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Periodic Patterns in the Binary Difference Field

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Abstract. The difference sequence of a binary sequence is the binary sequence representing the presence of a difference in value at two neighboring sites in the original sequence. The difference field is the ordered ensemble of all difference sequences aligned one under the other. It is equivalent to the space-time pattern of a one-dimensional cellular automaton under a simple asymmetric rule. Periodic boundary conditions imposed at the boundaries of the propagation net of changes, which is induced by a finite change of values in the initial state, give rise to periodic bands of tilings along these boundary lines. Width and period of these bands evolve in a well-defined way, exhibiting period and bandwidth doubling. A special kind of self-similarity is apparent, and the pattern has a fractal skeleton. Periodic boundary conditions may result from a conservation law imposed on the states in the propagation net.

1. The difference field of a binary sequence: basic properties

Consider a binary sequence in which, for convenience, the binary symbols are represented by 1 and 0. The *difference sequence* (in short, the *difference*) of such a sequence is itself a binary sequence, a 1 being generated whenever there is a change of value at consecutive sites in the original sequence and a 0 otherwise. The difference field is the well-aligned juxtaposition of successive differences in proper order (figure 1). It forms a top-down triangular field with a number of rows in each direction which is equal to the length of the basic sequence.

Note that the binary values are placed on the sites of a triangular lattice. They could have been placed along a square lattice, as is common in the study of the state-time pattern of one-dimensional cellular automata. Indeed, in that case (figure 2), the difference field emerges in the state-time pattern of a ternary one-dimensional cellular automaton with the basic binary sequence imbedded in the initial seed, evolving according to the asymmetric rule

$$\begin{aligned} a^k_{t+1} &= a^k_t \oplus a^{k+1}_t \quad \text{when} \quad a^k_t \neq * \quad \text{and} \quad a^{k+1}_t \neq * \\ &= * \qquad\qquad\quad \text{when} \quad a^k_t = * \quad \text{or} \quad a^{k+1}_t = *. \end{aligned} \tag{1.1}$$

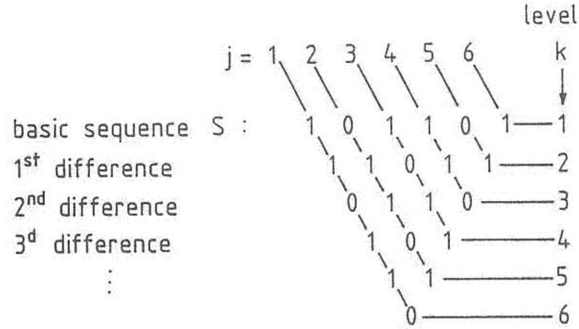


Figure 1: The difference field $DF(S)$ of binary sequence S. Ordering of elements a_k^j.

	k = 1	2	3	4	5	6	7	8
t = 0	*	1	0	1	1	0	1	*
1	*	1	1	0	1	1	*	*
2	*	0	1	1	0	*	*	*
3	*	1	0	1	*	*	*	*
4	*	1	1	*	*	*	*	*
5	*	0	*	*	*	*	*	*
6	*	*	*	*	*	*	*	*

Figure 2: Equivalence of the difference field to the state-time pattern of a cellular automaton (sequences considered circular).

a_t^k represents the value of site k at time t; $\oplus$ is the classical modulo 2 summation.

In the sequel, the triangular-lattice patterns will be used.

The evolution rule given by equation (1.1) is *non-additive*. It means that the difference field of a sequence composed of different subsequences cannot be obtained as a kind of superposition of the difference fields of these subsequences.

On the other hand, rule (1.1) shows *triangular symmetry* for all pure binary sequences (without the $*$-values). This implies left- and right-side *state-time exchangeability*, meaning that the left or right side of the field may be seen as initial state, while the columns parallel to the left or right side are the successive differences of the left- or right-side sequence. As a direct consequence, we formulate the *triangular determination property*: Knowledge of all values on any side of a top-down triangle in a difference

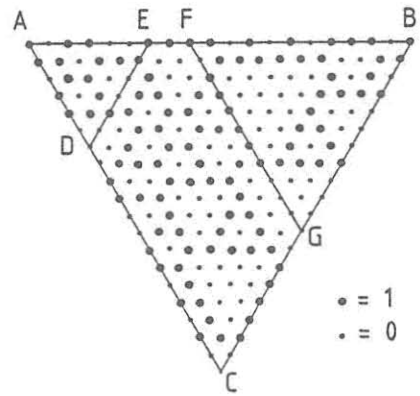


Figure 3: Triangular determination property. The values along line DEFG completely determine all values on or inside ABC.

field completely determines all values in the whole triangle.

In the next sections, boundary conditions along the broken line DEFG, as shown in figure 3, will be considered. It is a corollary of the *triangular determination property* that fixing the values along all sites on this line (being N in number) completely determines the circumscribed field ABC (side size N, also).

2. The parity invariant net

The *parity* of a sequence or a set of binary values (0,1) will be defined as the parity of the number of 1's in that set. We will relate this to the following theorem, the proof of which can be found in [1].

Theorem 1. *The value of a site j on the k^{th} level ($(k-1)^{\text{th}}$ difference) can be derived from the values at level $(k-m)$ as follows:*

$$a_k^j = \sum_{i \in A_m} a_{k-m}^{j+i} \quad \textit{(summation modulo 2)} \tag{2.1}$$

with the following recursive scheme for the set of indices A_m

$$A_{\ell+1} = [A_\ell \cup (A_\ell + 1)] \backslash [A_\ell \cap (A_\ell + 1)] \tag{2.2}$$

and

$$\begin{aligned} A_0 &= \{0\} \\ A_\ell + 1 &= \{\alpha + 1 : \alpha \in A_\ell\}. \end{aligned}$$

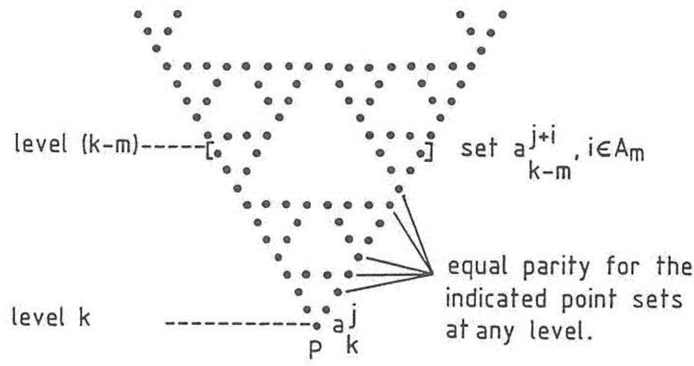


Figure 4: The simple parity invariant net.

A graphical interpretation is illuminating. The theorem means that the parity of any point P in a difference field equals the parity of the points on whatever level as indicated in the graph of figure 4. This also means that the parities of the point sets on any level are equal. Therefore, it seems appropriate to call this ordered set of points the *simple parity invariant net* (SPIN). The parity of a SPIN is the parity of the set of points on any level.

Notice that the SPIN can be constructed recursively, doubling the size of the graph at each step. In fact, it is a fractal set reminding of the so-called Sierpinsky gasket with fractal dimension $\log_2 3$ (see [2]). On each level in the SPIN, the number of points is of the form 2^k. Knowledge of the (invariant) parity completely determines the value of point P.

Now consider a second point P_2 on level k (figure 5.a). Its value is determined by the parity of the corresponding SPIN (P_2). The parity of the joint set (P_1, P_2) then equals the parity of all points on any level in both SPINs taken together, so that if a point belongs to both SPINs, it is counted twice. However, counting a point twice in parity determination is equivalent to deleting these points. Therefore, it is found that there exists a *compound parity invariance net* (CPIN) for the point set (P_1, P_2), which is, in fact, the EXOR composition of the separate SPINs. More points can be added, and the parity of their union may be considered. As EXORing has associative properties, it is clear that we have established the following theorem.

