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Classification of Semitotalistic Cellular Automata in Three Dimensions

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Abstract. This paper describes a mechanism by which three-dimensional semitotalistic cellular automata (CA) may be classified. The classification scheme is based upon the behavior of specific CA rules when originally configured (a) as isolated forms and (b) as random "primordial soup." The classifications of the behavior under these two separate schemes may then be orthogonally related.

Most of the simulations described herein were done in a universe of dense-packed spheres, where each cell has 12 touching neighbors. Results also apply to R^3, with 26 neighbors touching each (cubic) cell.

1. Introduction and notation

In a three-dimensional universe of dense-packed spheres, the spatial relationship between neighbors can be illustrated by placing a cell at the center of a cuboctahedron. The 12 corners then show the location of neighbor cells (see figure 1A); these 12 corners form four intersecting hexagons (e.g. (1,7,8,4,12,10), (6,10,11,3,8,9), etc.) which are parallel to four *coordinate planes.*

Let Ω denote a universe of dense packed spheres; R^3 will be used to refer to a three-dimensional Cartesian universe. For convenience, all of the simulations illustrated in this paper are for rules in Ω; nevertheless, the classification scheme herein described applies to rules in R^3 as well.

In general, a cell can have k states; in this paper, we will deal only with $k = 2$ and will be concerned with the 12 touching neighbors. The next generation state of a cell will be determined by the current state of the cell and its neighbors. The two states of a cell are called *living* and *dead*, which is also refered to as the *quiescent* or zero state, or as "space." Define *environment*, E, as the number of living neighbors of a given living cell, c, so that c remains alive the next generation. *Fertility*, F, is the number of living neighbors of a given dead cell so that the cell becomes alive next generation.

One way to describe rules governing the next generation state of a cell is to specify a range, given as $E_l \leq E \leq E_u$ and $F_l \leq F \leq F_u$. Then, all rules, R, under consideration can be written as 4-tuples of the form $R =$

(E_l, E_u, F_l, F_u). These rules are *semi-totalistic* [1] in that the next generation outcome depends solely upon the current state of c and the quantity of living neighbors and not their relative positions. The special cases of totalistic rules treat c itself as a neighbor; e.g., rules of the form $(x, y, x-1, y-1)$. Of course, other semi-totalistic rules exist; these can be specified as follows. Let $\mathbf{E} = (e_1, e_2, \ldots)$ where e_i, an environment state, specifies a number of living neighbors such that c remains alive next generation; and $\mathbf{F} = (f_1, f_2, \ldots)$ where f_i is a similarly described fertility state. We now use the notation $\mathbf{R} = [\mathbf{F}, \mathbf{E}] =$

$$[(e_1, e_2, \ldots e_m), (f_1, f_2, \ldots f_n)], \quad m \geq 0; n > 0$$

(note that there might be no valid environment state). The upper limits for m, n will depend upon the universe; for Ω, $m \leq 13$, $n \leq 12$. For convenience, we will also specify that $i < j <=> e_i < e_j$ and $f_i < f_j$. Note that we eliminate as invalid all rules where $f_1 = 0$. Such rules violate the "quiescient state" requirement [3]; that is, if all cells in a universe are zero, they must remain zero. Throughout, we will use both notations, supplying whichever is the most convenient for the purpose. Note that R is contained in $\mathbf{R}$ and that, for example, $(3,3,3,3)$ and $[(3),(3)]$ are equivalent, as are $(4,6,3,3)$ and $[(4,5,6),(3)]$; and, of course, $[(4,6,9),(3)]$ cannot be represented in the former notation. We might also write $[(),(x)]$ as (∞, ∞, x, x). These notations seem more appropriate for three dimensions than Wolfram codes [3, p. 127].

Most of our empirical simulations have been done in Ω. Computer experimentation required various finite universe sizes, occasionally wrapping coordinates in order to simulate an infinite universe. Because of the method of computer implementation, the finite values for Ω appear rhombohedric in shape and may be considered as being composed of successive layers; the number of such layers is referred to as the *universe size*. Figure 1B shows a size-5 universe where all cells are alive.

A *form* is a finite grouping of cells. Many rules yield forms called *oscillators*, that is they are periodic. An oscillator of period one is called a *stable* form.. An oscillator that translates through space is called a *glider*. A *convex hull*, h, is the minimal convex polyhedron that can wrap a form. It will include cells on the outer periphery of the form. An *Ω hull* is the minimal wrapping octahedron that can be constructed with eight planes parallel to the four coordinate planes; similarly, an *R^3 hull* wraps with a hexahedron constructed with planes parallel to the three coordinate planes. Let $D^{i,j}$ denote the minimal distance between the Ω hulls of forms i and j. If $D^{i,j} > 1$, then we say that forms i and j are *isolated*. A similar definition may be made for R^3.

