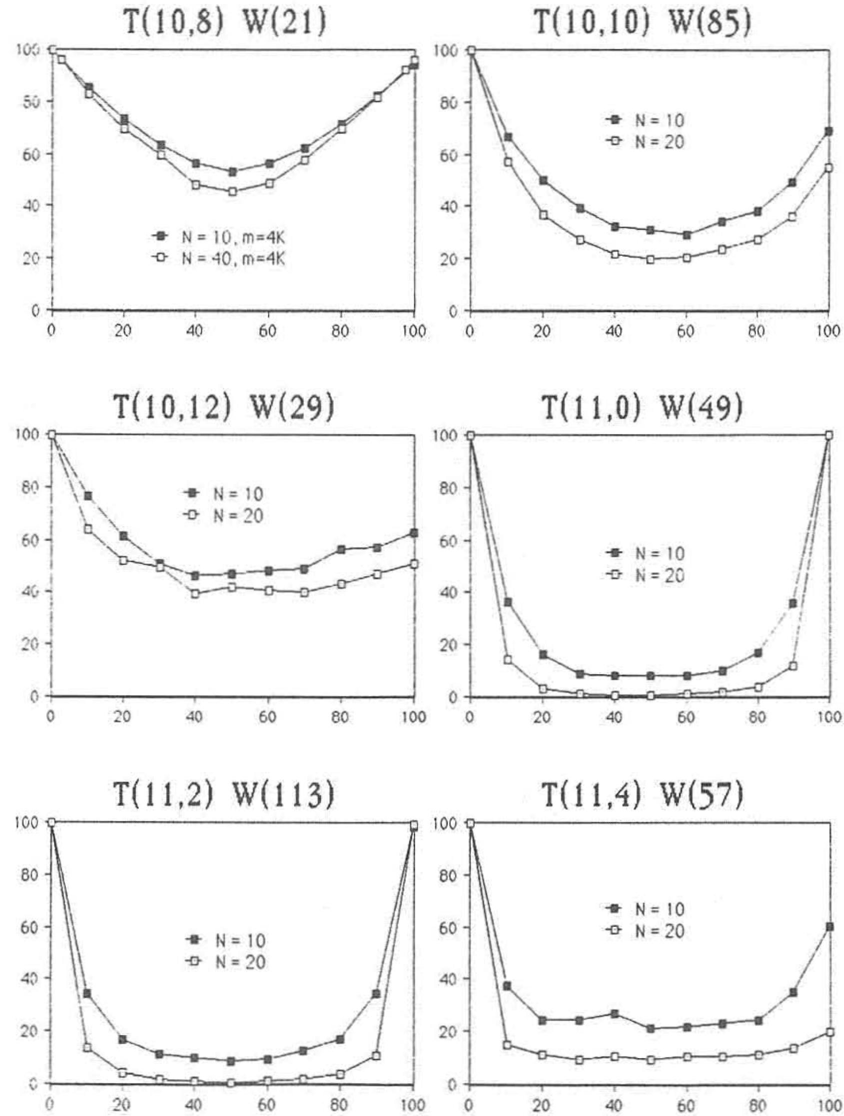


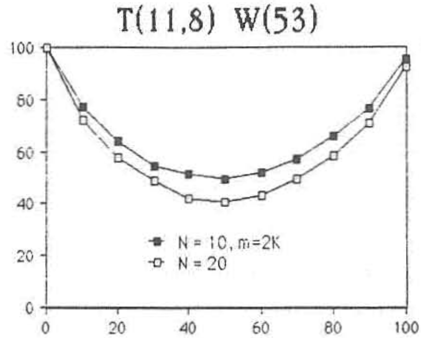

T(6,9) W(150)
N = 10, m=2K
N = 13, m=2K
T(6,10) W(86)
N = 10, m=5K
N = 20
T(6,14) W(94)
N = 10, m=4K
N = 20
T(7,0) W(50)
N = 10
N = 20
T(7,1) W(178)
N = 10
N = 20, m=2K
T(7,2) W(114)
N = 10
N = 20
0 20 40 60 80 100

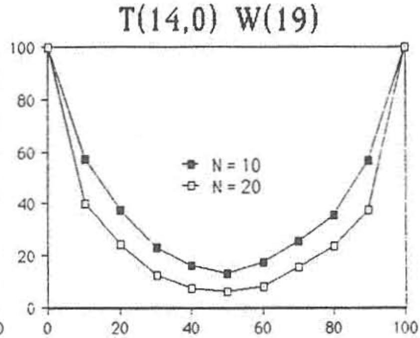

T(7,6) W(122)
N = 10, m=5K
N = 20, m=6K
T(7,8) W(54)
N = 10
N = 20
T(7,10) W(118)
N = 10
N = 20
T(7,14) W(126)
N = 10, m=5K
N = 20, m=5K
T(8,0) W(1)
N = 10
N = 20
T(8,2) W(65)
N = 10
N = 20

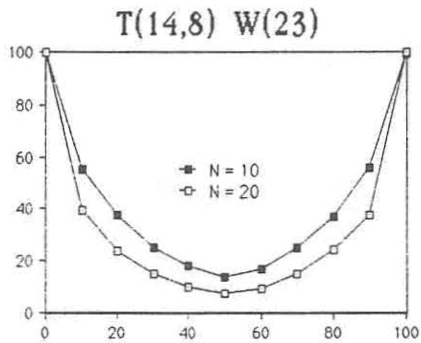

T(8,6) W(73)
N = 10
N = 20
T(8,8) W(5)
N = 10
N = 20
T(8,10) W(69)
N = 10
N = 20
T(8,14) W(77)
N = 10
N = 20
T(9,0) W(33)
N = 10
N = 20
T(9,2) W(97)
N = 10
N = 20

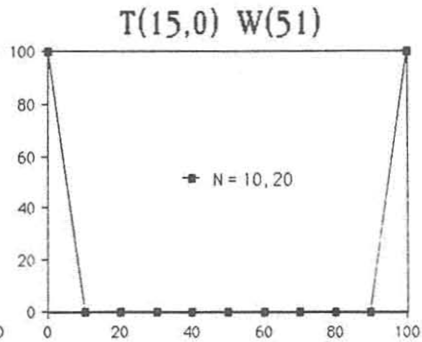

T(9,6) W(105)
N = 10
N = 13
T(9,8) W(37)
N = 10
N = 20
T(9,10) W(101)
N = 10
N = 20
T(10,0) W(17)
N = 10
N = 20
T(10,2) W(81)
N = 10
N = 20
T(10,4) W(21)
N = 10
N = 20


T(10,8) W(21)
N = 10, m=4K
N = 40, m=4K
T(10,10) W(85)
N = 10
N = 20
T(10,12) W(29)
N = 10
N = 20
T(11,0) W(49)
N = 10
N = 20
T(11,2) W(113)
N = 10
N = 20
T(11,4) W(57)
N = 10
N = 20









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