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On Synchronization and Phase Locking in Strongly Coupled Systems of Planar Rotators

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Abstract. Strongly coupled, dissipative systems of planar rotators are considered with the dynamics described by the autonomous system $d\underline{\varphi}/dt = \underline{\omega} + \gamma\underline{\Gamma}(\underline{\varphi})$ where the nonlinear coupling $\underline{\Gamma}$ depends on the phase differences $\varphi_i - \varphi_j$ of N rotators. It is pointed out that the simplest mechanism of the asymptotic synchronization in such systems can be described in terms of relative equilibria in the $N-1$ dimensional phase space transverse with respect to the barycentric motion. Particular examples of systems with short-range and all to all interaction are considered and the stability problems discussed. The simple entrainment picture based upon the presence of relative equilibria is excluded for large systems with all to all interactions and randomly distributed frequencies ω_i. Recently reported critical behavior of the synchronization phenomenon in these systems must then have more complicated dynamic origin.

1. Introduction

It is well-known that ensembles of structurally similar, non-linearly coupled subsystems may exhibit striking cooperative behavior in strongly non-equilibrium regimes. This self-organization effect bears some resemblance to much better understood ordering phenomena for equilibrium and near to equilibrium states. Open dissipative systems which never come to rest are often considered in this context, with the dynamics modeled by some postulated equations of motion involving only part of the system's degrees of

freedom, and an implicitly present external drive supplying energy to the system.

To this class belongs a recently popular model of N coupled rotators where each structural unit is described by an abstract phase variable φ_i. This variable in the absence of any interaction varies monotonically with time at constant rate (frequency) ω_i. Here the frequency parameter describes the "rotator's drive" (i.e., its intrinsic activity caused by some unspecified energy supply). The coupling term, usually depending only on the relative phase differences $\varphi_i - \varphi_j$, describes the energy redistribution throughout the system. The common features of such systems are the possibility of *phase locking* (an asymptotic correlation among phase variables of all rotators, largely independent on the initial conditions) and *synchronization* (or "entrainment"), where all rotators, or a macroscopic part of them, asymptotically move with a single common frequency Ω.

In the biological context such models were first considered by Winfree [1,2] for describing the synchronization in ensembles of "biological clocks". He was also first to indicate that large communities of such rotators with randomly distributed frequencies may exhibit a kind of "phase transition" between asynchronous and synchronous "phases". This idea has been pursued and developed into an interesting qualitative theory by Kuramoto and co-workers [3–7]. A most detailed study has been presented there for a model with "all to all" interaction, decreasing with the system size as $1/N$. As it is well-known in equilibrium statistical physics, this is the simplest model interaction for which the "mean field" approximation (single structural unit in self-consistently averaged environment) becomes exact in the $N \to \infty$ limit, with the still preserved possibility of a "classical" critical behavior. It has been shown that a dynamical rotator model of this type with random frequences also exhibits the critical behavior at the onset of synchronization in the $N \to \infty$ limit, although the corresponding "order parameter" behaves quite differently from that in thermodynamic phase transitions.

Non-random systems of planar rotators were earlier proposed by J. C. Neu [9] (as models of coupled chemical oscillators), and by other investigators [10–13], as models of temporally patterned sequences of signals generated by neural networks governing rhythmic activities of muscular groups. Typical systems considered in these papers are essentially one-dimensional arrays of rotators with generally asymmetric coupling constants, which may vary along the chain. In particular a profound mathematical analysis of the resulting "frequency plateaus" has been given in [12] for chains with a linear gradient in native rotator frequences and weak nearest-neighbor coupling.

General systems of rotators with weak, in some sense, and otherwise arbitrary coupling are, on the other hand, a well-known and long studied subject in non-linear mechanics since the time of Poincare (see V. I. Arnold [14] for a review of these beautiful results including ergodic theorems, structural stability and resonance problems). Although complete synchronization is usually absent for weak coupling, such systems still exhibit the locking phenomenon to the free frequencies, where the ratios φ_i/φ_j of the phase variables are

asymptotically equal to the corresponding frequency ratios ω_i/ω_j. In the context of neural networks, these features of rotator systems were investigated by F. C. Hoppensteadt [15], who came to the coupled rotator model via his model of a voltage-controlled oscillator neuron.

In this paper we concentrate on rotator systems with sufficiently strong interaction allowing, under specific circumstances, the complete synchronization and yet another type of phase locking with asymptotically constant phase differences of all rotators. We stress the role of coupling symmetry by introducing an elementary geometric description (section 2), together with the concept of an "entrained" solution. The examinations of entrained solutions becomes much simpler for gradient systems (section 3). There we show that fully synchronized states are nothing but the states in the basins of attraction of an asymptotically stable relative equilibrium. The presence or absence of these equilibria depends on the coupling strength and sign, and may not depend on the system size. The latter statement is illustrated in section 4 by an example of a strongly coupled two-dimensional periodic rotator array with two frequencies. It is shown that this system has at least one relative asymptotically stable equilibrium, and behaves in the neighborhood of this equilibrium like a system of two coupled rotators.

Finite, non-random systems with "all to all" $1/N$ interaction are reexamined for this point of view in the main part of section 5. By initially restricting the use of the "order parameter" concept to relative equilibria only, we carry out the analysis of the synchronization and the related mechanical stability problem without any further assumptions (like the one introduced in [7] where the rotator ensemble is partitioned into synchronized and asynchronized populations). This section contains a simple instability theorem for sufficiently large systems with "inhibitory" (accentuating the phase differences) coupling and finite frequency spread. On the other hand, systems with sufficiently strong "excitory" interaction are proved there to have at least one relative asymptotically stable equilibrium by a standard perturbation argument with respect to the inverse coupling parameter.Relative equilibria, if present, provide the dynamically simplest mechanism of the complete synchronization in finite rotator systems. However, their role becomes marginal for large systems with $1/N$ interaction and random, identically distributed frequences. We prove (section 5, proposition 3) that the asymptotic synchronization in the above described sense is an event with vanishing probability in the $N \to \infty$ limit. In conclusion, synchronization effects in such systems, if any, must have much more complicated (and interesting) dynamical origin, with an open possibility for the presence of strange attractors for $N \geq 4$ as well as an intermittent character of the synchronization phenomenon. When the complete dynamical analysis of partial entrainment effect is not available, then the approximate scheme proposed in [7] is certainly the best known theory.

The concluding section 6 contains remarks on applications and future work. The most immediate application of the planar rotator model (with strong internal coupling) could be made in the description of firing patterns

of neural networks. Some open problems are listed: in particular finite frustrated systems of planar rotators (very recently investigated by Daido [16], in the framework of the "mean field" approximation). These problems seem particularly promising, as they may exhibit a rich structure of relative equilibria. Like in the Hopfield-Little model (see, e.g. [15]) these equilibria can be used as information storages with each equilibrium manifesting itself by a specific vector of locked phases.

2. Model

Consider a system composed of N structural units where the state of the i^{th} unit (called a rotator) is described by a single real variable φ_i, and where some rotators exhibit spontaneous cyclic activity with characteristic frequencies ω_i.