2. A general classification scheme

Although one might like to utilize the classification system of Wolfram [3, p. 111], it is perhaps not unexpected that three-dimensional cellular automata

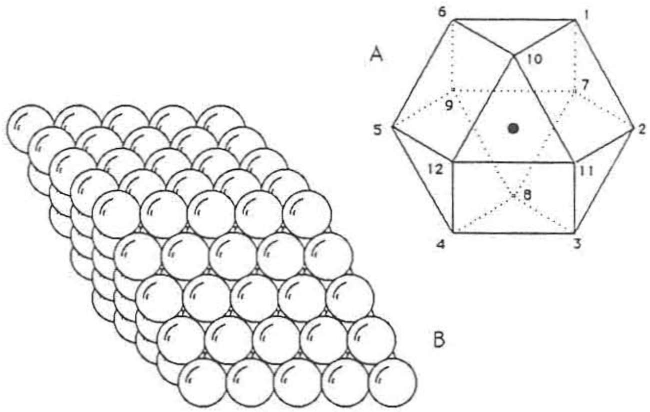

Figure 1: The twelve immediate neighbors in Ω can be depicted as the numbered corners of a cuboctahedron, each being a neighbor of the black spot in the center. The universe can be rhombohedric; an infinite universe may be simulated by wrapping, along traditional axes, the sides of a sufficiently large finite universe. (The size-5 universe in B would be too small for a useful simulation.) In figures 2 through 13 all rules specified are in Ω unless otherwise indicated.

behave somewhat differently than one- or two-dimensional systems. For one thing, chaos seems to prevail more frequently; moreover, there are obvious effects of cubic rather than linear growth. Nevertheless, it is possible to classify, at least in the broad sense, the rules under consideration in a manner somewhat similar to the types of Wolfram.

One should note that of the 2^{8192} possible rules in Ω, we are only classifying the 2^{27} totalistic and semi-totalistic rules. These classifications also apply to similarly constructed rules in R^3. (Note that the rules under consideration in Ω are contained in the $2^{2^{27}}$ rules of R^3—but cannot be represented in R^3 in the form $[\mathbf{E}, \mathbf{F}]$; that is, they are not in general totalistic or semi-totalistic in R^3).

With the above in mind, we can classify semi-totalistic CA in R^3 and Ω according to the behavior of (a) a universe consisting of one or more forms, all of which are isolated, and (b) a universe consisting of primordial "soup." The two classification schemes may then be orthogonally related in order better to describe the particular rule under consideration.

2.1 Classification according to behavior of isolated forms

All rules fall into one of three classes given below depending upon how isolated forms interact. For each, start at generation zero with a universe of isolated forms.

Class B–Bounded (Isolated forms can never interact.)

Some forms may disappear; others may fill to all live cells, depending upon the rule. All forms remain isolated and bounded in space by their original hulls. In other words, given a universe where $D^{ij} > 1$ for all forms i, j; the forms will remain isolated (see figure 2).

Class I–Interacting (Isolated forms may interact.)

Depending upon the rule and the forms involved, isolated forms may merge into single forms, or they may shrink and die; i.e., possibility exists for interaction. Growth of individual forms in most cases is bounded (probability $= 1$), but there may exist configurations which exhibit unbounded translation or growth (e.g. gliders, glider guns). This class somewhat resembles Wolfram type 4, although the existence of gliders is not guaranteed. A single finite primordial blob of appropriate density in an infinite universe will, with probability $= 1$, shrink to zero or more (considerably) smaller forms. For the rules examined, such starting configurations usually evolve into a (roughly) spherical quivering form which gradually shrinks and eventually either disappears or leaves one, two, or occasionally more small oscillators. To observe this behavior, we should start with a form whose primordial density is near the (relatively stable) density of the slowly contracting mass. Primordial forms of other starting densities seem to coalesce into isolated forms, which individually continue the shrinking process just described.

Class U–Unbounded (Isolated forms grow without bounds.)

Although some forms may shrink, most forms will grow and merge together. Some rules seem to have a finite but rather fuzzy "critical mass", M_c. If a finite isolated primordial blob is smaller than M_c, then it shrinks or meanders in a manner similar to type I behavior. If the primordial blob is larger, it grows without limit. The initial density of M_c should be somewhere around the final chaotic density. The size of M_c varies from rule to rule and (somewhat) from configuration to configuration within a given rule.

Just a few examples of rules for each of the three classes are given in table 1. Figure 5 depicts typical growth patterns for the three classes.

2.2 Classification according to primordial evolution

For each case, we wish to determine the ultimate destiny of random primordial soup configurations conducted in an infinite universe. Since we may choose to start with initial soup at a variety of densities (< 1), we designate

Class	Example rules
B	In Ω iff $f_1 > 3$ In R^3 iff $f_1 > 9$ (see [2])
I	The games of life; e.g., the Ω rule (3,3,3,3) and the R^3 rules (4,5,5,5) and (5,7,6,6). In general, Ω rules $[(x),(3)]$, $x > 2$; and R^3 rules $[(x_1, x_2),(y)]$; $x_1 > 4$, $4 < y < 10$. Others may be determined by testing individually.
U	All rules in Ω where $f_l < 3$; all rules in R^3 where $f_l < 5$ (see [2]). Certain rules in Ω of the form $(x, y, 3, 3)$; for example, (4,7,3,3). Many rules in R^3 of the form $[(e_1 \ldots e_m),(f_1 \therefore . f_n)]$; $f_1 > 4$, $f_n < 9$, $e_m < 9$; $m, n > 2$.