Theorem 2. *There exists a compound parity invariance net for any arbitrary point set on a given level in a difference field. It is the EXOR composition of the simple parity invariance nets for all separate points.*

An example is given in figure 5.b. Notice that the number of points on all levels, except the basic one, is always even. The self-similar properties of

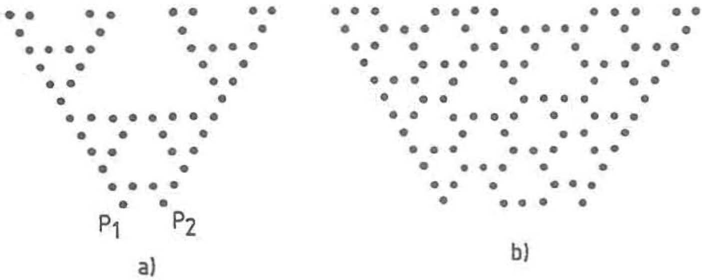


Figure 5: Compound parity invariant nets.

CPINs will be discussed in the next section.

3. The dual of the parity invariant net: the change propagation net (CPN)

Consider a binary sequence (possibly unbounded) and the corresponding difference field. A change in binary value on *one* site in the sequence is seen to induce changes at the underlying levels over an ever increasing region (figure 6.a). The set of sites for which the values change will be called the *simple change propagation net* (SCPN). This set is the upsidedown version of the SPIN. It is also the set of sites with value 1 in the difference field on a infinite binary sequence with a 1 on a single site. It has *additive* properties; i.e., the *compound change propagation net* (CCPN) according to a set of value changes in the basic sequence is the EXOR composition of the SCPNs of all points in that set (figure 6.b).

Lemma 1. *The values along columns parallel to the sides of the simple CPN form a periodic chain. The* k^{th} *column has minimal period* $2^{\lceil log_2 k \rceil}$ *(see figure 6).*

This follows from the recursive construction of the SCPN as discussed in section 2. As a consequence, there is a *doubling of the period length* by moving from column 2^k to column $2^k + 1$.

Theorem 3. *(Generalization of lemma 1.) The values along columns parallel to the sides of the compound CPN form a periodic chain. The* k^{th} *column has minimal period* $2^{\lceil log_2 k \rceil}$.

Proof. Column k is the EXOR composition of columns 1, $k - p_1 + 1 \ldots$, $k - p_m + 1$ of the SCPN, where p_m is the largest site number not exceeding k. It follows from lemma 1 and the superposition property that $2^{\lceil \log_2 k \rceil}$ is a

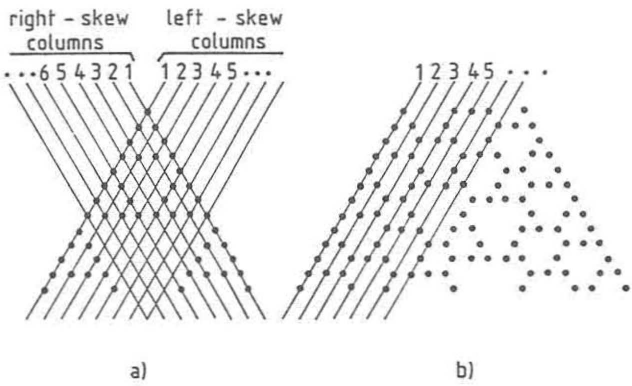


Figure 6: (a) Simple change propagation net (SCPN), (b) Compound change propagation net (CCPN).

period length of the resulting sequence. It is also the minimal period as a consequence of two observations. The first one is that the period sequence in column k and of length $2^{\lceil \log_2 k \rceil}$ has nonidentical first and second half parts. The second one is that if there would be a smaller period-length P, there must be a minimal period length which is a common divisor of $(P, 2^{\lceil \log_2 k \rceil})$, i.e. of the form 2^m (m integer $< \lceil \log_2 k \rceil$). But this would contradict the first observation. ∎

Theorem 4. *The compound change propagation net and its dual, the compound parity invariant net, have a fractal structure with fractal dimensions* $log_2 3$.

Proof. Consider a CCPN with base length ℓ up to level 2^k, with $\ell < 2^k$ (figure 7). The CCPN up to level 2^{k+1} is obtained by copying the hatched areas between levels 1 and 2^k on the levels $2^k + 1$ to 2^{k+1} as indicated. This is a consequence of theorem 3. As to the configuration of the UVW part, this is the *EXOR superposition* of the configuration in EFD and HCG. This is a consequence of the relationship of values on levels in the difference field which are a distance 2^k apart, as given by the SPIN: $a^j_{\ell_\circ + 2^k} = a^j_{\ell_\circ} \oplus a^{j+2^k}_{\ell_\circ}$ (see equation (2.1),(2.2)). Globally speaking, this means that the configuration on levels $2^k + 1$ to 2^{k+1} is obtained by EXORing two copies of the original configuration ABCD on these levels (one copy aligned to the left, the other to the right).

The same procedure applies when doubling the linear dimension once more. Notice that by the periodicity theorem, the parts which are to be EXORed every time, remain the same. This allows us to write the following recursion for the number V of 1-valued points up to the level 2^{k+1}:

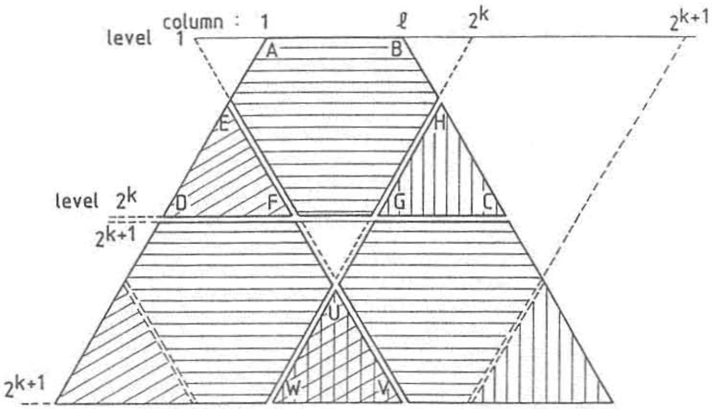


Figure 7: Recursive structure of the compound change propagation net.

$$V(2^{k+1}) = 3V(2^k) - V_c \quad \text{for } k \geq k_0 \text{ and } 2^{k_0} \geq \ell, \tag{3.1}$$

in which V_c = number of points in HGC + number of points in EDF - number of points in [HCG $\oplus$ EDF].

Solving the recursion (3.1) gives the growth rate equation.

$$\begin{aligned} V(2^{k+k_0}) &= 3^k[V(2^{k_0}) - \frac{V_c}{2}] + \frac{V_c}{2} \quad (3.2) \\ &= 2^{(k+k_0)\frac{\log_2\{3^k[V(2^{k_0})-\frac{V_c}{2}]+\frac{V_c}{2}}{k+k_0}} \quad (3.3) \end{aligned}$$

So, the growth rate is volume dependent. From equation (3.3), we find the fractal dimension as [3]:

$$\lim_{k\to\infty} \frac{\log_2\{3^k[V(2^{k_0}) - \frac{V_c}{2}] + \frac{V_c}{2}\}}{k + k_0} = \log_2 3. \tag{3.4}$$

∎

4. The partially extended and globally extended difference field

Consider the difference field of a basic sequence S. This field can be extended by adding two values on the second level, one at the left and one at the right boundary. This extends automatically the field on the third level, where once again two new values are added at the boundaries.

