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Method of Computation of the Reynolds Number for Two Models of Lattice Gas Involving Violation of Semi-detailed Balance

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Abstract. We show how the theory of lattice gases developed by Frisch, d'Humières, Hasslacher, Lallemand, Pomeau, and Rivet, can be extended to cases involving violation of semi-detailed balance. This allows further reduction of the viscosity. However, since the universality of the distribution is lost, the function $g(\rho)$ becomes dependent on the collision laws and has to be evaluated by a suitable generalization of the work of Hénon on viscosities. Cases with and without rest particles are considered. The lattice Boltzmann approximation is used.

1. Introduction

Lattice gas models (see [1], [2], and [3]) have become a very promising way to simulate some hydrodynamic phenomena. Indeed, the simplicity of its formalism allows the construction of codes where no integration of partial differential equations is necessary. It also led to the construction of specialized machines such as CAM [4] or RAP [5], which is able to simulate in real time the evolution of a fluid. Using these techniques, Rivet, Hénon, Frisch and d'Humières have already simulated fully three-dimensional external flows [6].

For sake of computational efficiency, it has become very important to work on optimizing the collision laws in the lattice gas models. One of the most important control parameters of all the simulation is indeed the Reynolds number. In the lattice gas models, the latter can be written as [7] $R = R_\star \times L \times V$ where $R_\star$ is a coefficient which depends on the collision

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laws, and L and V are respectively a characteristic length and a characteristic velocity expressed in units of lattice constant and sound velocity. Reaching a given Reynolds number at given $R_\star$ and V means choosing L. The problem is that the number of nodes goes as L^3 and then any given computational quantity goes as L^4. This also means that, at a given V, the computational resources (and therefore the efficiency) vary as $R_\star^{-4}$. That is why it has become of particular interest to increase $R_\star$. Since this coefficient is inversely proportional to the viscosity, the natural idea is to diminish the viscosity. The latter depends on the collision laws imposed on the gas via the mean free path: the more collisions there are, the smaller the mean free path is and the smaller the viscosity is.

Increasing the number of allowed collisions can be done in two different ways: either by increasing the number of particles involved or by setting fewer limitations on the collision rules. The first way has already been used in a very efficient manner by Hénon [8] who succeeded in raising $R_\star$ by a factor sixteen by an adequate choice of the collisions in a 25 particles model. However, this way of proceeding does not allow an indefinite increase in $R_\star$. The only way to raise it any more is to act on the collision rules themselves. The problem is that this can imply some drastic changes in the dynamics and in the macroscopic behaviour of the gas. It is then necessary to choose carefully which conditions on the collision rules are to be dropped. In all the models considered until now, the main three conditions are the following: the collisions must conserve mass and momentum and they must satisfy a sort of generalization of the micro-reversibility called the semi-detailed balance. The first two conditions are a very convenient way to ensure that the total energy of the gas is conserved and therefore, they seem difficult to give up. As to the third one, we know already from the work of Frisch et al. [7] and of Hénon its importance in the derivation of the universality of the equilibrium distributions. However, we show here that, even in models involving violation of semi-detailed assumption, it is possible to give an explicit development of the equilibrium distributions at low Mach number which can be seen as a generalization of the development given in [7]. This is done using the Boltzmann approximation (as described in [7]), the symmetries of the lattice and the properties of the collision matrix which enable us to explicitly solve the system of equations required (even when it involves 24 of more unknowns). Following these results, we also extend the expression of the viscosity using the work of Hénon [8]. Finally, we give the expression of the Reynolds number in our model and discuss briefly the possibilities of its maximization by a proper choice of the collision laws.

The outline of this paper is the following: in the second part, we summarize the lattice gas formalism and the notations we are going to use; in the third part, we compute the Reynolds number in a case without rest particle using the scheme described above; in the fourth part, we show how this computation can be extended to cases involving rest particles.

2. General formalism and notations

In this section, we summarize the general formalism of the lattice gas. We include the mathematical formulation of what we mean by "symmetry of the lattice" and its implication, namely the "*G-invariance*" , which will be the basis of all our computations. For a general review of the subject including the description of the most common lattice gas models, see the very detailed paper of Frisch, d'Humières, Hasslacher, Lallemand, Pomeau, and Rivet [7].