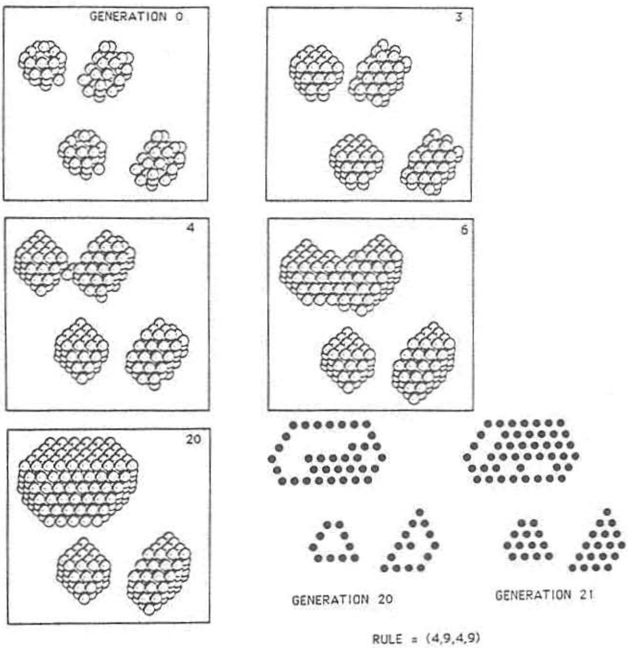

Figure 2: The Class B behavior of rule $(4,9,4,9)$ is shown here. At generation 0, the two top forms are not isolated in that $D^{ij} = 1$, but the two bottom ones are; i.e. $D^{ij} = 2$. At generation 4, the top forms begin to interact. By generation 20, they have merged into one form. The two bottom forms will never interact. Note that the three resulting forms are now oscillators with period > 1. A slice at generations 20 and 21 depicts the interior activity.

that a sizable portion of our starting configurations yield the intended result (i.e. probability $p > 0$; in some cases, $p = 1$). The five distinct groups are given below.

Group 0 (All cells disappear.)

No matter what we use for the starting configuration, all cells die after finite time.

Group 1 (Cells converge to isolated forms.)

Primordial soup contracts to isolated forms, at least some of which become oscillators.

Group 2 (Cells remain in chaotic turmoil.)

All cells in the universe have the property that they change states an infinite number of times. No stable forms or oscillators ever appear (see figure 6 for an example).

Group 3 (Cells remain chaotic, but stationary.)

No isolated forms are produced, but after finite time, all (or most) cells assume a state and remain there. Occasional attached oscillator pockets may appear. Chaos is essentially "frozen" in space. One could visualize Swiss cheese where from zero to all of the (not necessarily round) holes are in constant turmoil.

Group 4 (All cells become alive after finite time.)

Starting primordial configurations yield all live cells after a finite time (see figure 7).

Table two gives some examples. It is by no means complete, as many other rules will probably require empirical verification to determine their proper group.

Unfortunately, the group classification is not quite as clear-cut as the three Class B, I, and U. We must concede that many primordial configurations may yield debris similar to a lower group; that is, some primordial configurations for a Group 3 rule might yield Group 1 type debris. However, the converse will never be true; that is, the correct group is the highest numbered group such that primordial soups yield said group behavior with probability > 0.

Stated more exactly: specify a rule, R; there exist densities d_1, d_2 such that primordial soup densities between d_1 and d_2 will yield Group x behavior for R with probability one. For primordial densities outside this range, behavior is unpredictable in that residue may be similar to Group(s) $y, y \leq x$. (An exception: Group 0 rules yield no residue for all configurations; here,

Group	Examples
0	Rules in Ω where $f_1 > 10$, $e_1 = 12$.
1	R^3 life rules $(5,7,6,6)$ and $(4,5,5,5)$; many others. Ω rules $[(x),(3)]$; $x > 3$.
2	The Ω life rule $(3,3,3,3)$; most Ω rules (x,y,x,y) $1 \leq x, y \leq 7$; $y - x < 4$. The R^3 rule $(6,7,6,7)$, etc.
3	Most Ω rules (E_l, E_u, F_l, F_u) where $3 < E_l < 6$; $E_u > E_l + 5$ and $F_l > E_l$; $F_u < E_u$. The Ω rules $(0,12,x,12)$; $x \geq 8$. The Ω rule $(5,10,3,3)$, etc.
4	All Ω rules $(0,12,x,12)$; $x < 7$. (This can be proved in a manner similar to Classes B and U; see [2].)

$d_1 = 0+$; $d_2 = 1-$). For rules tested, $y \neq 2$ and is, in fact, usually 1 or 0 (if anything at all). A few tested rules have so far yielded inconclusive behavior.

Some rules are obviously more predictable than others. For example, the Group 2 Ω Life rule $[(3),(3)]$ follows Group 2 behavior as long as primordial densities are in the vicinity of 20% to 30%. Low or high densities tend to yield clumps, some of which stabilize into oscillators (see [1]).