We assume that the dynamics of the system are governed by the evolution equation

$$d\underline{\varphi}/dt = \underline{\omega} + \gamma \underline{\Gamma}(\underline{\varphi}), \tag{2.1}$$

where the C^1 vector field $\underline{\Gamma} : \mathcal{R}^N \to \mathcal{R}^N$ describes the coupling between rotators, γ measures the interaction strength, and $\underline{\omega} = (\omega_1, \omega_2, \ldots, \omega_N)$ is the vector of "innate" frequencies, $\underline{\omega} \neq \underline{0}$. We will additionally assume that the autonomous system (2.1) does not have equilibrium states:

$$\underline{\omega} + \gamma \underline{\Gamma}(\underline{\varphi}) \neq \underline{0}, \qquad \text{for all } \underline{\varphi} \in \mathcal{R}^N. \tag{2.2}$$

In other words, at any instant t at least one rotator of the system has non-zero instantaneous angular frequency, and the system "never dies" in biological terms.

Let $\underline{n} \in \mathcal{R}^N$ be an arbitrary vector with integer components ($\underline{n} \in \mathcal{Z}^N$), and $\underline{n_d}$ be a special vector of this type with all components equal to unity:

$$\underline{n_d} = (1, \ldots, 1). \tag{2.3}$$

We will restrict ourselves to vector fields $\underline{\Gamma}$ periodic in φ_i variables

$$\underline{\Gamma}(\underline{\varphi} + 2\pi \underline{n}) = \underline{\Gamma}(\underline{\varphi}), \qquad \text{for all } \underline{\varphi} \in \mathcal{R}^N,\ \underline{n} \in \mathcal{Z}^N, \tag{2.4}$$

and invariant with respect to the translations along the $\underline{n_d}$ direction,

$$\underline{\Gamma}(\underline{\varphi} + \alpha \underline{n_d}) = \underline{\Gamma}(\underline{\varphi}), \qquad \text{for all } \underline{\varphi} \in \mathcal{R}^N,\ \alpha \in \mathcal{R}. \tag{2.5}$$

Differentiating equation (2.5) with respect to the parameter α, one obtains

$$D\underline{\Gamma}(\underline{\varphi})\underline{n_d} = \underline{0}, \tag{2.6}$$

where $D\underline{\Gamma}(\underline{\varphi}) \in L(\mathcal{R}^N)$ is the linearization of $\underline{\Gamma}$, thus necessarily singular for all $\underline{\varphi}$.

The symmetry (2.5) is equivalent to the assumption that $\underline{\Gamma}$ is a function of the phase differences $\varphi_i - \varphi_j$. As usual [17], the presence of a one-parameter group of symmetries allows us to reduce by one the order of the system (2.1). Let us decompose vectors $\underline{\varphi}$ into "longitudinal" (parallel to $\underline{n}_d$) and transverse Ψ parts:

$$\underline{\varphi} = \theta \underline{n}_d + \underline{\Psi}, \quad (\underline{n}_d, \underline{\Psi}) = 0, \ \theta = N^{-1}(\underline{\varphi}, \underline{n}_d), \tag{2.7}$$

with θ having the meaning of a "barycentric" coordinate. Due to the symmetry (2.5) we have $\underline{\Gamma}(\underline{\varphi}) = \underline{\Gamma}(\underline{\Psi})$. If $t \mapsto \underline{\varphi}(t) \in \mathcal{R}^N$ is a solution to the system (2.1, then its longitudinal $\theta(t)$ and transverse $\underline{\Psi}(t)$ parts satisfy the equations:

$$d\theta/dt = \Omega_a + \gamma N^{-1}(\underline{\Gamma}(\underline{\Psi}(t)), \underline{n}_d), \tag{2.8}$$

where

$$\Omega_a := N^{-1}(\underline{\omega}, \underline{n}_d), \tag{2.9}$$

and

$$d\underline{\Psi}/dt = \underline{\omega} - \Omega_a \underline{n}_d + \gamma \underline{\Gamma}_{tr}(\underline{\Psi}(t)), \tag{2.10}$$

where

$$\underline{\Gamma}_{tr}(\underline{\Psi}) := \underline{\Gamma}(\underline{\Psi}) - N^{-1}(\underline{\Gamma}(\underline{\Psi}), \underline{n}_d)\underline{n}_d \tag{2.11}$$

is the transversal component of the field $\underline{\Gamma}$.

Conversely, let $t \mapsto \underline{\Psi}(t) \in \mathcal{R}^{N-1}$ be a solution to the system of the $n-1$ order:

$$d\underline{\Psi}/dt = \underline{\omega} - \Omega_a \underline{n}_d + \gamma \underline{\Gamma}_{tr}(\underline{\Psi}), \quad \underline{\Gamma}_{tr} : \mathcal{R}^{N-1} \to \mathcal{R}^{N-1}. \tag{2.12}$$

With this solution given, one may solve the equation

$$d\theta/dt = \Omega_a + \gamma/N(\underline{\Gamma}(\underline{\Psi}(t)), \underline{n}_d) \tag{2.13}$$

by simple quadrature, and thus construct by superposition (2.7), a solution to the system (2.1).

For a given $\underline{\varphi}$ the solutions of the equation (2.13) are parameterized by the initial value $\theta(0) = \theta_0$. Putting $\theta(t) = \theta_0 + \eta(t)$ one has $\underline{\varphi}(t) = \eta(t)\underline{n}_d + \theta_0 \underline{n}_d + \underline{\Psi}(t)$ where the vector $\theta_0 \underline{n}_d + \underline{\Psi}(t)$ has constant component along the $\underline{n}_d$ direction. This is why it will be often more convenient to work with yet another system of equations:

$$d\varphi/dt = \underline{\omega} - \Omega_a \underline{n}_d + \gamma \underline{\Gamma}_{tr}(\underline{\varphi}) \tag{2.14}$$

where $\underline{\omega} - \Omega_a \underline{n}_d + \gamma \Gamma_{tr}(\underline{\varphi})$ is a transverse vector field in $\mathcal{R}^N$ with the longitudinal component along $\underline{n}_d$ always equal to zero. We will denote the solutions of equation (2.14) by $\tilde{\underline{\varphi}} : \mathcal{R} \to \mathcal{R}^N$. The system (2.14) has an obvious first integral:

$$(\tilde{\underline{\varphi}}(t), \underline{n}_d) = \text{const.} \tag{2.15}$$

i.e., its phase trajectories are entirely contained in hyperplanes having common normal vector $\underline{n}_d$. The families of solutions belonging to two different hyperplanes are in one-to-one correspondence under the respective translation along the $\underline{n}_d$ direction, and $\underline{\Psi}$ components of the system (2.1) solutions are simply φ type solutions of the equation (2.13) contained in the plane $(\underline{\varphi}, \underline{n}_d) = 0$.

Below we select the constant in equation (2.15) in a most convenient way for a discussion of the singularities, and only later on we will pass to the $\underline{\Psi}$-type solutions. Formally, the above-described procedure corresponds to working with the N scalar equations of the type (2.10) written in the original coordinate system, including the "redundant" equation. Here one avoids the explicit introduction of new independent coordinates (like, e.g. phase differences) keeping in mind that such elimination may only overshadow system's symmetries.