This can be continued in an unbounded way. The field obtained in this way will be called the *partially extended field* (figure 8). It is completely determined by the basic sequence values and by the extra added values, which

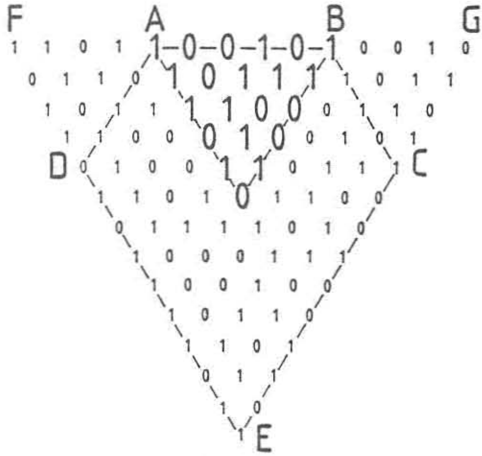


Figure 8: ABCDE: inner partially extended difference field of 100101. FGE: globally extended difference field. FAD, BGC: left and right outer extended field.

are actually placed along the boundary of any CCPN whose basis covers the basic sequence. By the corollary of the Triangular Determination Property of section 1, the circumscribed triangular field is completely determined. This will be called the *globally extended field.*

We will now consider *periodic boundary conditions* along columns AD and/or BC. This presupposes a potentially unbounded field size, although it is possible to interpret the properties on a more restricted field (sufficiently extended to incorporate a few periods for the lines considered). Boundary conditions along AD and BC may be taken totally independent of each other. Configurations on columns parallel to AD and to BC will be considered (these will be referred to as left- and right-skew columns). Because of the symmetry of the difference rule, all properties which are valid for left-skew columns are also valid for right-skew columns. Therefore, only left-skew columns will be considered in the derivations that follow.

Some notations:

$Sp(j)$ will denote the *minimal period sequence* on column j.

$P[Sp(j)]$ will denote the *period length* of $Sp(j)$.

$\mathrm{Par}[Sp(j)]$ will denote the *parity* of the sequence $Sp(j)$.

A single element in the field will be denoted by (r, c), r being the row number and c the column number.

$V(r, c)$ represents the value of element (r, c), $\bar{V}(r, c)$ its complement.

$\text{Par}[(a, b), (c, d), (c, f) \ldots]$ represents the parity of the set of elements mentioned between $[\ldots]$.

Theorem 5. *In the globally extended field, either all columns are periodic or all are non-periodic. Left-skew columns may show a period-doubling from left to right.*

Proof. Consider column j, with $Sp(j)$ and $P[Sp(j)] = p$. Then, column $(j-1)$ must be periodic too as it is the uniquely determined difference of column j which is periodic. Moreover, $P[Sp(j-1)] \leq P[Sp(j)]$. Column $(j+1)$ must also be periodic. With a given $Sp(j)$, two possible corresponding sequences which are each others complement may occur at column $(j+1)$. (As complementary sequences are the only different sequences with equal differences). So, column $(j+1)$ is a sequence of alternating complements, or a succession of either one of the two possible sequences. In the latter case, $P[Sp(j+1)] = P[Sp(j)]$. The former case implies $P[Sp(j+1)] = 2P[Sp(j)]$. ∎

We will now elaborate further on the evolution of the periods.

Lemma 2.

$$\begin{aligned} Par[Sp(j)] &= odd \begin{cases} \Leftrightarrow P[Sp(j+1)] = 2P[Sp(j)] \\ \Rightarrow Par[Sp(j+1)] = odd/even \text{ when} \\ \quad P[S(j)] = odd/even \end{cases} && (4.1) \\ Par[Sp(j)] &= even \Leftrightarrow P[Sp(j+1)] = P[Sp(j)]. && (4.2) \end{aligned}$$

Proof. Consider a period sequence $Sp(j)$ with $P[Sp(j)] = p$. According to the parity invariance net theorem 2:

$$\text{Par}[(1, j+1), (p+1, j+1)] = \text{Par}[Sp(j)].$$

If $\text{Par}[Sp(j)] = \text{odd}$, this means that $V(p+1, j+1) = \bar{V}(1, j+1)$. In consequence of this and theorem 5, there is a period doubling on column $(j+2)$. Also, $V(k+p, j+1) = \bar{V}(k, j+1)$ so that there is an equal number of 1' and 0's in $Sp(j+1)$. This number equals $P[Sp(j)]$ and this implies evenness of $Sp(j+1)$ in case $P[Sp(j)] = \text{even}$, and oddness in the opposite case. If $\text{Par}[Sp(j)] = \text{even}, V(p+1, j+1) = V(1, j+1)$ and there is no change in period length. ∎

Lemma 3. *$Par[Sp(j)] = odd$ and $\frac{P[Sp(j)]}{2^{k_0}} = odd$ for some integer $k_0 \geq 1$ implies that*

$$Par[Sp(j + 2^{k_0})] = Par[Sp(j)] = odd \tag{4.3}$$

and

$$P[Sp(j + 2^{k_0})] = 2P[Sp(j)]. \tag{4.4}$$

Proof. (See figure 9.) Partition $Sp(j)$ in $N_0 = P[Sp(j)]/2^{k_0}$ groups of 2^{k_0} successive elements each. Consider the same groups in the second period on column j. Select groups $1, 3, 5 \ldots N_0$ in the first period and groups $2, 4, \ldots (N_0 - 1)$ in the second period. Construct the parity invariant net on the elements of the selected groups and observe that this net completely covers all elements of a sequence S_α of length $2P$ in column $(j + 2^{k_0})$. (This is only true for $N_0 =$ odd, and follows from the overall PIN-structure). According to the PIN-properties:

$$\begin{aligned}\text{Par}[S_\alpha] &= \text{Par}[\text{group } 1 + \text{group } 2 + \text{group } 3 + \cdots + \text{group } N_0] \\ &= \text{Par}[Sp(j)]. \end{aligned} \tag{4.5}$$

Now, Par$[Sp(j)]$ is odd (premise), and so is Par$[S_\alpha]$. This means that sequence S_β on column $(j + 2^{k_0} + 1)$ with length $(S_\beta) = 2 \cdot$length (S_α), is a concatenation of 2 complementary sequences of length $2P$ (lemma 2). These have identical differences (on column $(j + 2^{k_0})$), implying that $S_\alpha = Sp(j + 2^{k_0})$. Together with equation (4.5) this gives equation (4.3). As length $(S_\alpha) = 2P[S(j)]$, equation (4.4) follows.

Lemma 4. *Par*$[Sp(j)] =$ *odd and* $\frac{P[Sp(j)]}{2^{k_0}} =$ *odd (for some* $k_0 \geq 1$*) implies that*

$$P[Sp(j + \alpha)] = 2P[Sp(j)] \text{ for } 1 \leq \alpha \leq 2^{k_0}. \tag{4.6}$$

Proof. Lemmas 2 and 3 imply that both $P[Sp(j + 1)]$ and $P[Sp(j + 2^{k_0})]$ equal $2P[Sp(j)]$. By the monotic evolution of periods implied by theorem 5, the period lengths on all intermediate columns must be constant and equal to $2P[Sp(j)]$. ∎

As a direct consequence of the lemmas 3 and 4, the following global theorem can now be formulated.