It has been observed that rules with a large environment range surrounding a small fertility range tend to be Group 3; as the two ranges approach equality, oscillating pockets begin to appear (see figure 8). On the other hand, rules supporting mostly interconnected chaotic turmoil, with just a few stationary configurations, have not yet been observed. That is, Group 2 behavior seems to be "pure," involving all cells. Only the density of the chaotic turmoil varies–this depends upon the rule in question.

Putting it more precisely, a rule R is a Group 3 rule only if there exists an n_0 such that for all generations $> n_0$ some cells do not change state; whereas R is a Group 2 rule if and only if no such n_0 exists. Unfortunately, it is possible that some Group 3 rules behave like Group 2 rules for long periods before eventually settling down, and therefore may not yield easily to an empirical investigation to determine n_0.

2.3 Combining class and group

Since every rule has a class and a group, we may combine both schemes orthogonally (when we are certain where the rule belongs). We will use a subscript for the group. For example, The Ω Life rule $(3,3,3,3)$ is Class I_2 and the R^3 Life rules are Class I_1 (see figures 3 and 4).

It is tempting to ask why two classification schemes are necessary. The answer is that many rules behave in a manner indistinguishable in one of the categories, but distinguishable in the other. For example, Ω rules $[(3),(3)]$

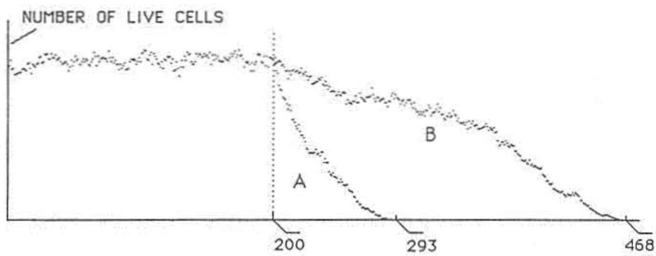

Figure 3: The Class I and Group 2 behavior of Ω rule $(3,3,3,3)$ is depicted. From generations 0 to 200 an "infinite" universe (size = 20) was used. In this infinite universe, chaotic turmoil would have persisted forever. At generation 200, the simulation was split into two separate simulations. In A, we unwrapped all three axes, leaving essentially an isolated blob. For simulation B, only one axis was unwrapped. The idea here was to emulate an exceedingly large, isolated primordial blob in an attempt to determine whether $(3,3,3,3)$ was Class U or Class I. If primordial chaos had persisted for simulation B (i.e. the form was trying to grow), one would have concluded that $(3,3,3,3)$ was a Class U rule requiring an extremely large "critical mass" to initiate unbounded behavior. Since the form shrank (for all experiments tried), one can conclude that $(3,3,3,3)$ is Class I.

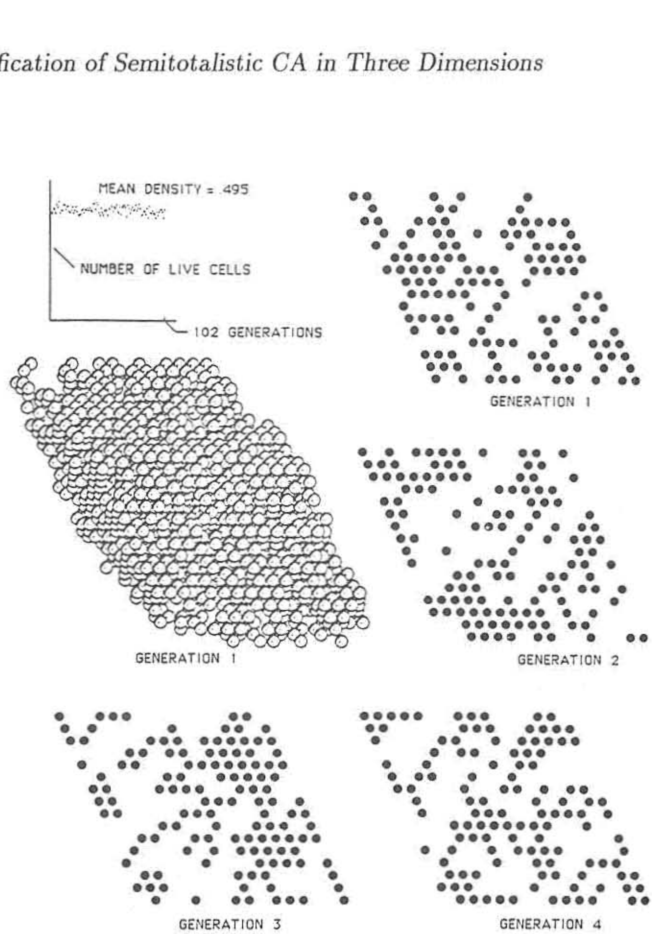

Figure 4: Here are the live cells at generation 200 (see figure 3). Simulation A: at generation 226, the blob had shrunk in all dimensions; by generation 258, it had gotten quite small. The form disappeared a short time later (generation 294). Simulation B: the form shrank more slowly, and only along the unwrapped y-axis. By generation 393, the blob, still "infinite" in x and z directions, had gotten quite thin. At generation 469, the form finally disappeared.