A lattice gas consists of particles of mass unity moving with speed c on a D-dimensional lattice $\mathcal{L}$ (typically, D equals two or four) which is required to obey some symmetries that we will describe below. Each node in $\mathcal{L}$ is connected to its b nearest neighbors by a set of b velocity vectors $\mathbf{c}_i\,(i = 1,\dots b)$, so called because they represent the actual velocities of the particles moving from a node to its neighbor. Each $\mathbf{c}_i$ has spatial components $\mathbf{c}_{i\alpha}(\alpha = 1,\dots,D)$.[1]

We adopt an "exclusion principle" which forbids two particles to be at the same node with the same velocity. Therefore, each node can assume only a finite number of states, each state being described by a b-bit binary word $s = \{s_i;\ i = 1,\dots b\}$ where s_i is equal to one if the node contains a particle with speed $\mathbf{c}_i$ and zero otherwise. Given an initial configuration $s(.) = \{s(\mathbf{r}_*);\ \mathbf{r}_* \in \mathcal{L}\}$, where $s(\mathbf{r}_*)$ represents the state of the node $\mathbf{r}_*$, the gas will evolve in time through collisions occurring at each node. A collision is defined as the passage from an input state s to an output state s'. The collisions are governed by the set of a $2^b \times 2^b$ collision matrix A, whose elements $A(s \to s')$ represent the transition probability from the state s to the state s'. This matrix is supposed to satisfy certain properties that we will specify later. Thanks to this collision matrix, the evolution of the gas can be described in a statistical way using a probability distribution $P(s(.))$ which gives the probability of occurrence of a configuration $s(.) = \{s(\mathbf{r}_*);\ \mathbf{r}_* \in \mathcal{L}\}$. The evolution of $P(s(.))$ in time is then described by a Liouville equation (see [7], section 3.3)

$$P\big(t_* + 1, \mathcal{S}s'(.)\big) = \sum_{s(.)\in\Gamma} \prod_{\mathbf{r}_*\in\mathcal{L}} A\big(s(\mathbf{r}_*) \to s'(\mathbf{r}_*)\big) P\big(t_*, s(.)\big), \quad (2.1)$$

where Γ denotes the set of all possible configurations in the lattice and $\mathcal{S}$ is the streaming operator $\mathcal{S} : s_i(\mathbf{r}_*) \mapsto s_i(\mathbf{r}_* - \mathbf{c}_i)$.

This probability distribution enables us to define physical quantities of interest, namely

(a) the mean population in the i^{th} direction:

$$N_i(t_*, \mathbf{r}_*) = \sum_{s(.)\in\Gamma} s_i(\mathbf{r}_*) P\big(t_*, s(.)\big), \quad (2.2)$$

(b) the density:

[1] In this paper, Greek and Roman indices refer respectively to coordinate and velocity labels. Summation over repeated Greek indices, but not Roman ones, is implicit.

$$\rho(t_*, \mathbf{r}_*) = \sum_i N_i(t_*, \mathbf{r}_*), \tag{2.3}$$

(c) the mean velocity:

$$\mathbf{u}(t_*, \mathbf{r}_*) = \frac{1}{\rho(t_*, \mathbf{r}_*)} \sum_i \mathbf{c}_i N_i(t_*, \mathbf{r}_*). \tag{2.4}$$

The above defines a very general class of lattice gas models. We now need to specify the symmetries of the lattice and the properties of the collision matrix A in order to make the computations more tractable.

2.1 Symmetries of the lattice

Given the definition of the lattice, specifying its symmetries is equivalent to specifying the symmetries of the velocity set $V = \{\mathbf{c}_i;\ i = 1, \ldots b\}$. In the following we will therefore impose two conditions on V (see [8]):

i. V must be isotropic to fourth order. The meaning of this assumption is the following: if we define a tensor of order n ($n \in \mathbf{N}$) by

$$T_{\alpha_1 \ldots \alpha_n} = \sum_i \mathbf{c}_{i\alpha_1} \ldots \mathbf{c}_{i\alpha_n}, \tag{2.5}$$

then any tensor up to n equal four should be isotropic, i.e. their components should be invariant by any rotation of the coordinate axes. The T are then shown to be given by:

$$T_\alpha = \sum_i \mathbf{c}_{i\alpha} = 0, \tag{2.6}$$

$$T_{\alpha\beta} = \sum_i \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta} = \frac{bc^2}{D}\delta_{\alpha\beta}, \tag{2.7}$$

$$T_{\alpha\beta\gamma} = \sum_i \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta}\mathbf{c}_{i\gamma} = 0, \tag{2.8}$$

$$\begin{aligned} T_{\alpha\beta\gamma\delta} &= \sum_i \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta}\mathbf{c}_{i\gamma}\mathbf{c}_{i\delta} \\ &= \frac{bc^4}{D(D+2)}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}), \end{aligned} \tag{2.9}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

ii. The velocities must be interchangeable in the following sense: defining G as the group of isometries of the lattice (therefore leaving V globally invariant), for any set of velocities $(\mathbf{c}_i, \mathbf{c}_j)$ there is an isometry of G mapping $\mathbf{c}_i$ on $\mathbf{c}_j$. A set of b tensors $T = \{T_i;\ i = 1 \ldots b\}$ will then be said *G-invariant* if any isometry of G which maps $\mathbf{c}_i$ on $\mathbf{c}_j$ maps T_i on T_j.