Theorem 6. *(odd parity - even period theorem). If* $\frac{P[Sp(j)]}{2^{k_0}} =$ *odd (for some* $k_0 \geq 1$*) and Par*$[Sp(j)] =$ *odd, then*

$$P[Sp(j + \alpha)] = 2^m \cdot 2^{k_0} \text{ with } m = \lceil \log_2(\frac{\alpha}{2^{k_0}} + 1) \rceil. \tag{4.7}$$

Remark. $m = \lceil \log_2(\frac{\alpha}{2^{k_0}} + 1) \rceil$ is equivalent to $m = \lfloor \log_2(\frac{\alpha-1}{2^{k_0}} + 1) + 1 \rfloor$. Both equalities arise from the condition $(2^{m-1} - 1)2^{k_0} + 1 \leq \alpha \leq (2^m - 1)2^{k_0}$, $m = 1, 2, \ldots$, resulting from application of lemma 4. Notice that the conditions of the theorem always imply that $P[Sp(j)] =$ even. For $P[Sp(j)] =$ odd, following theorem emerges:

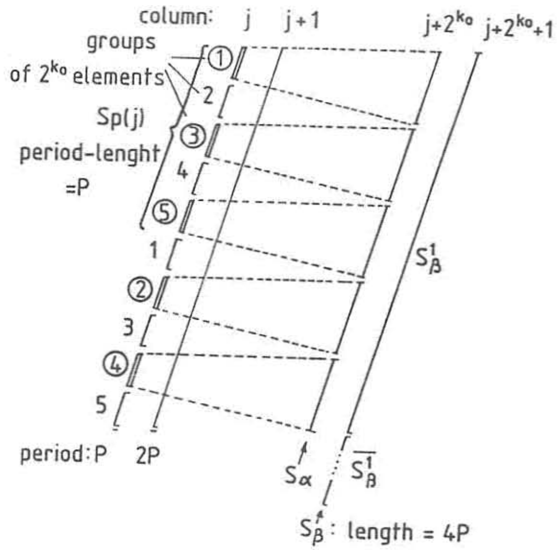


Figure 9: Relationship between parity and period length of columns j and $j + 2^{k_0}$.

Theorem 7. *(odd parity - odd period theorem). If* $P[Sp(j)] = odd$ *and* $Par[Sp(j)] = odd$, *then*

$$P[Sp(j + \alpha)] = 2^{m+1} \text{ with } m = \lfloor \log_2 \alpha \rfloor. \tag{4.8}$$

Proof. This is a direct consequence of lemma 2 which implies that $P[Sp(j+1)] = 2P[Sp(j)] =$ even and $\text{Par}[Sp(j+1)] =$ odd under the given conditions. This then makes $Sp(j + 1)$ satisfy the conditions of theorem 6 with $k_0 = 1$. ∎

Theorems 6 and 7 both deal with the conditions that the first period sequence considered had an *odd* parity. This fixes completely the further periodicity evolution in strict dependence on the length of that period sequence, but *totally independent of any other boundary condition* in the field. In what follows, *even* parity for $Sp(j)$ will be considered. It will be shown that a given period length may persist over a band of arbitrary width, by fixing certain boundary conditions. Of course, in view of theorems 6 and 7, the parity within this fixed-period band remains even up to the (rightmost) last column with the same period length. From then on, the odd-parity theorems begin to work.

Theorem 8. *(Even parity-odd period theorem) For* $P[Sp(j)] = odd$ *and* $Par[Sp(j)] = even$, *it is always possible to construct* $Sp(j + 1)$ *in such a way that* $P[Sp(j + 1)] = P[Sp(j)]$ *and* $Par[Sp(j + 1)] = odd$ *or even.*

Proof. Consider a completely specified $Sp(j)$ satisfying the conditions of the theorem. Set $p = P[Sp(j)]$. From lemma 2 it follows that $P[Sp(j + 1)] = p$.

Pick out *one* arbitrary place $(\alpha, j+1)$ with $1 \leq \alpha \leq p$. Due to the oddness of p, it is always possible to construct a PIN relating $\mathrm{Par}[\{(\mu, j+1) : \mu = 1 \ldots p, \mu \neq \alpha\}]$ to the parity of a set $\Omega_{j,\alpha}$ of elements in $Sp(j)$. So, we have

$$\begin{aligned} \mathrm{Par}[Sp(j+1)] &= \mathrm{Par}[(\alpha, j+1), \{(\mu, j+1)\ \mu = 1 \ldots p, \mu \neq \alpha)\}] \\ &= \mathrm{Par}[(\alpha, j+1),\ \Omega_{j,\alpha}]. \end{aligned} \tag{4.9}$$

As $\mathrm{Par}[\Omega_{j,\alpha}]$ is given, equation (4.9) implies that $\mathrm{Par}[Sp(j+1)]$ can be set to either parity by appropriately choosing $V(\alpha, j+1)$. Notice that fixing a single element on $(j+1)$ completely determines column $(j+1)$. ∎

Remark. A repeated application of this theorem implies the possibility of constancy of period length over an arbitrary number of columns (possibly leading to periodic behavior over columns).

Theorem 9. *(Even parity-even period theorem.)*

A. *$P[Sp(j)] = 2^k (k \geq 1)$ and Par $[Sp(j)]$ = even $\Rightarrow$ $Sp(j)$ is a column in a band of width 2^{k-1} according to the period doubling scheme of theorem 6.*

B. *$P[Sp(j)] = P$ = even, but $\neq 2^k$ $(k \geq 1)$ and $\mathrm{Par}[Sp(j)]$ = even*

$$\Rightarrow \begin{cases} \mathit{Par}[Sp(j+1)] = \mathit{even} \overset{a}{\Leftrightarrow} \mathit{Par}[(1,j),(3,j)\cdots(p-1,j)] = \mathit{even} \\ \qquad\qquad\qquad\qquad \overset{b}{\Leftrightarrow} \text{it is always possible to construct } Sp(j+2) \text{ such that } \mathit{Par}[Sp(j+2)] = \text{odd or even.} \end{cases}$$

Proof.

A. Observe that according to the PIN property : $\mathrm{Par}[Sp(j)] = \mathrm{Par}[V(1, j - 2^k)]$. As this parity is odd, $V(1, j-2^k) = 0$. As $Sp(j)$ is periodic, any period has the same parity and so all elements on column $(j - 2^k)$ have a value $= 0$. So $P[Sp(j-2^k)] = 1$, and all columns of rank less than $(j - 2^k)$ are identically zero. Now, either all first elements of all columns of rank $> j - 2^k$ are zero and then all elements are zero which implies that $P[Sp(j)] \neq 2^k (k \geq 1)$), or there is a column $(j - \rho) > j - 2^k$ with $V(1, j - \rho) = 1$. It is a column with all elements equal to 1, and it satisfies the conditions of the odd parity-odd period theorem 7. It even follows that if $P[Sp(j)] = 2^k$ is the minimal period of $Sp(j)$, ρ satisfies $2^{k-1} \leq \rho < 2^k$. The actual value of ρ depends on the parity of a certain subset of $Sp(j)$ and can be derived using the PIN-properties. If the elements in $Sp(j)$ can be chosen freely, it is always possible to make $Sp(j)$ have an arbitrary rank in the band of width 2^{k-1} in which it is embedded.

B. $\overset{a}{\Leftrightarrow}$: is directly clear from the PIN connecting the elements on columns $j+1$ and j.

$\overset{b}{\Leftrightarrow}$: consider column $(j+2)$. Par$[Sp(j+2)]$ = even iff Par$[(1, j+1), (3, j+1), \ldots (p-1, j+1)]$ = even. The last parity also equals Par$[(p-1, j+1)+$ some elements on columns of rank $\leq j$ as can be found from the PIN covering the points $[(3, j+1) \ldots (p-1, j+1)]$. So, it suffices to choose $V(p-1, j+1)$ in an appropriate way to make the parity above even. Notice that this choice completely determines $Sp(j+1)$ and sets Par$[Sp(j+2)]$ = even. Then $Sp(j+1)$ and $Sp(j+2)$ satisfy the conditions of the theorem, so that it can be repeatedly applied in order to provide a band of columns of arbitrary width. ∎

Until now, we have considered the possible evolution of periodicities for left-skew columns when moving from left to right when starting with an initial $Sp(j)$. In case $P[Sp(j)]$ = odd, this evolution of periodicities is fixed in a way which solely depends on Par$[Sp(j)]$. When $P[Sp(j)]$ = even, some arbitrariness in the possible width of bands of a given period length is possible by properly choosing some boundary conditions. For the sake of completeness, we still have to consider the evolution of left-skew columns when moving to the left. As moving to the left of $Sp(j)$ means that we consider the difference field of column j, it is completely determined by $Sp(j)$. In fact, the evolution of some $Sp(j+\alpha)$ in *horizontal* direction to the left is the state-time evolution of a fixed length circular cellular automaton. It is known that these evolve towards some limit cycle of a restricted set of states. Hence, when moving to the left, after crossing a transient zone, a region will eventually be entered in which there is also periodicity in the horizontal direction. We state some specific results for the periodic evolution for extension of left-skew columns to the left.