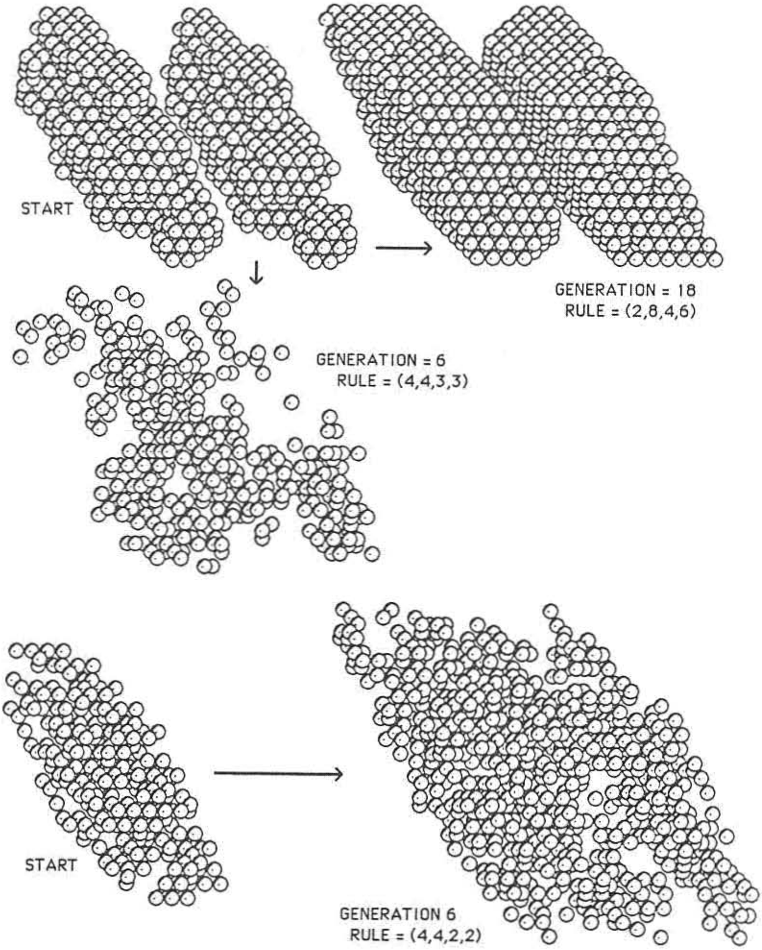

Figure 5: Here we see the three classes. Start with two isolated blobs ($D^{ij} = 2$). When we apply the Class B rule $(2, 8, 4, 6)$, the forms grow into long period oscillators, but never interact, as they are bounded by their Ω hulls. Class I rule $(4, 4, 3, 3)$ shows the gradual confused merging (and shrinking) of the two forms. For the Class U rule $(4, 4, 2, 2)$, if we start with a single smaller blob, it rapidly grows without bound.

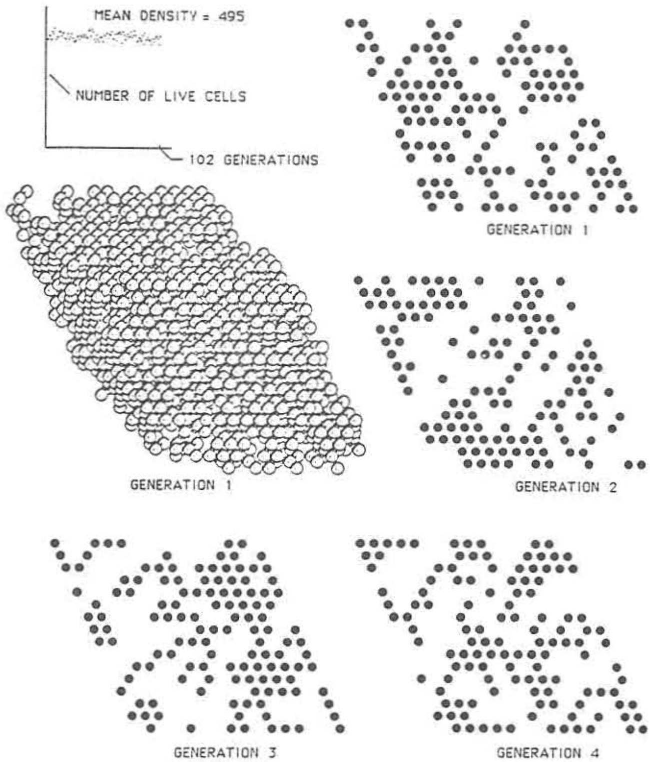

Figure 6: The chaotic turmoil of Group 2 rule $(5,7,5,7)$ is depicted. This rule is Class B, hence designated as Class B_2. The successive slices show that all cells are continually changing values. Here, the universe was wrapped; the size was 16. The steady state density of the chaos (upper left corner) was a relatively stable .495, which should be no surprise.