It can be shown (see [7]) that the combination of i) and ii) leads to the two following properties:

P1 Any set of G-invariant tensors of 1st order is given by

$$T_{i\alpha} = \lambda \mathbf{c}_{i\alpha} \quad \forall \alpha, \tag{2.10}$$

P2 Any set of G-invariant tensors of 2nd order is given by

$$T_{i\alpha\beta} = \lambda \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta} + \mu\delta_{\alpha\beta} \quad \forall (\alpha, \beta). \tag{2.11}$$

These two properties form the basis of all our computations.

2.2 Properties of A

Since A is composed of transition probabilities, it is positive in the sense that all its elements are positive. Due to the normalization of the probabilities, it also obeys the relation:

$$\sum_{s'} A(s \to s') = 1 \quad \forall s. \tag{2.12}$$

Furthermore, it is supposed to have three more properties, namely to conserve mass and momentum and to be invariant under any isometry of G. Since the particles are of mass unity, the mass, $m(s)$, and the momentum, $\mathbf{P}(s)$, in each state s can be defined as

$$m(s) = \sum_i s_i,$$

$$\mathbf{P}(s) = \sum_i s_i \mathbf{c}_i.$$

Conserving the mass means that for any input state s and output state s', the corresponding matrix element will be non zero if and only if $m(s)$ equal $m(s')$. This can be written as

$$(m(s) - m(s'))A(s \to s') = 0 \quad \forall (s, s'),$$

or equivalently

$$\sum_i (s_i - s'_i)A(s \to s') = 0 \quad \forall (s, s'). \tag{2.13}$$

In a similar way, the momentum conservation can be written as:

$$\sum_i (s_i - s'_i)\mathbf{c}_{i\alpha}A(s \to s') = 0 \quad \forall \alpha \quad \forall (s, s'). \tag{2.14}$$

The invariance of A under the isometries of G will be formulated as:

$$A\big(\mathcal{G}(s) \to \mathcal{G}(s')\big) = A(s \to s') \quad \forall \mathcal{G} \in G, \forall (s, s'). \tag{2.15}$$

Now comes the difference between our class of model and those described in [7]. Usually a fourth constraint is imposed on the collision rules, the semi-detailed balance. This constraint can be written as: [2]

$$\forall s \quad \sum_{s'} A(s' \to s) = \sum_{s'} A(s \to s') = 1, \tag{2.16}$$

In this paper, *we are not going to impose such a relation*. What can be expected from such a difference? As stated above, the first consequence is that the number of possible collisions will be greatly increased. More importantly, we allow now some configurations to be privileged compared with others. To understand this point, it is useful to make an analogy with quantum mechanics. Suppose that the states s refer to some "energy states" and then, label the energy levels. $A(s \to s')$ is then the transition probability from the s-level to the s'-level. The semi-detailed balance would ensure that the depopulation of the s-level ($\sum_{s'} A(s \to s')$ term) is balanced exactly by its population via the other levels ($\sum_{s'} A(s' \to s)$ term). If we drop this assumption, nothing prevents some levels from being systematically "emptied" asymptotically achieving a zero probability of occurrence after a finite time.

3. Monospeed model

In this section, we will give a method of computing the Reynolds number in the model defined above. In general, the Reynolds number R is written as $R = \frac{L \cdot U}{\nu}$ where L and U are respectively a characteristic length and a characteristic speed of the fluid under consideration and ν its kinematic shear viscosity. In the lattice gas case, it has been shown in [7] that in the case of low-speed equilibria and when the Boltzmann approximation is valid, the Reynolds number involves a rescaled viscosity $\nu'(\rho) = \frac{\nu(\rho)}{g(\rho)}$ where ρ is the mean density. The scaling factor $g(\rho)$ is linked to the development to second order of the mean equilibrium populations N_i^{eq} through the equation:

$$N_i^{eq}(\rho, \mathbf{u}) = d(1 + \frac{D}{c^2}\mathbf{c}_{i\alpha}u_\alpha + Ng(\rho)Q_{i\alpha\beta}u_\alpha u_\beta) + O(\mathbf{u}^3), \tag{3.1}$$

where

$$d = \frac{\rho}{b}, \quad N = \frac{D(D+2)}{2c^4} \quad \text{and} \quad Q_{i\alpha\beta} = \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta} - \frac{c^2}{D}\delta_{\alpha\beta}.$$

We proceed therefore in the following way: first we compute the mean equilibrium populations using the Boltzmann approximation to find the expression for $g(\rho)$. Then, using the expression of the shear viscosity found by Hénon [8] which is valid even without the semi-detailed balance, we find the expression for the Reynolds number. Finally, we briefly discuss the maximization of this Reynolds number by a proper choice of the collisions.