Theorem 10. *If $P[Sp(j)]$ = odd, then $\forall k \geq 1$: $P[Sp(j-k)] = P[Sp(j)]$ and Par$[Sp(j-k)]$ = even.*

Proof. Suppose Par$[Sp(j-\alpha)]$ = odd for some $\alpha \geq 1$, then there must be a period doubling on line $j-\alpha+1$ (theorem 5). This would imply $P[Sp(j)]$ = even, what contradicts the premises. ∎

Theorem 11. *A. $P[Sp(j)] = 2^k$(= even) implies that j is a column in a band of columns of constant periodicity $P[Sp(j)]$ and bandwidth 2^{k-1}. With ρ the rank of j in this band (which depends on the parity of some subset in $Sp(j)$), halving of the period must occur at columns $j+\rho-(2^\alpha-1)\cdot 2^{k-\alpha}$, $\alpha = 1, 2, \ldots, (k-1)$. Column $j+\rho-2(2^{k-1}-1)$ is full of 1's, and all columns further to the left are 0.*

B. $P[Sp(j)]$ = even $\neq 2^k$, and $\frac{P[Sp(j)]}{2^{k_0}} = p_0$ = odd implies that either there is:

a) an unlimited band of columns with period $P[Sp(j)]$:

1. *sufficient is that $Sp(j)$ already belongs to a band of width $> 2^{k_0-1}$ (this is possible by theorem 9.B).*
2. *if $Sp(j)$ is the first column of a band of width $= \omega < 2^{k_0-1}$, it is necessary that the parity of the elements of $Sp(j)$ covered by the PIN which covers the first $\frac{P[Sp(j)]}{2}$ elements of column $(j+\omega-2^{k_0-1}-1)$ is even. Or,*

b) a band of width 2^{k_0-1} and period $P[Sp(j)]$ surrounding $Sp(j)$: sufficient is that the parity of the elements of $Sp(j)$ covered by the PIN which covers the first $\frac{P[Sp(j)]}{2}$ elements of column $(j+\omega-2^{k_0-1}-1)$ is odd.

Proof.

A. This is the only way to extend the difference field to the left without contradicting theorem 9.A.

B. Suppose that under condition a)1, there would be a halving of period. The parity of the first halve period must then be odd and the odd-parity theorems 6 or 7 apply. These would be in contradiction with the premises. For a)2 and b): if not so, theorems 6 or 7 should become violated again. ∎

The results obtained in this section are summarized in configuration table 1 showing period sequence parity (Py), period length (Pl), and bandwidth (Bw). P is the length of the initial period sequence. 2^{k_0} is such that $P/2^{k_0} =$ odd (for some $k_0 \in \{0,1,2,\ldots\}$).

Corollaries to theorems 10 and 11:

1. The occurrence of one period-doubling induces an infinite series of period doublings. The bandwidths are also doubling. As a consequence, it is impossible to find identical period sequences on two different columns (identical even under shift).

2. In bands of constant *irreducible* period length P (irreducible: the period is not a multiple of a smaller one), the bandwidth never exceeds $P/2$, unless it is unbounded.

3. For bands of constant (but possibly reducible) period length P, starting with an odd parity period P_0 (from left for left-skew bands), the bandwidth never exceeds $1+(2^{\alpha}-1)2^{k_0}$, (*with* $k_0 : P_0/2^{k_0} =$ odd and $\alpha = \log_2 P/P_0$). The absolute upper bound occurs for $P_0 = 1$ and equals P.

5. Some examples of extended binary fields

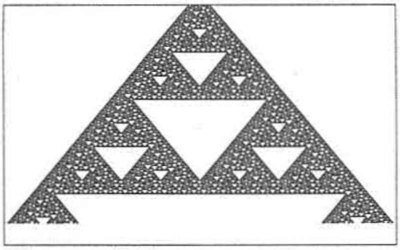
Any change propagation net is a special case of an extended binary field with initial state the sequence of initial value-changes and with periodic left- and right-skew columns full of 1's. (figure 10.a)

```
A. Odd parity of initial period-sequence
  a. Extensions to the right

     Py   [O] E...E  O  E...E  O  E...E  O  E..
     Pl   [P] 2P.....2P 4P.....4P 8P.....8P 16P..

     Bw          2^k0       2^(k0+1)      2^(k0+2)

  b. Extensions to the left
    1. P: odd
     Py   ....E  E  E  E [O] O  E  O  E  E  E  O
     Pl   ....P  P  P  P [P] 2P 4P 4P 8P 8P 8P 8P

     Bw       unbounded        according to A.a.

    2. P= even = 2^k (k= 0,1,2 ... )

     Py ..E  E  E  O  O  E  O  E...E...O    E...[O]
     Pl ..1  1  1  1  2  4  4  8...P/2...P/2 P...[P]

     Bw   unbounded     1  2   4    2^(k-1)     2^k

    3. P= even ≠ 2^k
     Py ...E..E  E  E [O]      ..O   E  E...E [O]
     Pl ...P..P  P  P [P]  or  ..P/2 P        [P]

     Bw    unbounded           b.1 or b.3  2^(k0-1)

B. Even parity of initial period - sequence
   This sequence can only coincide with an (E,P)- sequence from
   the configurations under A. The corresponding difference-field
   configuration must be consistent with one of the possibilities
   mentioned under A.
```

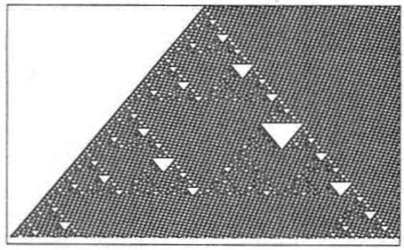
Table 1: Configuration of possible parities, period lengths, and bandwidths.

An astonishing variety of patterns emerges, depending on the specific choice of periodic (or aperiodic, chaotic or random) boundary sequences (figures 10 and 11). For any periodic boundary (left/right, or both), a difference in patterns between the inner partially extended field and the (left/right or both) outer extended fields becomes apparent. Whereas the (inner) partially extended field mostly shows a very complex pattern, the outer extensions become periodic (for increasing periods, the structure of a period may become complex too). For an aperiodic boundary, such a clear distinction in patterns between inner and outer extension does *not* seem to exist. It could be worthwhile to investigate this different behavior in view of aperiodicity and randomness characterizations of sequences.

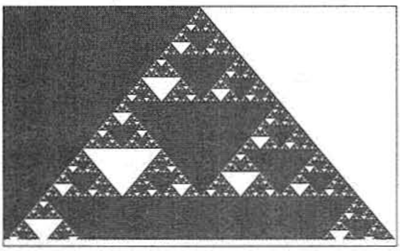
In the next section, we will have a closer look at the global structure.