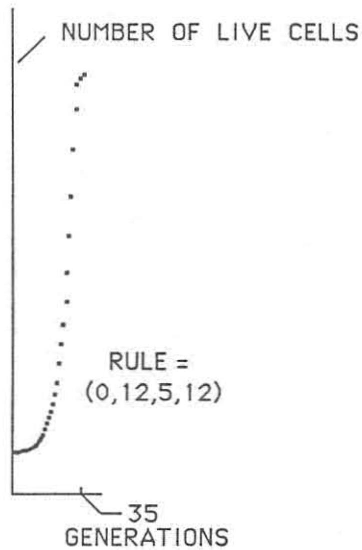

Figure 7: Here we see the typical rapid growth of a Class U_4 rule. All cells became live after 35 generations.

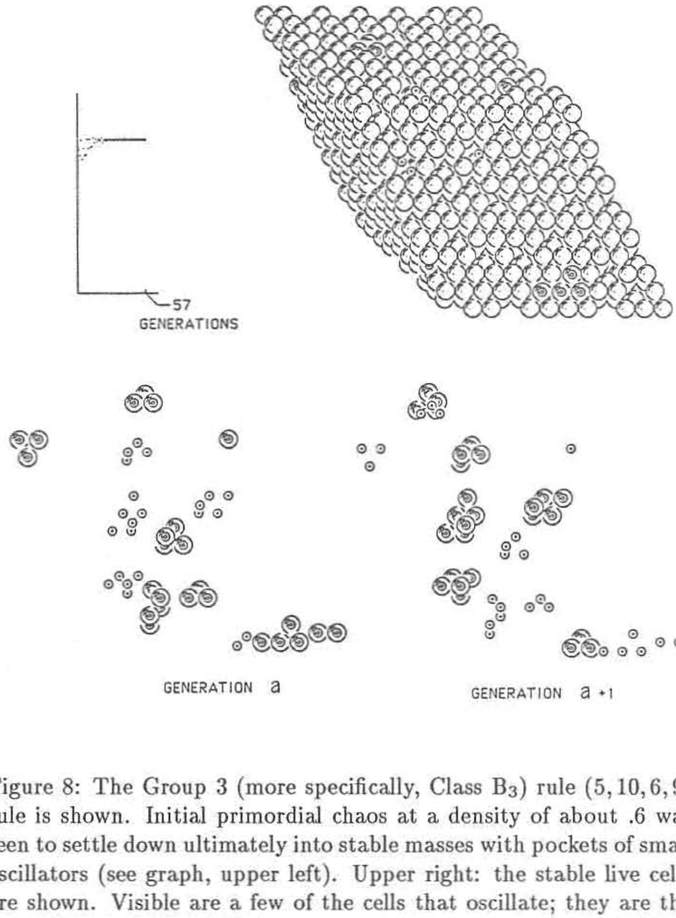

Figure 8: The Group 3 (more specifically, Class B_3) rule $(5, 10, 6, 9)$ rule is shown. Initial primordial chaos at a density of about .6 was seen to settle down ultimately into stable masses with pockets of small oscillators (see graph, upper left). Upper right: the stable live cells are shown. Visible are a few of the cells that oscillate; they are the live cells containing bullseyes. Bottom: here the stable live cells have been removed. The remaining cells oscillate forever with (in this case) a period of two. The currently live ones contain bulls-eyes and the currently dead ones are shown as just bulls-eyes.

(Class I), and [(2),(2)] (Class U) exhibit Group 2 behavior: random primordial soups (over certain initial density ranges) appear to exhibit chaotic turmoil forever. As another example, the Ω rules $(1,12,4,11)$ and $(4,5,4,5)$ are both Class B, but the former usually merges to all live cells (Group 4) while the latter remains in chaotic turmoil (Group 2). Hence, the most precise way to indicate behavior is to specify both class and group. Figure 9 illustrates the various categories.

Observe that not all possibilities are present. For example, no rule has yet been discovered in the I_4 category. This is probably because the larger environment range required for growth to all live cells tends to lead to Group U rather than Group I behavior.

Class I_3 rules, scarce in Ω, are interesting in that they require rather high primordial densities in order to observe the Group 3 behavior. Apparently, the initial dense soup presents a hostile environment to Class I fertility rules, which like lower densities, and lower starting densities tend to yield separate clumps.

Not all orthogonally constructed classes make sense (e.g. Class U_0). On the other hand, if we consider other rules, it may turn out that some combinations that make no sense for totalistic rules do in fact cover others. For example, consider Conway's Life, which can be represented in R^3 as a certain non-totalistic rule. One could reasonably place this rule in Class U_1. That is, one might observe that many Conway forms seem to exhibit expansive, almost unbounded growth, but as the forms expand, they break apart into small (contracting) isolated blobs, which then expand, interact, etc. It seems there is a "tug of war" between the two classifications–Class U behavior and Group 1 behavior–with Group 1 eventually winning out. Naturally, a purist would argue that Conway's Life falls into Class I_1, but the tug of war concept is rather romantic.