[2] Note that this semi-detailed balance is a generalization of the detailed balance (also called micro-reversibility in collision theory) $A(s' \to s) = A(s \to s') \quad \forall s, s'$.

3.1 Mean equilibrium populations

To derive the expression of the mean equilibrium populations, we make use of the Boltzmann approximation. This means that we assume that the particles entering a collision process have no prior correlations. This is a very crude assumption which works only for mean quantities. It is then shown by Frisch et al in [7] that this approximation leads to the lattice Boltzmann equation

$$N_i(t_* + 1, \mathbf{r}_* + \mathbf{c}_i) = N_i(t_*, \mathbf{r}_*) + \Delta_i{}^{Boltz}$$

$$\Delta_i{}^{Boltz} = \sum_s \sum_{s'} (s'_i - s_i) A(s \to s') \prod_j N_j{}^{s_j} (1 - N_j)^{(1-s_j)} \quad \forall i. \tag{3.2}$$

The equilibrium populations are therefore the solutions of the system of b equations

$$\Delta_i{}^{Boltz} = 0 \quad \forall i. \tag{3.3}$$

Once more, this is a mathematically well-defined problem but still not very tractable. Fortunately, the symmetries of the lattice (and especially the G-invariance) lead us to postulate an expansion of the solutions of the form

$$N_i(\rho, \mathbf{u}) = d + \lambda \mathbf{c}_{i\alpha} u_\alpha + \mu Q_{i\alpha\beta} u_\alpha u_\beta + \eta \delta_{\alpha\beta} u_\alpha u_\beta + O(\mathbf{u}^3), \tag{3.4}$$

where

$$Q_{i\alpha\beta} = \mathbf{c}_{i\alpha} \mathbf{c}_{i\beta} - \frac{c^2}{D} \delta_{\alpha\beta}, \tag{3.5}$$

and where d, λ, μ and η are some constants to be determined as functions of ρ, the density of the gas.

By definition of ρ and $\mathbf{u}$,

$$\rho = \sum_i N_i, \tag{3.6}$$

$$\rho \mathbf{u} = \sum_i N_i \mathbf{c}_i, \tag{3.7}$$

d, λ, and η are constrained to satisfy

$$d = \frac{\rho}{b}, \eta = 0 \quad \text{and} \quad \lambda = \frac{dD}{c^2}.$$

The only free parameter left is μ. In order to determine it, we inject the expansions of the N_i (3.4) into the system of equation (3.3) and set Δ_i equal to zero up to the second order. We obtain now three systems of b equations corresponding to the setting to zero of the order zero, one, and two. In fact, since the Δ_i are G-invariant, all the directions are equivalent. Therefore, these three systems simplify to three independent equations (one for each order). The first two equations (corresponding to the zero and the first order) have no free parameters and we will have to check that they are indeed null

without any further condition. At this point, we expect the properties of A to play an important role. As to the third equation, it will give us one condition which will enable us to determine μ.

Using the expansion of the N_i given above, it is shown in appendix A that the Δ_i can be expanded as a function of zero-, first- and second-order tensors as

$$\Delta_i = r_i + \frac{\lambda}{d(1-d)} t_{i\alpha} u_\alpha + \left(\frac{\lambda^2}{2d(1-d)}[v_{i\alpha\beta} + w_{i\alpha\beta} + (2d-1)z_{i\alpha\beta}] + \mu z_{i\alpha\beta}\right) u_\alpha u_\beta, \tag{3.8}$$

where r, w, v, and z are respectively written:

$$r_i = \sum_s \sum_{s'} (s'_i - s_i) A(s \to s') \left(\frac{d}{1-d}\right)^p (1-d)^b, \tag{3.9}$$

$$t_{i\alpha} = \sum_s \sum_{s'} (s'_i - s_i) A(s \to s') \left(\frac{d}{1-d}\right)^p (1-d)^b P_\alpha(s), \tag{3.10}$$

$$v_{i\alpha\beta} = \tag{3.11}$$

$$\sum_s \sum_{s'} (s'_i - s_i) A(s \to s') d^{p-1} (1-d)^{b-p-1} \sum_k \left((2d-1)s_k - d^2\right) \frac{c^2}{D} \delta_{\alpha\beta},$$

$$w_{i\alpha\beta} = \tag{3.12}$$

$$\sum_s \sum_{s'} (s'_i - s_i) A(s \to s') \left(\frac{d}{1-d}\right)^p (1-d)^b \frac{1}{d(1-d)} P_\alpha(s) P_\beta(s),$$

$$z_{i\alpha\beta} = \sum_s \sum_{s'} (s'_i - s_i) A(s \to s') \left(\frac{d}{1-d}\right)^p (1-d)^b \frac{1}{d(1-d)} Y_{\alpha\beta}(s). \tag{3.13}$$

where p is the number of particles in the state s: $p = \sum_i s_i$.