3. "Entrained" solutions

Definition 1. *We say that an integral curve $t \mapsto \underline{\varphi}(t)$ of the system (2.1) is an entrained solution if there exist such a constant Ω and vector $\underline{\Psi}_I$ so that*

$$\underline{\varphi}(t) = (\Omega t + \delta(t))\underline{n}_d + \underline{\Psi}(t) \tag{3.1}$$

where

$$\lim_{t \to +\infty} \underline{\Psi}(t) = \underline{\Psi}_I, \tag{3.2}$$

$$\lim_{t \to +\infty} \delta(t)/t = 0. \tag{3.3}$$

Ω is then called the *entrained frequency* of the system, and $\underline{\Psi}_I$ is the *vector of locked phases.* It is easy to see that properties (3.1) and (3.2) are not independent as any solution to the system (2.12) with the property

$$\lim_{t \to +\infty} \underline{\Psi}(t) = \underline{\Psi}_I, \tag{3.4}$$

(or an equivalent solution to the system(2.13)) induces an entrained solution of the system (2.1). Indeed, constructing the function:

$$\theta(t) = \Omega_a t + \theta_0 + \gamma N^{-1} \int_0^t (\underline{\Gamma}(\underline{\Psi}(\tau), \underline{n}_d) d\tau. \tag{3.5}$$

We see that

$$\lim_{t\to+\infty} \theta(t)/t = \Omega_a + \gamma N^{-1}(\underline{\Gamma}(\underline{\Psi}_I), \underline{n}_d) \tag{3.6}$$

using nothing but L'Hopital's rule. Hence

$$\Omega = \Omega_a + \gamma N^{-1}(\underline{\Gamma}(\underline{\Psi}_I), \underline{n}_d) \tag{3.7}$$

is the corresponding entrained frequency.

If in addition to equation (3.3)

$$\lim_{t\to+\infty} d\underline{\Psi}(t)/dt = \underline{0}, \tag{3.8}$$

then $\underline{\Psi}_I$ is necessarily an equilibrium point of the system (2.14). It can be proven that this is always the case for gradient systems (see section 3).

Below we will concentrate on entrained solutions induced by singularities of the vector fields in equation (2.12) (or equation (2.14)). Obviously, any singularity $\underline{\Psi}_s$ of the vector field $\underline{\omega} - \Omega_a \underline{n}_d + \gamma \underline{\Gamma}_{tr}(\underline{\Psi})$ induces at least one entrained solution of the form:

$$\theta(t) = \Omega t + \theta_0, \tag{3.9}$$

$$\underline{\Psi}(t) = \underline{\Psi}_s. \tag{3.10}$$

Following the terminology already used by Poincarè, we will call such solutions *relative equilibria* of the system (2.1). If $\underline{\Psi}_s$ is an asymptotically stable equilibrium of the system (2.12) then each integral curve of the system entering the basin of attraction of $\underline{\Psi}_s$ is an entrained solution of the system (2.1); see figure 1. One may call such entrained solutions *relatively asymptotically stable* (r.a.s).

Returning to our original assumption of the non-singular character of the vector field $\underline{\omega}+\gamma\ \underline{\Gamma}(\underline{\varphi})$, we see that it can be trivially satisfied, e.g., for a large class of systems with an initially transverse vector field $\underline{\Gamma}$ and nonvanishing average frequency Ω_a. The presence of true equilibria in the original system (2.2) additionally complicates the analysis of possible entrained solutions and this problem is excluded here from consideration.

4. Entrainment in gradient systems

An important subclass of systems with the dynamics described by equation (2.1) are gradient ones, where the vector field $\underline{\Gamma}$ is a gradient of a C^2-scalar function $U_0 : \mathcal{R}^N \to \mathcal{R}$:

$$\underline{\Gamma}(\underline{\varphi}) = -\text{grad}\ U_0(\underline{\varphi}). \tag{4.1}$$

Consequently, the vector field $\underline{\omega} + \gamma\ \underline{\Gamma}(\underline{\Psi})$ is then the gradient field of the function

$$U(\underline{\varphi}) = -(\underline{\omega}, \underline{\varphi}) + \gamma U_0(\underline{\varphi}). \tag{4.2}$$

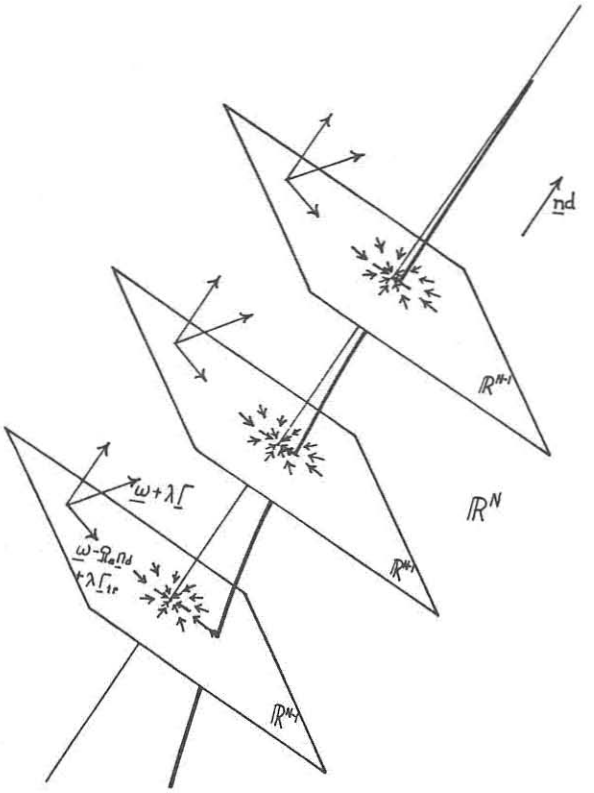

Figure 1: Schematic presentation of an entrained solution. The vector field $\underline{\omega} + \gamma\,\underline{\Gamma}(\underline{\varphi})$ is invariant with respect to translations along the $\underline{n}_d$ direction. Its projection $\underline{\omega} - \Omega_a\underline{n}_d + \gamma\,\underline{\Gamma}_{tr}(\underline{\varphi})$ looks identical on transverse hyperplanes $(\underline{\varphi}, \underline{n}_d)$. If the projected field has an asymptotically stable equilibrium then an entrained solution (dark line) is possible.

Below both the functions U_0 and U will be called potentials.