Some Class U_2 rules support gliders (for example, the R^3 rules Life.001 4544 and Life.110 4544 discussed in [2]) and therefore possible universal computation. Here, forms must be constructed carefully, for if configurations are too large, they will grow without limit. No glider supporting semi-totalistic rule other than the Class I_2 rule $(3,3,3,3)$ has yet been discovered in Ω. This author speculates that no "glider gun" will be found–and hopes to be proven wrong.

Certain Class U_2 rules appear to pulsate chaotically between two densities, occasionally heading toward a relatively stable density for a time, then resuming pulsating behavior (see figure 10). One of these, the Ω rule $(1,2,1,2)$, has been examined in some detail. In order to determine if this behavior eventually quieted down, a universe in the shape of a long cylinder was constructed. The cylinder was 15 by 15 cells across and 200 cells long, and was wrapped on all axes to simulate a kind of infinite (albeit weird) universe. After selected generations, plots were produced which gave the number of live cells in each slice of the long axis. Alternate generations showed little variation in the short run, but gradually changed over a longer period. If we were to make a movie composed of frames where each frame was a composite

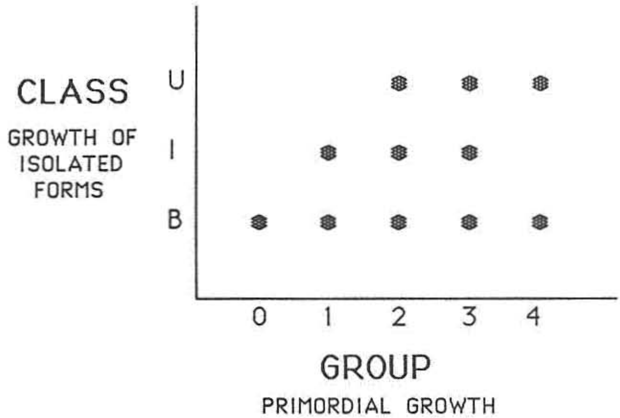

Figure 9: The classes and groups are depicted. Classes U_0 and U_1 do not exist, and rules for Classes I_0 and I_4 have not been discovered. In Ω, Class U rules are most common (more than 3×2^{22}); next are Class B rules (about 2^{21}). Class I rules are scarce, with fewer than 2^9. Class I_3 is probably the rarest category; more than likely, there are fewer than 100 such rules.

of, say, ten consecutive odd generation numbers, our movie would present a profile of the surface of a choppy lake (see figure 11). Clearly, more study of this unusual behavior is necessary.

The Class I_3 Ω rule $(5,10,3,3)$ is very interesting. First of all, several tests were necessary in order to determine that this rule was Class I and not Class U. In an infinite universe, this rule, with appropriate starting primordial densities (about .5 to .6), yields a large stable mass with small pockets of oscillators (see figure 8, top plot of figure 12). However, the really unusual behavior is observed when we unwrap one of the axes in order to see if the mass begins to contract. If the prior mass has evolved to the stable form already mentioned, then although shrinking starts immediately, there is considerable resistance to the ultimate demise of the form–almost as if a tough "outer skin" from time to time affords protection against further decomposition (see figure 12, bottom plot). This possibility has not been determined with certainty and needs to be further investigated; if true, it might then just be possible for certain Class I rules to ward off their own decomposition and even grow for a time–all this from large random primordial forms.

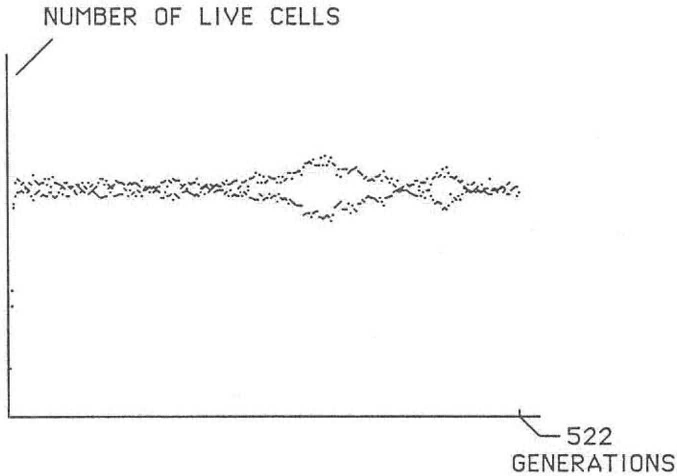

Figure 10: Unusual "pulsating" behavior exhibited by Class U_2 rule $(1,2,1,2)$. Chaotic turmoil would stabilize (more or less) at a given density, only to become unstable again, with successive generations swinging wildly between two quite distinct values. This behavior seems to go on endlessly. Here, a universe of size 15 was wrapped in order to simulate infinite space. For this pulsating behavior, the use of such a small size probably affected the outcome.