In the above equations, we have used the first and second-order momentum in a state s, $P(s)$, and $Y(s)$, defined respectively as as (see [8]):

$$P_\alpha(s) = \sum_i s_i \mathbf{c}_{i\alpha}, \tag{3.14}$$

$$Y_{\alpha\beta}(s) = \sum_i s_i Q_{i\alpha\beta}. \tag{3.15}$$

Note that Y is has a null trace:

$$\sum_\alpha Y_{\alpha\alpha} = 0. \tag{3.16}$$

Note also that the tensor $w_{i\alpha\beta}$ is identically zero when the semi-detailed balance is assumed. To prove that, we use momentum conservation to write $w_{i\alpha\beta}$ as

$$w_{i\alpha\beta} = \sum_s \left(\frac{d}{1-d}\right)^p (1-d)^b \frac{1}{d(1-d)} P_\alpha P_\beta s_i [\sum_{s'} A(s' \to s) - 1],$$

which is obviously zero if the semi-detailed balance (2.16) is valid.

We check now that the two first orders in the expansion of the Δ_i are null.

r_i is a zero-order tensor which is G-invariant. Its value is then independent of the direction i and can be written as the mean over all the directions of its components, that is,

$$\begin{aligned} r &= \frac{1}{b}\sum_i r_i \\ &= \frac{1}{b}\sum_s \sum_{s'} \left(\frac{d}{1-d}\right)^p (1-d)^b \sum_i (s'_i - s_i) A(s \to s'). \end{aligned}$$

Since A has been assumed to verify mass conservation (2.13), r is identically zero.

$t_{i\alpha}$ is a first-order tensor which is G-invariant. Thanks to the property **P1** mentioned above (equation (2.10), it can then be written as

$$t_{i\alpha} = \psi \mathbf{c}_{i\alpha}. \tag{3.17}$$

We multiply the two members of (3.17) by $\mathbf{c}_{i\alpha}$ and we sum over i. We obtain then

$$\frac{bc^2}{D}\psi = \sum_s \sum_{s'} \left(\frac{d}{1-d}\right)^p (1-d)^b P_\alpha \sum_i (s'_i - s_i)\mathbf{c}_{i\alpha} A(s \to s').$$

Using now the relation of momentum conservation (2.14), we see that ψ is zero and that $t_{i\alpha}$ is identically null as expected.

We now make use of the G-invariance to simplify the expression of the second-order term of Δ_i. v, w, and z being G-invariant second-order tensors, the property **P2** (equation (2.11)) enables us to write them as:

$$v_{i\alpha\beta} = \eta \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta} + \eta' \delta_{\alpha\beta}, \tag{3.18}$$

$$w_{i\alpha\beta} = \phi \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta} + \phi' \delta_{\alpha\beta}, \tag{3.19}$$

$$z_{i\alpha\beta} = \psi \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta} + \psi' \delta_{\alpha\beta}. \tag{3.20}$$

Summing these three relations over i and using the relation of mass conservation gives us three relations between the coefficients, namely

$$\eta' = -\frac{c^2}{D}\eta,$$

and two similar equations between the four other coefficients. This means that the three tensors v, w, and z are in fact proportional to the $Q_{i\alpha\beta}$. In the case of v, it is easy to show that the coefficient of proportionality η is zero by multiplying the two members of the relation of proportionality by $Q_{i\alpha\beta}$, summing over i, α, and β and using the fact that the $Y(s)$ are traceless tensors.