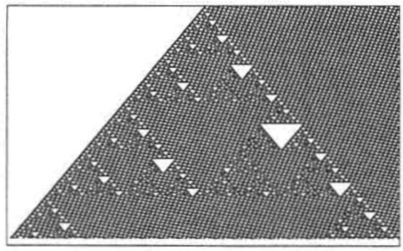
(a) A change propagation net (CCPN) BF(0, 10010111100101011010, 0)



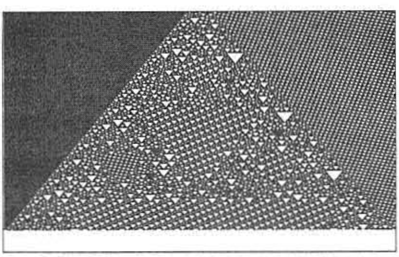
(b) A right half-infinite CCPN BF(0, , 11110)



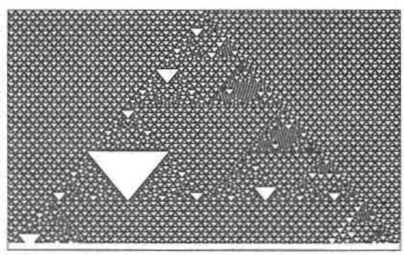
(c) A left half-infinite CCPN BF(101, , 0)



(d) BF(101, , 11110) = figure (b) $\oplus$ figure (c) (theorem 12)

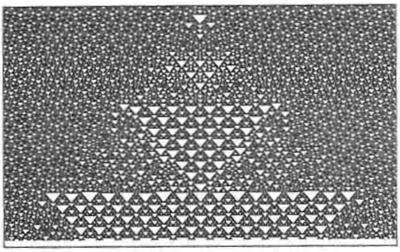


(e) BF(101, 10010111100101011010, 11110) = figure (a) $\oplus$ figure (b) $\oplus$ figure (c) (theorem 12)

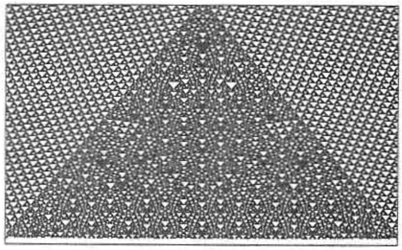


(f) BF(1111110, , 1110111)

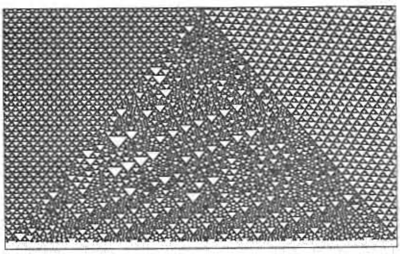
Figure 10: Examples of difference fields (number of levels $\simeq$ 315).



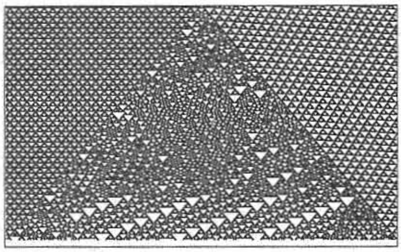
(a) BF(101011100110100,, 101011100110100)
Identical left and right period sequences.



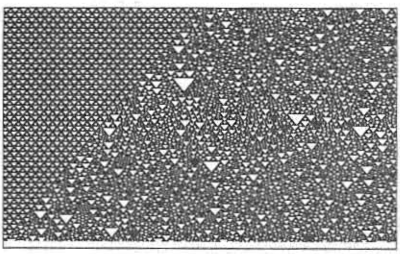
(b) BF(1110101110,,1110101110)
Identical left and right period sequences.



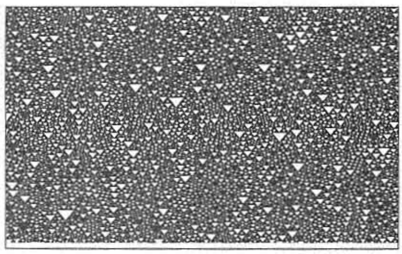
(c) BF(1010110,,011010101)



(d) BF(0110101,,011010101)
Compare to 12(c): shift in left sequence.



(e) BF(1010101,,random): there is still left-skew periodicity.



(f) BF(random,,random): completely unstructured.

Figure 11: Examples of difference fields (continued).

6. Recursive and fractal properties of the inner extended field

6.1 A decomposition property

In section 3, we discussed the fractal properties of the CCPN, which is a special case of an extended difference field. Its fractal dimension was derived on basis of the growth rate equation (3.3) which was easily determined because the whole structure could be built up recursively starting with small-sized building blocks of known mass (number of 1's in it).

In the more general case, such an exact recursive procedure based on finite building blocks does not seem to exist. However, a growth rate equation can be found with some bounded unknown parameters. This shows that the overall structure of the inner extended field can be decomposed in two subsets. One has a fractal nature with a fractal dimension which seemingly depends on specific boundary conditions and that may possibly differ from $log_2 3$. The other subset has dimension 0, 1, or 2.

We will first introduce a notation for characterizing a specific (extended) difference field. A field will be denoted by $BF(P_L, I_S, P_R)$, in which P_L, I_S, P_R, respectively refer to the left periodic sequence, the interconnecting sequence at the basic level and the right periodic sequence (figure 12). I_S may be the empty sequence; the corresponding notation is $BF(P_L,, P_R)$. All fields and nets will be considered relative to a *reference point* in the lattice : i.e. the point at which the first element of P_L is located. A shift of ℓ places to the right will be represented by multiplication of $BF(.,.,.)$ with z^{ℓ}, a shift to the left by multiplication with $z^{-\ell}$. Notice that any CCPN is a $BF(1, I_S, 1) = z^{-1} \cdot BF(0, 1 + I_S + 1, 0)$. (+ stands here for concatenation). $BF(P_L, I_S, 1) = z^{-\infty} BF(., H(P_L) + I_S, 1)$ will be called a *left half-infinite CCPN*. $BF(1, I_S, P_R) = BF(1, I_S + H(P_R), .)$ is a *right half-infinite CCPN*. $H(P_L), H(P_R)$ are the equivalent boundary conditions on the basic level corresponding to P_L, P_R respectively. After a *transient zone*, these sequences become periodic too.

As a consequence of the additive property of CCPN, we can now state:

Theorem 12. *(Decomposition theorem). Any extended binary difference field can be decomposed as a sum (modulo 2) of a left and right half-infinite field and of a finite field as follows:* ($0[len(I_S)]$ *denotes a sequence of 0's of length equal to the length of* I_S)

$$\begin{aligned} BF(P_L, I_S, P_R) &= BF(P_L, 0[len(I_S)], 0) \oplus BF(0, 0[len(I_S)], P_R) \\ &\quad \oplus BF(0, I_S, 0) \\ &= BF(P_L,, 0) \oplus z^{[1+len(I_S)]} BF(0,, P_R) \\ &\quad \oplus BF(0, I_S, 0). \end{aligned} \tag{6.1}$$

6.2 The half-infinite CCPN

Now consider a left half-infinite CCPN: $BF(P_L,, 0)$ in which the reference point has value 1 (figure 13). Distinguish between F_L, the left outer extended field with transient zone included, and F_L^P, the two-dimensional periodic part

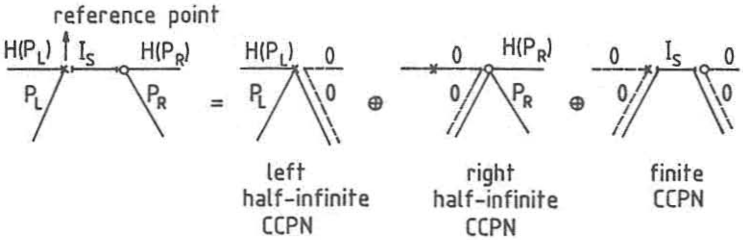


Figure 12: The decomposition theorem.