3. Conclusion: a comparison to the Wolfram types

There is no exact correspondence to the four Wolfram types, but one should remember that his classification scheme included all CA in one dimension and not just totalistic and semi-totalistic rules. Nevertheless, we can observe that Wolfram type 1 corresponds to Class B_0, and type 2 matches Classes B_1, B_3,and (somewhat) U_3 with infinite primordial configurations. Wolfram type 3 is, naturally, similar to Class U_2 and U_3. Although experiments have not yet been performed, some non-totalistic rules undoubtedly exist in Ω which could be categorized as Wolfram type 4. These would likely fall into Classes I_1 and I_2. It appears to be possible that the new classification scheme just might cover all rules, whether totalistic or not. (Admittedly, things get complicated when we allow our cells to have more than two states.)

There might be a temptation to say that we do not need another classification system; that any rule could be classified by one of the four Wolfram types. For example, if we consider the two cell states (living and dead) as equals, then one might argue that Classes B_1 and U_3 are equivalent. However, note that in terms of growth behavior, B_1 is bounded but U_3 is not. As another example, we might say that any rule which supports gliders should

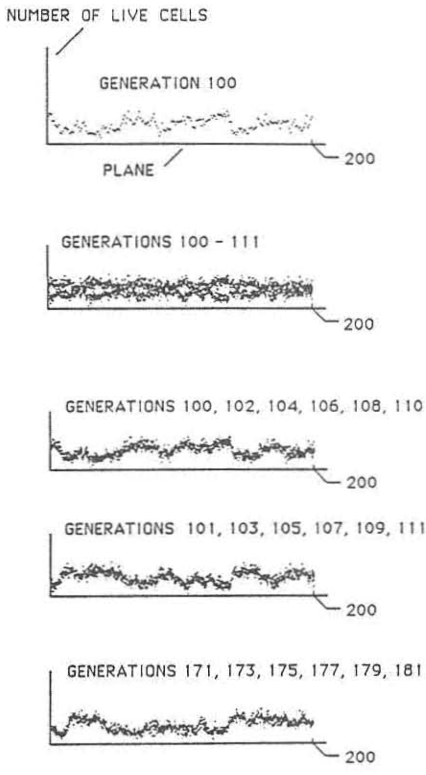

Figure 11: The same behavior depicted in figure 10 is again illustrated here, but in a different manner. Here, a $200 \times 15 \times 15$ universe was utilized. The plots show the number of live cells contained in the 200 successive slices along the long axis. Note that over short ranges of generations the odd generations formed a pattern, and the even generations formed a sort of complementary pattern. After a time, the pattern tended to shift around. If we made a movie where each frame was composed of successive groups of, say, ten consecutive odd generations, the appearance would likely be that of a lake on a rough day.

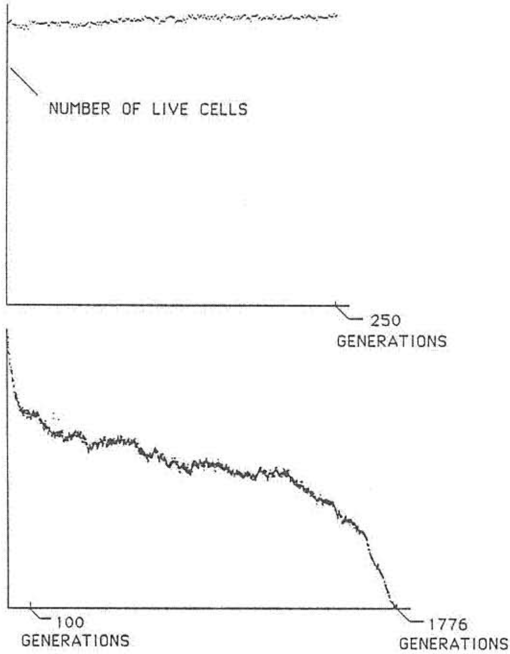

Figure 12: The Ω rule $(5, 10, 3, 3)$ is shown here to be a Class I_3 rule. For the graph at the top, the (size 16) universe was wrapped; the chaos eventually stabilized into frozen "swiss cheese" containing small pockets of oscillators and/or empty space. When one dimension was unwrapped (simulating an extremely large but finite primordial form), the shrinking process began rapidly, but halted periodically. Apparently a tough "outer skin" tended to protect the form from demise–at least temporarily.

Figure 13: The rule $(5, 10, 3, 3)$ was employed on a primordial size 12 wrapped universe at an initial density of .6. The snapshot at the top shows the relatively stable "swiss cheese" at generation 102; we see just the cells that are continually changing. The form stabilized with between 1022–1055 cells alive at any given time. 42 cells were periodic and 1018 remained alive without change. The other $(1728 - 1060 = 668)$ cells remained dead. At the bottom, we can observe the effects of unwrapping the y coordinate (creating an infinite "pancake" that was 12 cells thick). After 30 generations, the form had shrunk–note that many of the cells on the top and bottom have begun to die or oscillate. The spheres containing bulls-eyes are cells that have died at least once but have come back to life. They will eventually die permanently. The cells containing no spheres and only bulls-eyes are cells that were once alive. The empty space in the middle is occupied by mostly live cells that are not changing states; for clarity they have not been shown.

fall into Wolfram type 4. However, a characteristic of type 4 is that any cell in the universe might be hit by a glider passing through space; In three-dimensional space, even with several streams of gliders, the probability is zero that any particular cell would be "hit" by one of them.

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