We are then left with a very compact expression for the Δ_i:

$$\Delta_i = \Big(\frac{\lambda^2}{2d(1-d)}\phi + \Big[\mu + \frac{(2d-1)}{2d(1-d)}\lambda^2\Big]\psi\Big)Q_{i\alpha\beta}u_\alpha u_\beta.$$

Since the $Q_{i\alpha\beta}u_\alpha u_\beta$ terms are not identically zero, setting to zero each Δ_i is equivalent to requiring the coefficient in front of these terms to be zero, that is, to write:

$$\Big(\frac{\lambda^2}{2d(1-d)}\phi + \Big[\mu + \frac{(2d-1)}{2d(1-d)}\lambda^2\Big]\psi\Big) = 0. \tag{3.21}$$

Using this equation and the explicit expression of λ, we obtain μ and then $g(\rho)$ in this case:

$$\mu \equiv d\mu_0 g(\rho) = d\mu_0 g_{sdb}(\rho)\Big(1 - \frac{1}{1-2d}\frac{\phi}{\psi}\Big), \tag{3.22}$$

where $\mu_0 = \frac{D(D+2)}{2c^4}$ and $g_{sdb}(\rho)$ is the value of $g(\rho)$ when the semi-detailed balance holds and can be written as [7]:

$$g_{sdb}(\rho) = \frac{D}{D+2}\frac{1-2d}{1-d}. \tag{3.23}$$

When the collisions obey the semi-detailed balance, $w_{i\alpha\beta}$ and therefore ϕ are zero and $g(\rho)$ equals $g_{sdb}(\rho)$ as expected.

The expression we obtained for $g(\rho)$ is quite simple but is still not very tractable since ϕ and ψ have been obtained only implicitly through the G-invariance. We give now an explicit expression of $g(\rho)$ involving two quantities clearly dependent on the collision laws.

We multiply (3.19) and (3.20) by $Q_{i\alpha\beta}$ and sum over i, α and β to get

$$\begin{aligned} N\phi &= \sum_s\sum_{s'}\sum_i (s'_i - s_i)Q_{i\alpha\beta}A(s\to s')\Big(\frac{d}{1-d}\Big)^p(1-d)^b \\ &\quad \frac{1}{d(1-d)}P_\alpha(s)P_\beta(s) \\ &= \sum_s\sum_{s'} A(s\to s')\Big(\frac{d}{1-d}\Big)^p(1-d)^b \\ &\quad \frac{1}{d(1-d)}P_\alpha(s)P_\beta(s)[Y_{\alpha\beta}(s') - Y_{\alpha\beta}(s)], \end{aligned} \tag{3.24}$$

$$N\psi = \sum_s \sum_{s'} \sum_i (s'_i - s_i) Q_{i\alpha\beta} A(s \to s') \Big(\frac{d}{1-d}\Big)^p (1-d)^b \qquad (3.25)$$
$$\frac{1}{d(1-d)} Y_{\alpha\beta}(s)$$
$$= \sum_s \sum_{s'} A(s \to s') (\frac{d}{1-d}) (1-d)^b$$
$$\frac{1}{d(1-d)} Y_{\alpha\beta}(s)[Y_{\alpha\beta}(s') - Y_{\alpha\beta}(s)] \qquad (3.26)$$

where

$$N = \sum_i \sum_\alpha \sum_\beta (Q_{i\alpha\beta})^2. \qquad (3.27)$$

Thanks to the properties of isotropy of the lattice, N is easily computable. It is

$$N = \frac{bc^4(D-1)}{D}. \qquad (3.28)$$

Following Hénon in [8], we make use of the following quantities:

$$\mu_1 = \frac{D}{2(D-1)bc^4} \sum_s \sum_{s'} A(s \to s') d^{p-1} \qquad (3.29)$$
$$(1-d)^{b-p-1} \sum_\alpha \sum_\beta {Y_{\alpha\beta}}^2(s),$$
$$\mu_5 = \frac{D}{2(D-1)bc^4} \sum_s \sum_{s'} A(s \to s') d^{p-1}$$
$$(1-d)^{b-p-1} \sum_\alpha \sum_\beta P_\alpha(s) P_\beta(s) Y_{\alpha\beta}(s),$$
$$\mu_6 = \frac{D}{2(D-1)bc^4} \sum_s \sum_{s'} A(s \to s') d^{p-1}$$
$$(1-d)^{b-p-1} \sum_\alpha \sum_\beta P_\alpha(s') P_\beta(s') Y_{\alpha\beta}(s'),$$
$$\mu_7 = \frac{D}{2(D-1)bc^4} \sum_s \sum_{s'} A(s \to s') d^{p-1}$$
$$(1-d)^{b-p-1} \sum_\alpha \sum_\beta Y_{\alpha\beta}(s) Y_{\alpha\beta}(s'),$$

where P_α and $Y_{\alpha\beta}$ are given respectively in (3.14) and (3.15).