If the vector field $\underline{\Gamma}$ has the symmetry (2.5) and has the potential U_0, then integrating $\underline{\Gamma}$ along a segment with endpoints $\underline{\varphi}$ and $\underline{\varphi} + \alpha \underline{n}_d$ one obtains

$$U_0(\underline{\varphi} + \alpha \underline{n}_d) = U_0(\underline{\varphi}) - \alpha(\underline{\Gamma}(\underline{\varphi}), \underline{n}_d). \tag{4.3}$$

Let us define the "transverse" potential $U_{tr} : \mathcal{R}^N \to \mathcal{R}$ by

$$U_{tr}(\underline{\varphi}) = -(\underline{\omega} - \Omega_a \underline{n}_d, \underline{\varphi}) + \gamma[U_0(\underline{\varphi}) + N^{-1}(\underline{\varphi}, \underline{n}_d)(\underline{\Gamma}(\underline{\varphi}), \underline{n}_d)]. \tag{4.4}$$

Putting, as usual, $\underline{\varphi} = \underline{\Psi} + \theta \underline{n}_d$, we have from equation (4.3):

$$U_{tr}(\underline{\varphi}) = U_{tr}(\underline{\Psi}) = -(\underline{\omega} - \Omega_a \underline{n}_d, \underline{\Psi}) + \gamma U_0(\underline{\Psi}) \tag{4.5}$$

i.e., U_{tr} behaves identically on all $(\underline{\varphi}, \underline{n}_d) = \text{const.}$ planes, and its gradient field is purely transverse (U_{tr} has a vanishing directional derivative along the $\underline{n}_d$ direction). Using (2.6), one can check that U_{tr} is the potential of the vector field defining the system (2.14) thus being also a gradient system.

If a gradient field $\underline{\Gamma}$ is initially transverse and has the symmetry (2.5) then the potential U_0 is invariant with respect to the translations along the $\underline{n}_d$ direction. Conversely, any system (2.1) with the translationally invariant potential U_0.

$$U_o(\underline{\varphi} + \alpha \underline{n}_d) = U_o(\underline{\varphi}) \text{ for all } \underline{\varphi} \text{ and } \alpha \tag{4.6}$$

is obviously described by a transverse vector field. The entrained solutions, if any, for such systems have necessarily only one entrained frequency: $\Omega_a = \Omega$.

Let us restrict the potential U_{tr} to the plane $(\underline{\varphi}, \underline{n}_d) = 0$, and let $\underline{\Psi}_m$ be an isolated minimum of the restricted potential. Then $\underline{\varphi}_m = \underline{\Psi}_m + (c/N)\underline{n}_d$ are the isolated minima of the potential U_{tr} restricted to the planes $(\underline{\varphi}, \underline{n}_d) = c$, even if the potential U_o is not translationally invariant along the $\underline{n}_d$ direction.

Exploring standard results for gradient systems (see, e.g., [18]), we see that each such minimum is an asymptotically stable equilibrium of the gradient system (2.14), and thus induces a family of r.a.s. entrained solutions.

Let us also note that for gradient systems one may prove that the vector of locked phases is an equilibrium point of the system (2.14). Indeed, if $\lim \underline{\Psi}(t) = \underline{\Psi}_I$ for some trajectory $\varphi(t)$, then $\underline{\Psi}_I$ is the (unique) ω-limit point of the solution $t \to \underline{\Psi}(t)$ to the system (2.14), and all ω-limit points of a gradient flow are equilibria [18].

In conclusion, the search for families of r.a.s. entrained solutions of gradient systems is equivalent to the finding of all isolated minima of the potential $U_{tr}(\underline{\varphi})$, restricted to one of the planes $(\underline{\varphi}, \underline{n}_d) = \text{const}$. This is equivalent to finding all the sinks of the vector field $\underline{\omega} - \Omega_a \underline{n}_d + \gamma \underline{\Gamma}_{tr}(\underline{\varphi})$ in $\mathcal{R}^{N-1}$. Restricted isolated extrema of the potential U_{tr} can be easily identified for sufficiently large $|\ \gamma\ |$, if the potential U_o itself has an isolated extremum. By the well-known perturbation lemma U_{tr} will also have an isolated extremum at $\underline{\Psi}_I(1/\gamma)$ for sufficiently large values of the coupling parameter, with $\underline{\Psi}_I$ being a continuous function of the parameter $1/\gamma$.

Interaction graphs

Consider a finite system of N rotators where the components of the vector field $\underline{\Gamma}$ can be written in terms of binary interactions

$$\Gamma_j(\underline{\varphi}) = \sum_{k,j} J_{jk}\, g_{jk}\,(\varphi_j - \varphi_k) \tag{4.7}$$

and where parameters J_{jk} are not necessarily symmetric in j, k indices. Abscribing to a given j^{th} rotator a vertex j and the set of oriented (ingoing) edges $\{\,(j,k)\,:\, J_{jk}\, g_{jk}\,(\varphi_j - \varphi_k) \neq 0\}$, one obtains a digraph structure of interacting rotators.

Particular topological features of this digraph may further facilitate the search for entrained solutions.

In most of the published studies the interaction between two rotators i and j has been chosen in a specific functional form:

$$g_{jk}(\varphi_j - \varphi_k) = \sin(\varphi_j - \varphi_k). \tag{4.8}$$

It can be shown [3,10,13] that this type of interaction can be considered general for systems of weakly coupled structural units where each unit (with possibly complicated "internal dynamics") has a limit cycle. The coordinate φ_j $(j = 1, \ldots, N)$ is then a "position" coordinate on the j^{th} cycle, and sin-type interaction corresponds to the leading term in perturbation series.

We will consider the interaction (4.8) as a generic one for coupled rotator systems allowing arbitrary values for the coupling constant γ. All systems with symmetric J_{jk} and interactions of the type (4.8) are, of course, gradient ones with the potential:

$$U_o(\underline{\varphi}) = \sum J_{jk}\, \cos(\varphi_j - \varphi_k) \tag{4.9}$$

where summation is over all of the edges of the interaction graph. Hence the related vector field $\underline{\Gamma}$ is here necessarily tranverse. Introducing unit vectors $\underline{s}_j = \cos\varphi_j \underline{e}_1 + \sin\varphi_j \underline{e}_2$, attached to each vertex of the interaction graph, one may write the potential U_o in the form $\sum J_{jk}(\underline{s}_j, \underline{s}_k)$. This potential is obviously invariant with respect to the uniform rotations of all vectors $\underline{s}_j$ through the same angle $\alpha - a$ symmetry operation equivalent to the above discussed translations in the $\underline{\varphi}$ space along the $\underline{n}_d$ direction. In this representation U_o coincides with the configurational part of the Hamiltonian of an intensively studied and well-known equilibrium model of classical planar magnets. Resulting analogies were exploited to some extent in [4]. However, the dynamics of such planar magnetic systems could be quite different from that postulated in equation (2.1), as such magnetic systems are typically considered as conservative.

One more equivalent description of the rotator system is given in terms of the complex variables $z_j = \exp(i\varphi_j)$. In particular, the arithmetic mean of these variables, $\mu = N^{-1}\sum_i z_j$, is a useful quantity for systems with uniform, all-to-all, interactions (see section 6).

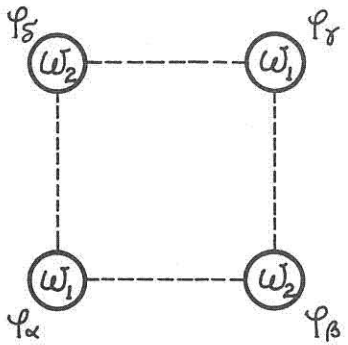

Figure 2: Two-dimensional periodic system of planar rotators with two native frequences.