of it. $F_L(k), \alpha_L(k), \beta_L(k+1)$ refer to the triangular fields ABD, BCD, EIJ as shown in figure 13. k refers to the exponent of the linear field size 2^k. This CCPN satisfies the periodicity properties of difference fields with periodic boundary conditions (section 4). There are right-skew bands of width 2^k with periodicity 2^k (k integer, these bands start from the rightmost 1-sequence). Left-skew sequences have a bandwidth which is smaller than the period length $P[P_L]$. Consider a basic triangular field with linear dimension 2^k and volume (number of 1's it contains) $V(2^k)$. Doubling the linear scale to 2^{k+1} gives a structure which can only be partly reconstructed from the extended field of linear dimension 2^k, as shown in figure 13. Notice the appearance of small top-down triangular fields at levels $mP[P_L]+1$, $(m = 1, 2, \ldots)$. These are doubling in size for increasing $m = 2^\ell$. The growth rate of V from linear dimension 2^k to 2^{k+1} satisfies:

$$V(2^{k+1}) = 2V(\alpha_L(k)) + V(\beta_L(k+1)) + V(F_L(k)). \tag{6.2}$$

Now, field $\beta_L(k+1)$ is of the same nature as field $\alpha_L(k)$, and so we write

$$V(\beta_L(k+1)) = \mu_{k+1} V(\alpha_L(k)) = \mu_{k+1} V(2^k) \tag{6.3}$$

with μ_{k+1} uniformly bounded above.

Also,

$$V(F_L(k)) = 2^{k-1}(2^k+1)(\rho_\infty + \nu_k) \tag{6.4}$$

in which ρ_∞ is the limit density over the unbounded two-dimensional periodic region F_L. ν_k is a correction factor for considering only a finite-dimensional triangle $F_L(k)$ in F_L; it is uniformly bounded (will $\to 0$ for $k \to \infty$). The recursion therefore becomes:

$$V(2^{k+1}) = (2 + \mu_{k+1})V(2^k) + 2^{k-1}(2^k+1)(\rho_\infty + \nu_k). \tag{6.5}$$

Defining

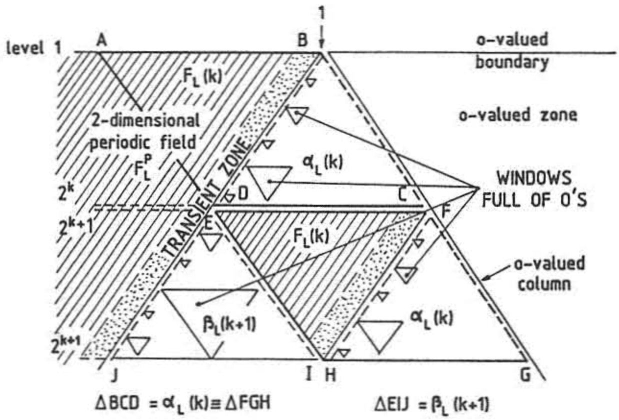


Figure 13: The left half-infinite CCPN.

$$\varphi_\ell = \sqrt[\ell]{(2+\mu_{k+1})(2+\mu_{k+2})\cdots(2+\mu_{k+\ell})} \tag{6.6}$$

and starting with a linear dimension $V(2^{k_0})$, the solution of (6.5) is given by:

$$\begin{aligned} V(2^{k_0+\ell}) &= \{V(2^{k_0}) + 2^{k_0-1}\cdot 2^{k_0}\tfrac{1+2^{-k_0}}{\varphi_1}(\rho_\infty+\nu_{k_0}) \\ &\quad +\tfrac{4(1+2^{-(k_0+1)})}{\varphi_2^2}(\rho_\infty+\nu_{k_0+1}) \\ &\quad +\tfrac{4^2(1+2^{-(k_0+2)})}{\varphi_3^3}(\rho_\infty+\nu_{k_0+2})\cdots \\ &\quad +\tfrac{4^{\ell-1}(1+2^{-(k_0+\ell-1)})}{\varphi_\ell^\ell}(\rho_\infty+\nu_{k_0+\ell-1})\} \qquad (6.7) \\ &= \varphi_\ell^\ell V(2^{k_0}) \\ &\quad +2^{2k_0-1}(\rho_\infty+\nu_{k_0})(1+2^{-k_0}) \\ &\quad [\textstyle\sum_{j=0}^{\ell-1}(\frac{\rho_\infty+\nu_{k_0+j}}{\rho_\infty+\nu_{k_0}})(\frac{1+2^{-(k_0+j)}}{1+2^{-k_0}})\frac{\varphi_\ell^\ell}{\varphi_{j+1}^{j+1}}]. \end{aligned}$$

Now there are two possibilities:

Possibility 1. $\rho_\infty = 0$ iff $\mathrm{len}[P_L] = 2^\ell$ *for some integer* ℓ. $\rho_\infty = 0$ *also implies that the fractal dimension of the inner extended field is equal to* $\log_2 3$.

The "iff" follows from the periodicity properties of section 4. $\rho_\infty = 0$ implies that the zone to the left of P_L becomes a pure 0-valued area, possibly after crossing a transient band (which implies $\nu_j \neq 0$). In case $\nu_j = 0$, the transient zone does not exist and the inner extended field is identical to the SCPN, and it has fractal dimension $\log_2 3$. When $\nu_j \neq 0$, the condition $\rho_\infty = 0$ produces a field that is identical to a CCPN, so it has fractal dimension $\log_2 3$, too. Limiting this CCPN to the inner extended field under consideration does not alter the fractional dimension, because of the constant width of the transient band. This implies that the volume contribution of this band is bounded above by linear proportionality to the linear scale, so it is dominated by $\log_2 3$.

Possibility 2. $\rho_\infty \neq 0$: *the fractal dimension of the inner extended field is 2.* In this case, it follows from equation (6.7) that $V(2^{k_0+\ell})$ is bounded below by

$$V(2^{k_0+\ell}) \geq$$

$$4^\ell[\frac{V(2^{k_0})}{2^\ell} + 2^{2k_0-2}(\rho_\infty + \nu_{k_0})(1+2^{-k_0})(1-(\frac{1}{2})^\ell)]. \tag{6.8}$$

The right-hand part is the expression in (6.7) for $\mu_\ell = 0$, i.e., $\varphi_\ell^\ell = 2^\ell$. The second term tends to a constant and dominates. Therefore, the resulting dimension is 2.

Notice that the difference between the two cases above does not seem to be a structural one. Where we have growing white spaces (overall zero-value) in the first case, we find the left-hand periodic texture in the second case. It is this fact that brings the pattern to full dimensionality, as is clear from equation (6.7). Emptying the zone in which this texture appears produces a pattern that now satisfies

$$V(2^{k_0+\ell}) = (\varphi_\ell')^\ell V(2^{k_0})$$

with

$$\varphi_\ell' = \sqrt[\ell]{(2+\mu_{k_0+1}')(2+\mu_{k_0+2}')\cdots(2+\mu_{k_0+\ell}')}.$$

It follows that the fractal dimension is $\lim_{\ell\to\infty}(\log_2 \varphi_\ell') = \log_2(\lim_{\ell\to\infty}\varphi_\ell')$.

It is not clear how μ_k', the ratio between the resulting volumes of the left lower triangle and the top triangle in the linear scale doubling scheme of figure 13 evolves. Unless $\mu_\ell \to 1$ for $\ell \to \infty, \varphi_\ell' \neq 3$.

So, if we squeeze out the periodic-field pattern from the global difference field pattern, we are left with a fractal object. We will call this *the fractal skeleton* of the difference field. The difference field patterns will then be considered as a *submerged fractal skeleton* (figure 14).