We can then write ϕ and ψ as

$$\phi = 2(\mu_6 - \mu_5), \qquad (3.30)$$

$$\psi = 2(\mu_7 - \mu_1). \qquad (3.31)$$

μ_1 was already computed in [8]; it is

$$\mu_1 = \frac{1}{2}. \tag{3.32}$$

μ_5 is computed in appendix B; it is

$$\mu_5 = \frac{1}{2}(1-2d). \tag{3.33}$$

Note that μ_5 vanishes for $d = \frac{1}{2}$ which is easily understandable. At this point, the density of particles equals the density of holes and μ_5 is just proportional to the sum over all the states s of the moment of fourth-order $\sum_\alpha \sum_\beta P_\alpha(s)P_\beta(s)Y_{\alpha\beta}(s)$. This quantity is independent of the collision laws and therefore, when the density of particles equals the density of holes, it should be invariant by duality (hole-particle exchange). Through this transformation a state $s = \{s_i;\ i = 1 \ldots b\}$ goes to $\tilde{s} = \{1 - s_i;\ i = 1 \ldots b\}$ and therefore, $P_\alpha(s)$ goes to $-P_\alpha(s)$ and $Y_{\alpha\beta}(s)$ goes to $-Y_{\alpha\beta}(s)$. So, to be invariant by duality, μ_5 must be null. The second-order tensor $P_\alpha(s)P_\beta(s)Y_{\alpha\beta}(s)$ (no summation) will henceforth be denoted PPY.

Using (3.32) and (3.33), we can now write ϕ and ψ as

$$\phi = \frac{(1-2d)}{\mu_5}(\mu_6 - \mu_5), \tag{3.34}$$

$$\psi = 2\mu_7 - 1. \tag{3.35}$$

We can now give a physical interpretation of ϕ in term of the collision laws. We introduce the following notation:

$$\begin{aligned} \Delta(Q) \;=\; & \frac{D}{2(D-1)bc^4} \sum_s \sum_{s'} A(s \to s') d^{p-1} \\ & (1-d)^{b-p-1} \sum_\alpha \sum_\beta [Q_{\alpha\beta}(s) - Q_{\alpha\beta}(s')], \end{aligned} \tag{3.36}$$

and

$$\bar{Q} = \frac{D}{2(D-1)bc^4} \sum_s \sum_{s'} A(s \to s') d^{p-1}(1-d)^{b-p-1} \sum_\alpha \sum_\beta Q_{\alpha\beta}(s) \tag{3.37}$$

where Q represents any s dependent second-order tensor. $\Delta(Q)$ represents the transfer of Q through the collisions and $\bar{Q}$ is a sort of mean value of Q over the whole lattice. Using this notation, we can then write:

$$\phi = (1-2d)\frac{\Delta(PPY)}{\overline{PPY}}. \tag{3.38}$$

We see that ϕ is proportional to the amount of PPY transferred through the collisions. This amount is zero when $d = \frac{1}{2}$ because of the duality invariance at this point; it is also null when the collisions obey semi-detailed balance. Reciprocally, we can infer that if we chose the collision laws so that they transfer totally PPY, we will obtain the same expansion for the mean equilibrium populations as those obtained with the semi-detailed balance assumption. Therefore, the N_i will obey (to second-order) the Fermi-Dirac distribution (see [7]):

$$N_i = \frac{1}{1 + \exp(\mathbf{h} + \mathbf{qc}_i)}. \tag{3.39}$$

Expressing ϕ and ψ in term of μ_6 and μ_7, we can now write $g(\rho)$ as

$$g(\rho) = \frac{2g_{sdb}(\rho)}{1-2d}\Big(\frac{(1-2d)\mu_7 - \mu_6}{2\mu_7 - 1}\Big). \tag{3.40}$$

3.2 Viscosity

Making some small adjustment required by the violation of semi-detailed balance, we find after Hénon in [8] that the viscosity is

$$\nu = \frac{\tau c^2}{(D+2)} \tag{3.41}$$

$$\Big(- \frac{1}{\frac{1}{N}\sum_s \sum_{s'} A(s \to s')\Big(\frac{d}{1-d}\Big)^p (1-d)^b \frac{1}{d(1-d)} Y_{\alpha\beta}(s)[Y_{\alpha\beta}(s') - Y_{\alpha\beta}(s)]} - \frac{1}{2}\Big)$$

which can be written using ψ as

$$\nu = \frac{\tau c^2}{(D+2)}(-\frac{1}{\psi} - \frac{1}{2}). \tag{3.42}$$

This can therefore be written using μ_7:

$$\nu = \frac{\tau c^2}{2(D+2)} \frac{1+2\mu_7}{1-2\mu_7}. \tag{3.43}$$

We can note that, contrary to the case when semi-detailed balance holds, we can not insure that the viscosity should be always positive. We will come back to this point in the conclusion.