5. Two-dimensional lattice of rotators

As a simple example of an arbitrarily large system exhibiting perfect synchronization, we consider first a finite two-dimensional array of rotators with two characteristic frequences ω_1 and ω_2 occupying vertices of a rectangular set $\Lambda \subset \mathcal{Z}^2$ in a chessboard fashion (figure 2).

We will assume that only nearest neighbors interact and introduce periodic boundary conditions, for which points of Λ can be viewed as located on a two-dimensional torus. We will show that for sufficiently large γ such systems have at least one class of entrained solutions irrespective of the system size. These are solutions induced by the "ground-state" of the potential

$$U_{tr}(\underline{\varphi}) = -\Delta\omega(\sum_{j\in I_1}\varphi_j - \sum_{k\in I_2}\varphi_k) + \gamma\sum\cos(\varphi_j - \varphi_k), \tag{5.1}$$

where I_1 and I_2 are subsets of Λ occupied by ω_1- and ω_2-type rotators respectively. Here $\Delta\omega = \omega_1 - \Omega_a = (\omega_1 - \omega_2)/2$, and the summation in the interaction term goes over all pairs of nearest neighbors.

To find the ground state of U_{tr} let us divide the system into square plaquettes (figure 3), ascribing to each plaquette the potential

$$U_p(\varphi_\alpha, \varphi_\beta, \varphi_\gamma, \varphi_\delta) = -(\Delta\omega/4)[(\varphi_\alpha + \varphi_\gamma) - (\varphi_\beta + \varphi_\delta)]$$

$$+(\gamma/2)\cos(\varphi_\alpha+\varphi_\beta)+\cos(\varphi_\beta+\varphi_\gamma)+\cos(\varphi_\gamma+\varphi_\delta)+\cos(\varphi_\delta+\varphi_\alpha)]. \tag{5.2}$$

The total potential U_{tr} can then be considered as a sum of plaquette potentials with suitably chosen sets of plaquette variables. The plaquette potential U_p itself is evidently invariant with respect to the uniform shift

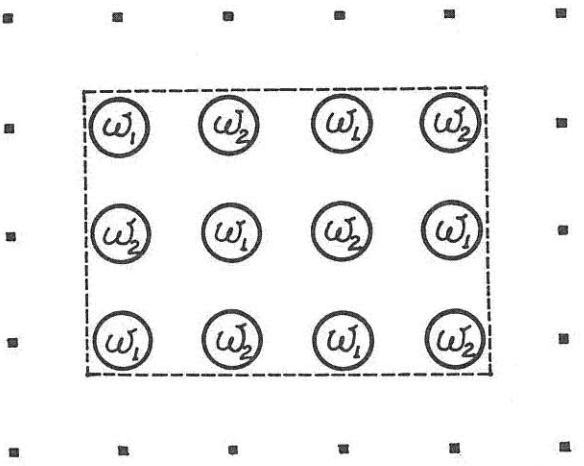

Figure 3: Single plaquette of four rotators and its vertex variables.

in all its variables (translations along the $(1,1,1,1)$ direction in the corresponding $\mathcal{R}^4$ space). Let us assume that for some ratio $\Delta\omega/\gamma$ (a unique parameter of the system) the potential U_p has an isolated absolute minimum at $(\varphi_\alpha^m, \varphi_\beta^m, \varphi_\gamma^m, \varphi_\delta^m)$ in the $\varphi_\alpha + \varphi_\beta + \varphi_\gamma + \varphi_\delta = 0$ plane. If plaquettes with minimal Ψ^m variables can be tiled into a pattern on Λ, then this pattern will be a ground state of U_{tr}. It remains to show that such a pattern is indeed an isolated minimum in the $\sum \varphi_i = 0$ plane, as so far we can only assure the existence of an isolated minimum on the subset of this plane with vertex variables of each plaquette summing to zero. Let $\underline{\Psi}$ be a point which does not belong to this subset. Then there exists a plaquette such that $\Psi_\alpha + \Psi_\beta + \Psi_\gamma + \Psi_\delta = c \neq 0$. Subtracting $c/4$ from each variable we see that

$$U_p(\Psi_\alpha, \Psi_\beta, \Psi_\gamma, \Psi_\delta) > U_p(\Psi_\alpha^m, \Psi_\beta^m, \Psi_\gamma^m, \Psi_\delta^m)$$

unless $\Psi_\alpha = \Psi_\alpha^m + c/4$, $\Psi_\beta = \Psi_\beta^m + c/4$, $\Psi_\gamma = \Psi_\gamma^m + c/4$, $\Psi_\delta = \Psi_\delta^m + c/4$. Thus possible degeneracy may only happen when some plaquettes have assigned to them uniformly shifted values of the Ψ^m plaquette variables, with $c \neq 0$. However, a set Λ of plaquettes with uniformly shifted variables is compatible on Λ if and only if all plaquettes have the same shift $c/4$. This, however, contradicts our assumption $(\underline{\varphi}, \underline{n}_d) = 0$.

Stationary points of the plaquette potential U_p in the $(\underline{\varphi}, \underline{n}_d) = 0$ plane satisfy the equations:

$$2\sin(\Psi_\alpha - (\Psi_\beta + \Psi_\delta)/2)\cos((\Psi_\delta - \Psi_\beta)/2) = -\Delta\omega/2\gamma,$$

$$2\sin(\Psi_\beta - (\Psi_\alpha + \Psi_\gamma)/2)\cos((\Psi_\gamma - \Psi_\alpha)/2) = -\Delta\omega/2\gamma,$$

$$2\sin(\Psi_\gamma - (\Psi_\beta + \Psi_\delta)/2)\cos((\Psi_\delta - \Psi_\beta)/2) = -\Delta\omega/2\gamma,$$

$$2\sin(\Psi_\gamma - (\Psi_\alpha + \Psi_\gamma)/2)\cos((\Psi_\gamma - \Psi_\alpha)/2) = -\Delta\omega/2\gamma,$$

$$\Psi_\alpha + \Psi_\gamma = -(\Psi_\beta + \Psi_\delta). \tag{5.3}$$

It follows from equations (5.3) that for $\Delta\omega \neq 0$

$$\sin(\Psi_\alpha + (\Psi_\alpha + \Psi_\gamma)/2) = \sin(\Psi_\gamma + (\Psi_\alpha + \Psi_\gamma)/2),$$

$$\sin(\Psi_\beta + (\Psi_\beta + \Psi_\delta)/2) = \sin(\Psi_\delta + (\Psi_\beta + \Psi_\delta)/2), \tag{5.4}$$

For $0 <| \Delta\omega/4\gamma |< 1$ one may show, after some algebra, that U_p has exactly two extrema given by $\sin 2\Psi_\alpha = -\Delta\omega/4\gamma$, $\Psi_\beta = -\Psi_\alpha$, $\Psi_\gamma = \Psi_\alpha$, $\Psi_\delta = -\Psi_\alpha$.