6.3 The general difference field with periodic boundary conditions

From the decomposition property (section 6.1) and the considerations about the half-infinite CCPN, it follows that a general difference field $BF(P_L, , P_R)$ is the EXOR superposition of two submerged fractal skeletons. The resulting skeleton is now filled with an EXOR mixture of the left- and right-hand periodic "liquids" (figure 15). This mixture is "locally" periodic. Different mixture compositions are possible, due to phase shifts in the superpositions of the left and right periodic fields. Multicomponent mixtures may occur,

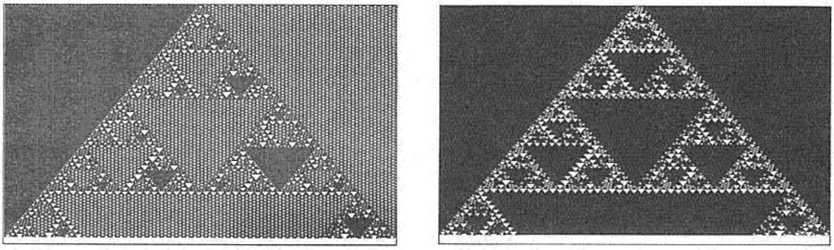


Figure 14: Fractal (b) and submerged (a) fractal skeletons.

in which the components are different shifted versions of the left and right periodic fields. The resulting volume (global as well as for the skeletons) is bounded above by the sum of the volumes of the two half-infinite fields. The resulting dimensionality is thus bounded above by the maximal dimensionalities of these two component fields.

As far as $BF(P_L, I_S, P_R)$ is concerned, its fractal skeleton dimension is the same as that of the fractal skeleton of a corresponding $BF(P'_L, , P'_R)$ (figure 16). Both patterns differ in the width of the transient bands and in some extra initial volume. But this becomes eventually dominated by the $BF(P'_L, , P'_R)$ inner volume.

The last point also shows that seen at sufficiently large scale, the BF (P_L, I_S, P_R) patterns are similar in nature, independent of I_S. Only the occurence of specific mixtures will be influenced (compare figures 17a.b.c.d).

7. A conservation law that generates periodic boundary conditions

Consider a difference field. The vector $(n_1, n_2, n_3 \ldots n_\ell)$, the components of which represent the number of 1's on each difference level is a *characteristic* of this field. Consider a *balanced* set of places at the basic level, i.e. a set of places in which the number of 0-values equals the number of 1-values. Take this set of points as the generating sequence for a CCPN (figure 18). The value-changes induced by this CCPN will not alter n_1 (because of the balancedness of the generating sequence), but will generally alter the other components n_j, as the CCPN will not cover balanced subsets at all levels. However, we can search for difference fields which are *balanced everywhere under a given CCPN* (up to the level where the CCPN completely covers the difference field, which should be potentially unbounded). Exploratory computer experiments show that this forces periodicity at the CCPN boundaries (from a certain level on), and hence overall periodicity. It has not yet become clear why this is so. The reverse fact that periodic boundary conditions imply balancedness is certainly not true. As a matter of fact, looking

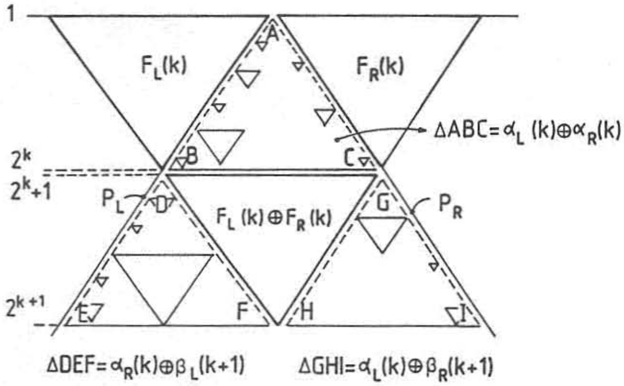


Figure 15: Structure of $BF(P_L,,P_R)$.

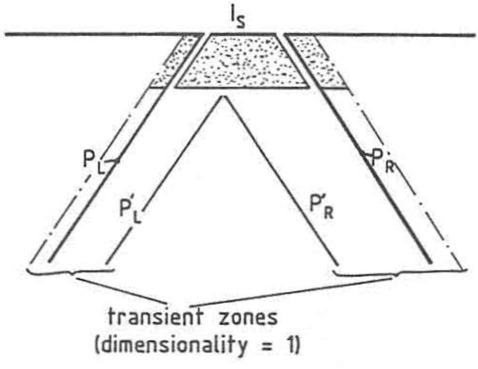
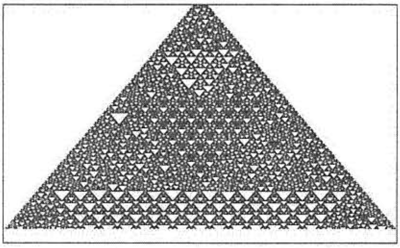
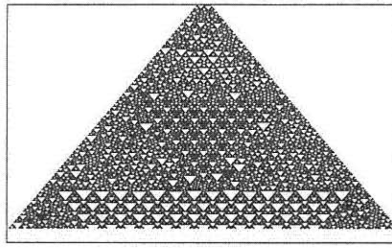


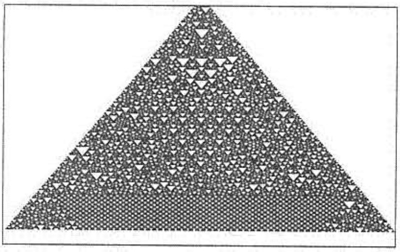
Figure 16: Relationship between a general difference field $BF(P_L, I_S, P_R)$ and $BF(P'_L,, P'_R)$.



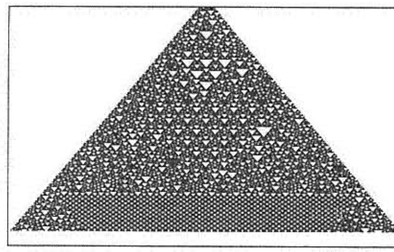
(a) BF(P_L, 0100101110, P_R)



(b) BF(P_L, 0000011111, P_R)



(c) BF(P_L, 11001111111, P_R)



(d) BF(P_L, 01001011101, P_R)

Figure 17: Fields with identical P_L and P_R sequences but with different I_S sequences. $P_L = P_R = 101011100110100$. Compare also to figure 12.a showing the outer field.

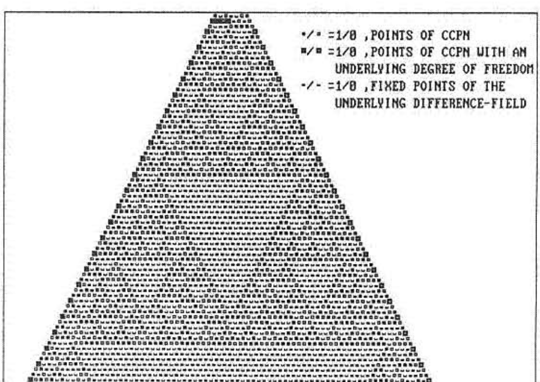


Figure 18: A balanced CCPN.

for the set of sequences with the same $(n_1, n_2, n_3 \ldots)$ was at the origin of the material which was developed here.

8. Conclusions

The binary difference field originates from the most simple non-trivial local interaction rule: a modulo-2 summation of just two neighboring cells. A complete analysis has been presented of the overall pattern resulting from periodic boundary conditions along a left- and/or right-skew column. These patterns exhibit a fascinating and unlimited variety in details embedded in an invariant overall structure. Immediately, the question arises whether similar results exist in more general cases involving other interaction-rules between multiple-valued cells. Even more intriguing is the fact that this behavior is induced by a kind of conservation law, as mentioned under section 7. Further work needs to be done in order to come to a full understanding and generalization of the conditions under which such a structured behavior emerges.

References

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