The $\partial^2 U_p^t/\partial\Psi_I\partial\Psi_k$ matrix at these stationary points becomes diagonal in the coordinate basis $\underline{e}_d = 1/2(1,1,1,1)$, $\underline{e}_{tr}^{(1)} = 1/2(-1,-1,1,1)$, $\underline{e}_{tr}^{(2)} = 1/2(1,-1,-1,1)$, $\underline{e}_{tr}^{(3)} = 1/2(1,-1,1,-1)$. $\underline{e}_d$ belongs, as it should, to the zero eigenvalue, and remaining eigenvalues are, respectively, equal to: $-\gamma\cos 2\Psi_\alpha$, $-\gamma\cos 2\Psi_\alpha$, $-2\gamma\cos 2\Psi_\alpha$. Hence for both signs of the interaction γ the plaquette potential has exactly one minimum for $0 <| \Delta\omega/4\gamma |< 1$. In conclusion, the whole system considered has at least one r.a.s. equilibrium where the phase difference between any two ω_1 and ω_2 rotators is equal to $2\Psi_\alpha$, and Ψ_α is this solution of the equation $\sin 2\Psi_\alpha = -\Delta\omega/4\gamma$ for which $-\gamma\cos 2\Psi_\alpha > 0$.

The method used above gives no information on possible local minima of U_{tr} which are different from the ground state.

6. Systems with complete interaction graph

Consider a finite system where each rotator interacts with all others with equal strength via the coupling,

$$\Gamma_j(\underline{\varphi}) = N^{-1}\sum_k \sin(\varphi_j - \varphi_k),\ j = 1,\ldots,N. \tag{6.1}$$

Obviously, this is a gradient system where the potential Uo can be chosen as

$$U_o(\underline{\varphi}) = (2N)^{-1}\sum_{j,k}\cos(\varphi_j - \varphi_k), \tag{6.2}$$

with an unrestricted summation over all pairs of indices j, k.

The vector field (6.1) and its potential (6.2) have particular features simplifying the system's analysis. Let us introduce, following Kuramoto [7], the complex "order parameter":

$$\mu = N^{-1}\sum_j \exp(i\varphi_j), \qquad |\,\mu\,| \leq 1. \tag{6.3}$$

The potential U_o can then be written as

$$U_o = (\frac{1}{2})N\ |\,\mu\,|^2$$

and its level surfaces are given by the equations $|\,\mu\,| = \text{const}$. Moreover, the potential U_o has an absolute maximum for $|\,\mu\,| = 1$, and an absolute minimum for $\mu = 0$. Putting $\mu = |\,\mu\,| \exp(i\zeta)$, one may write the components of the $\underline{\Gamma}$ field in the form:

$$\Gamma(\underline{\varphi}) = |\,\mu\,| \sin(\varphi_i - \zeta) \tag{6.4}$$

with an explicit dependence on the φ_i variable and an implicit dependence on all phase variables through the quantities $|\,\mu\,|$ and ζ. All components of the $\underline{\Gamma}$ field are uniformly bounded: $|\,\Gamma_j(\varphi)\,| \leq |\,\mu\,| \leq 1$.

The transverse motion of the system with an arbitrary frequency vector will be governed by the equations :

$$d\varphi_j/dt = \omega_j - \Omega_a + \gamma\,|\,\mu\,| \sin(\varphi_j - \zeta). \tag{6.5}$$

The transverse potential of the system is

$$-(\underline{\omega} - \Omega_a \underline{n}_d, \underline{\varphi}) + (\frac{1}{2})N\gamma\ |\,\mu\,|^2 \tag{6.6}$$

where the presence of the linear form $(\underline{\omega} - \Omega_a \underline{n}_d, \underline{\varphi})$ modifies the potential $\gamma U_o = (\frac{1}{2})N\gamma\ |\,\mu\,|^2$. Together with μ, one may introduce the transverse order parameter defined on the system's solutions:

$$\mu_I(\varphi(t)) = N^{-1}\sum_j \exp(i\tilde{\varphi}_i(t)) = \exp(-i\,\Omega_a t)\mu(\varphi(t)). \tag{6.7}$$

Transverse order parameters of two solutions of the $\tilde{\varphi}$-type residing in different hyperplanes differ by a constant phase factor,

$$\mu_I(\varphi(t)) = \exp(i\alpha)\mu_I(\underline{\Psi}(t)) \text{ for } (\varphi(t), \underline{n}_d) = \alpha \tag{6.8}$$

In particular, if $\underline{\varphi}_s$ is an equilibrium point of the system (6.5) with a complex value of $\mu_I(\underline{\varphi}_s(t)) = \mu_s$ then there always exists a transverse hyperplane where the respectively shifted equilibrium will have a real and non-negative value of the order parameter μ_s. Such hyperplanes are most convenient in finding the coordinates of relative equilibrium $\varphi_{s,j}$ being the solutions to the system of equations

$$\omega_j - \Omega_a + \gamma\,\mu_s \sin(\varphi_{s,j}) = 0, \tag{6.9}$$

where

$$\mu_s = N^{-1} \sum_j^N \cos(\varphi_{s,j}) \text{ and } N^{-1} \sum_j^N \sin(\varphi_{s,j}) = 0. \tag{6.10}$$

Obvious necessary conditions for the existence of relative equilibria are

$$| (\omega_j - \Omega_a)/\gamma | \leq 1 \qquad \text{for all } j. \tag{6.11}$$

If at least one frequency of the rotator system is different from other frequencies, the parameter μ_s is necessarily different from zero. We will first discuss this more "realistic" case, which turns out to be simpler than the analysis of the phase locking in the frequency degenerate case (all ω_j equal).

With $\mu_s \neq 0$, singular points satisfy the equations

$$\sin(\varphi_{sj} = (\omega_j - \Omega_a)/\gamma\, \mu_s \tag{6.12}$$

where μ_s, in turn, are solutions of the family of implicit equations

$$\mu_s = N^{-1} \sum_j \nu_j [1 - (\omega_j - \Omega_a)^2/(\gamma^2 \mu_s^2)]^{1/2} \tag{6.13}$$

where $\nu_j \in \{-1, 1\}$ are the sign factors. The equation (6.13) can be rewritten as

$$\mu_s^2 = N^{-1} \sum_j \nu_j [\mu_s^2 - (\omega_j - \Omega_a)^2]^{1/2} \tag{6.14}$$

and hence

$$\mu_s \geq \max_j | \omega_j - \Omega_a | /|\gamma| = |\Delta\omega|/|\gamma| \tag{6.15}$$

Equation (6.14) may have solution satisfying the condition $\mu_s > |\Delta\omega|/|\gamma|$ only for some of the 2^N possible sign factor choices (e.g., it is clear that for all $\nu_j = -1$ there are no positive solutions of equation (6.14)).

If such solutions exist for a certain choice of sign factors ν_j, it will lead to an *isolated singularity* in the respective transverse plane with φ_{sj} uniquely determined (modulo 2π) by the pair of equations:

$$\sin(\varphi_{sj}) = \omega_j - \Omega_a)/\gamma\, \mu_s,$$

$$\cos(\varphi_{sj}) = \nu_j [1 - (\omega_j - \Omega_a)^2/\gamma^2\, \mu_s^2]^{1/2} \tag{6.16}$$

Angular coordinates Ψ_{sj} of this singularity projection onto the $(\underline{\varphi}, \underline{n}_d) = 0$ plane (a candidate for the vector of locked phases of the corresponding relative equilibrium) can be found by subtracting from each φ_{sj} the sum $N^{-1}\sum\varphi_{sj}$.

γU_o can be considered the "true" potential of a system with all frequencies equal. For $\gamma < 0$ (an excitatory coupling), the corresponding absolute minimum of γU_o is obviously attained at a single point $\underline{\Psi}_s = 0$ in the transverse space. For sufficiently large $|\gamma|$, the frequency term in the U potential can be considered as a perturbation to U_o producing a small shift in $\underline{\Psi}_s = 0$ minimum position. As a result, for sufficiently large $|\gamma|$ there will be at least one class of entrained solutions which is induced by the basin of attraction of this minimum.

For $\gamma > 0$ (an inhibitory coupling) and $N \geq 4$ there is a nontrivial degeneracy in the γU_o absolute minimum position in the transverse space. Indeed, the equation

$$N^{-1} \sum_{j=1,N} \exp(i\,\varphi_j) = 0$$

has then infinitely many solutions $\{\varphi_j\}$ not related by a "rigid" rotation (see figure 4).

The presence of the non-zero frequency term (leading to $\mu_s \neq 0$) will partly remove this degeneracy, leading to isolated singularities (as discussed above). However, the position of the $\underline{\Psi}_s$ singularities will remain extremely sensitive with respect to small changes in the system frequencies for arbitrary large γ. For systems with negative and sufficiently large (in absolute value) values of γ these points will act as repellers for the system's trajectories, once again indicating the possibility of a very complicated dynamical behavior.

The situation is not so complicated if one is interested only in the entrainment phenomenon when the positive coupling strength remains constant and the system size increases. Indeed, an elementary stability analysis shows that all singularities $\underline{\Psi}_s$ are then necessarily unstable equilibria.

To discuss stability of equilibria we shall consider the linearization of the vector field $\underline{\omega} - \Omega_a \underline{n}_d + \gamma\,\underline{\Gamma}$ coinciding with the derivative of the vector field $\gamma\,\underline{\Gamma}$.

The matrix elements of this operator are

$$(\gamma\, D\,\underline{\Gamma}(\underline{\varphi}))_{ii} = \gamma\, N^{-1} \sum_{j(\neq i)} \cos(\varphi_i - \varphi_j) \tag{6.17}$$

$$(\gamma\, D\,\underline{\Gamma}(\underline{\varphi}))_{ii} = -\gamma\, N^{-1} \cos(\varphi_i - \varphi_j),\ i \neq j \tag{6.18}$$

We have already noticed that $D\,\underline{\Gamma}(\underline{\varphi})$ is singular equation (2.6). Hence

$$\det D\underline{\Gamma}(\underline{\varphi}_s) = 0 \tag{6.19}$$

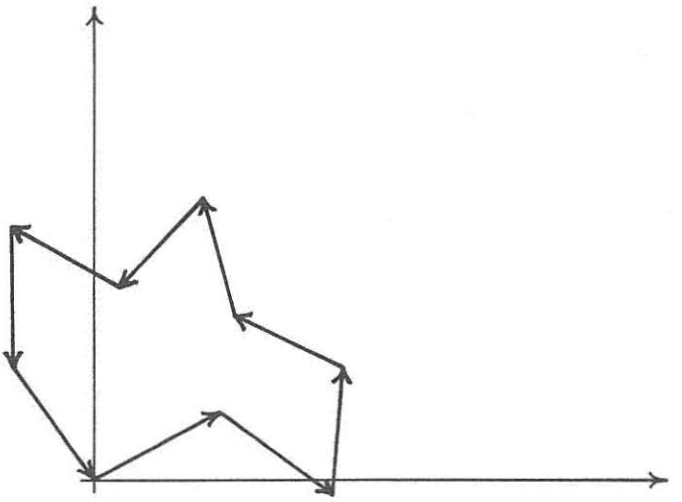

Figure 4: For $N \geq 4$ there are infinitely many sets of unit vectors in the complex plane summing up to the zero vector, resulting from the deformation of the polygon shown. The same argument can be used for an arbitrary fixed value of the order parameter μ, which shows that the level surfaces of the U_0 potential could be quite intricate for $N \geq 4$.

Clearly, $D\underline{\Gamma}(\underline{\varphi}) = D\underline{\Gamma}(\underline{\Psi})$ for all $\underline{\varphi}$. Moreover, $\underline{\varphi}_s$ is an asymptotically stable equilibrium of the system (2.14) if and only if $-D\,\underline{\Gamma}(\underline{\varphi}_s))$ is positively definite when restricted to the transverse space. Hence, the positive semi-definiteness of $-D\underline{\Gamma}(\underline{\varphi}_s)$ is the necessary condition for the existence of r.a.s. entrained solutions.

In turn, the conditions

$$-(\gamma\, D\,\underline{\Gamma}(\underline{\varphi}_s))_{ii} \geq 0, \qquad \text{for all } i, \tag{6.20}$$

are necessary for the positive semi-definiteness of $D\,\underline{\Gamma}(\underline{\varphi}_s)$. These two simple observations allow us to exclude a class of unstable solutions for different signs of the interaction parameter γ.

Let $\underline{\varphi}_s$ be a singularity residing on a hyperplane with real $\mu_s \geq 0$. Then, by equations (6.10),

$$-(\gamma\, D\,\underline{\Gamma}(\underline{\varphi}_s))_{ii} = \gamma(N^{-1} - \mu_s \cos(\varphi_{si})). \tag{6.21}$$

Consider, first, the case $\gamma < 0$. It is clear that all the solutions with some of the (φ_{si}) negative, can not be stable. Thus, the only solution which could be stable is that obtained for all $\nu_j = +1$, corresponding to the maximal possible value of μ_s. For sufficiently large γ, this solution will be stable, as obtained by perturbing the $\mu = 1$ minimum of the potential $\gamma\, U_o$. We obtain

Proposition 1. *Rotator communities with purely excitatory ($\gamma < 0$) all to all $1/N$ interaction may only have one class of r.a.s. solutions with a single vector of locked phases.*

For $\gamma > 0$ conditions (6.18) narrows the interval for ms to

$$0 < \mu_s \leq N^{-2} \tag{6.22}$$

Taking into account the existence condition (6.15) we are led to the restrictive inequality

$$|\Delta\omega|/|\gamma| \leq \mu_s \leq N^{-1/2} \tag{6.23}$$

This can be summarized in Proposition Two:

Proposition 2. *Large ($N \to \infty$) systems of rotators with purely inhibitory type of $1/N$ interaction (fixed $\gamma > 0$) and finite frequency spread cannot have entrained solutions.*

On the other hand, for fixed N entrained solutions are not excluded if γ is sufficiently large.

The distinction between excitatory ($\gamma < 0$) and inhibitory ($\gamma > 0$) interactions persist for rotator systems with randomized (in the spirit of the Mattis model of spin glass) interaction signs (see [16]). The evolution equations contain then additional random variables $\tau_j \in \{-1, 1\}$:

$$\varphi_j = \omega_j - \Omega_a + \gamma N^{-1} \sum_j \tau_j \tau_j \sin(\varphi_j - \varphi_k) \tag{6.24}$$

Introducing after [16] new variables $\varphi_j \to \varphi_j + \phi\, \tau_j/2$, one may gauge-out the τ_j randomness from the evolution equations, and thus systems of this type again separate into two classes with different behaviors depending on the sign of γ. For $\gamma < 0$ the τ_j configurations will be stored in the vector of locked phases, which is different for each configuration.

Partial entrainment and large random systems with $1/N$ interaction

In our simple analysis of the entrainment phenomenon we concentrated on the case when the tranverse space may have sink-type singularities forcing *all* rotators to move with the same asymptotic frequency Ω_a. However, the system may exhibit some amount of organization if some of the equilibria are of the saddle type. To be more precise, consider the simplest and possibly the generic situation (a lack of other obvious symmetries) when the equilibrium $\underline{\Psi}_s$ is hyperbolic in the transverse space (and it is *not* a sink). According to the Stable Manifold Theorem (see, e.g., [19]) the solutions $\underline{\varphi}(t)$ sufficiently close to this point will have an asymptotically constant component in the subspace of dimension less than $N - 1$ and a complementary "escaping" component. Diagonalizing the operator $D\,\underline{\Gamma}(\underline{\varphi}_s)$ one will obtain

a local equivalence of the original system to a new system of non-interacting abstract rotators, where a *part* of them will move with the same asymptotic frequency Ω. However, conditions (6.11) remain necessary for this type of behavior.

The very presence of equilibria may not be necessary for the partial entrainment if more complicated trapping sets (strange attractors for $N \geq 4$) are present in the transverse space. One may also think about a partial entrainment with finite time scale, where the system lives for a long time in some "small" sets in the transverse space with intermittent short periods of asynchronous behavior. These ideas are at the heart of the Kuramoto approach [7] when one works ab initio with averaged equations of motion (6.5), assuming the possibility of the non-zero average order parameter and latter on taking into account the fluctuation around this average. We will show below that such behavior must be of crucial importance for the possible partial entrainment in large systems with random frequencies. Indeed, it is not difficult to estimate the probability that a system with random frequencies has relative equilibria, which leads to the Proposition 3.

Proposition 3. *Almost all infinite rotator systems with identically normally distributed independent frequencies and arbitrary γ/N interactions do not have relative equilibria. In particular, perfectly entrained solutions are excluded with probability one.*

Proof. A frequency vector of a system with entrained solutions is necessarily located inside a cylinder set in the frequency space (compare equation (6.11)).

$$T_N(\gamma) = \{\underline{\omega} :_{j=1,\ldots,N} |\omega_j - \Omega_{N,a}| < |\gamma|,\ \Omega_{N,a} \in R\} \tag{6.25}$$

For the Gaussian frequency distribution consider the probability $P_N(\omega \in T_N(\gamma))$. Chosing the average frequency $\Omega_{N,a}$ and frequency deviations $\mu_j = \omega_j - \Omega_{N,a}$, $j = 1, \ldots, N-1$ as new integration variables (Jacobian factor equal to N), one obtains

$$\begin{aligned} P_N(\underline{\omega} \in T_N(\gamma)) = N(2^{1/2}\pi\sigma)^{-1} \int_{-\infty}^{+\infty} d\,\Omega_{N,a} \exp(-N(\Omega_{N,a} - \Omega)^2/2\,\sigma^2) \star \\ \int \cdots \int \qquad \exp(\sum_{k=1}^{N-1} u_j^2 - (u_1 + u_2 + \ldots + U_{N-1})^2/2\,\sigma^2) \\ |u_j| \leq |\gamma|,\ |\sum u_j| \leq |\gamma| \\ j = 1, \ldots, N-1 \end{aligned} \tag{6.26}$$

where $\overline{\Omega}$ and σ are the distribution parameters. Hence

$$P_N \leq (N^{1/2}/(2\pi\sigma)^{N-1})(\int_{-|\gamma|}^{|\gamma|} \exp(-u^2/2\sigma^2) d\,u)^N$$

and $\lim_{N\to\infty} P_N = 0$ for all finite $|\gamma|$. ∎

Comment: It is interesting to note that replacing the necessary condition (6.11) by a weaker condition

$$(1/|\gamma|) \parallel \underline{\omega} - \Omega_a \underline{n}_d \parallel = (1/|\gamma|)[\sum_{j=1}^{N} (\underline{\omega}_j - \Omega_a \underline{n}_d)^2]^{1/2} \leq N^{1/2} \tag{6.27}$$

one can obtain a critical behavior in the $N \to \infty$ limit when the corresponding probability tends to zero for all values of $|\gamma|$ less than some threshold value (proportional to σ) and has a non-zero value above this threshold. However, we were unable to find a reasonable dynamical interpretation of the inequality (6.27) besides an obvious observation that it ensures the existence of at least one rotator satisfying the condition $|\underline{\omega} - \Omega_a \underline{n}_d| \leq |\gamma|$.

Concluding Remarks

We sketched above a picture of the complete synchronization phenomenon as generically caused by the presence of relatively stable asymptotic equilibria in finite systems with translationally invariant coupling. We limited our discussion to relatively simple examples, and we are unable to prove the existence of multiple entrained states. In this respect higher-dimensional systems with frustrated interactions are interesting, as they have largely degenerate ground states.

Equally interesting are the dynamics of finite systems with random frequencies, constant interaction strength, and random interaction graphs. We are considering systems of this type as a model qualitatively explaining experimental data derived from the recorded electrical activity of single neurons in highly interconnected random neural networks studied *in vitro* [20–22]. These systems exhibit both chaotic and quasi-periodic bursting patterns, depending on the synaptic connection type and strength. The synaptic interaction can be influenced chemically or with electric stimulation, giving us the possibility to control the model's coupling parameter γ. Adopting a threshold activation picture for a single neuron [10], we carry out computer simulations of the spiking pattern produced by entrained and asynchronous neurons. These results will be reported in a separate publication.

Another class of problems is related to pattern storage in rotator systems. It should be mentioned that the idea itself is not new. E. Bienenstock [23] was first to propose the use of the rotator model in studies of memory and learning. However, the pattern retrieval has been carried out by Bienenstock by a "simulated annealing" procedure. The use of dissipative evolution equations (like equation (2.1) remains for us an interesting possibility.

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Note added in proof: Large systems of planar rotators with random frequencies have been thoroughly investigated in a recent paper by S. H. Strogatz and R. E. Mirollo [24]. The basic method is the same as we employed

proving proposition 3 (the use of necessary conditions for entrainment and probability estimates). However, as presented by those authors, the non-existence theorems for entrainment have much wider range of applicability (type of interactions and frequency distributions). Moreover, they proved that the entrained clusters, if any, in an asymptotically infinite system must have a "sponge-like" structure in an interesting analogy to percolation clusters.

Systems with all-to-all interactions and random frequencies have also been investigated in a paper previously unknown to us by G. B. Ermentrout [25]. We are grateful to the referee for pointing this out and for all other comments.

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