3.3 Reynolds number

In order to find the Reynolds number, we need now to define the characteristic length and the characteristic velocity of the flow. A natural unit of length is the lattice constant (distance between adjacent nodes). As in [7], we then have a natural unit of velocity: the speed necessary to travel the lattice constant in a unit time. In these units, we denote the characteristic length and velocity of the flow l_0 and u_0. Moreover, since we operate in an incompressible regime, the velocity u_0 should be small compared to the sound velocity c_s. It is therefore useful to express the Reynolds number in terms of the Mach number

$$M = \frac{u_0}{c_s}. \tag{3.44}$$

Combining all the results above, we obtain

$$R = M l_0 R_\star(\rho_0), \tag{3.45}$$

where

$$R_\star(\rho_0) = c_s \frac{4(D+2)}{\tau c^2} \frac{g_{sdb}(\rho_0)}{1-2d_0} \left(\frac{\mu_6 - (1-2d_0)\mu_7}{2\mu_7 + 1} \right). \tag{3.46}$$

The function g_{sdb} is given by (3.23). μ_6 and μ_7 are given by (3.28).

As in the case with the semi-detailed balance assumption, $R_\star$ contains all the local information.

3.4 Maximization of the Reynolds number

From the equations (3.45) and (3.46), it is clear that the Reynolds number is strongly dependent on the collision laws *via* the two coefficients μ_6 and μ_7. To try to maximize R, it is therefore tempting to write it as

$$R = k(d_0) F(\mu_6, \mu_7), \tag{3.47}$$

where $k(d_0)$ is a function of d_0 only and F is the function of two variables:

$$F(X,Y) = \left(\frac{X - (1-2d_0)Y}{2Y+1} \right), \tag{3.48}$$

and then to study F to find the values of X and Y maximizing this function. There are two problems to this approach: first, this function turns out to have no actual extrema (there are no values of X and Y such that the two first partial derivatives of F are zero). Second, μ_6 and μ_7 may not vary independently with respect to the collision laws. In other words, certain choices of these collision laws may fix *both* μ_6 and μ_7. For example, if we try to maximize F by setting μ_7 equal to the critical value $-\frac{1}{2}$ with a set of collisions verifying:[3]

$$Y_{\alpha\beta}(s) + Y_{\alpha\beta}(s') = 0, \tag{3.49}$$

we will also consequently fix the value of μ_6 to be $-\frac{1}{2}(1-2d)$. Then, the value of F in that case is just $-\frac{1}{2}(1-2d)$.

Such a difficulty arose because (3.49) does not allow any further conditions on the collision laws.[4] To fix independently μ_6 and μ_7, we need therefore to be able to impose some more global conditions, acting on groups of configurations. This means that there will be probably no obvious choice of the collisions and that only some numerical simulations can show in which direction to go.

[3]Note that this condition corresponds to the collision laws maximizing the Reynolds number in the cases where semi-detailed balance holds (see [8]).

[4]Remember that A, the collision matrix, is 2^b by 2^b and that the relations of normalization and conservation of mass and momentum impose already $2^b + 2b$ conditions.

However, it seems reasonable to think that the variations of the Reynolds number are governed mainly by the viscosity. A first approach to the problem of maximization of R could then consist of focusing only on minimizing the viscosity and hoping that this procedure will not minimize $g(\rho)$ at the same time. Such a procedure was applied recently with an optimized collision table computed by Hénon which minimizes the viscosity in the model described in this section. It led to a $R_*^{\max}$ equal to 17.22 obtained for a density of 0.42. This represents already a gain of a factor 2.5 compared with the value $R_*^{\max} = 7.57$ obtained with optimized collision rules obeying the semi-detailed balance. This proves the possibilities offered by our model. However, its implementation using these optimized rules has not yet been done and is still in progress.

4. Model involving a rest particle

In this section, we present the method of computation of the Reynolds number when one rest particle is present. For this purpose, we first need to generalize the model described in the previous section into a model including two different velocities. Then we compute the scaling factor $g(\rho)$ and the viscosity to generalize the expression of the Reynolds number obtained before. At the same time, we show that the results we obtained in the present section are consistent with those obtained in the third section and by other authors who made the semi-detailed balance assumption.

4.1 Formalism

We use the same lattice as in section 3. In addition to the b physical directions, we add a fictitious direction linked to the presence or the absence of a rest particle. Γ, the set of all the configurations, is then formed with $b+1$ bit words. For convenience, we note the different states S:

$$S = (s_*, s), \tag{4.1}$$

where

$$s = \left\{ s_i, i = 1 \ldots b; s_i = 0 \text{ or } 1 \right\}$$

labels the moving particles and

$$s_* = 0 \text{ or } 1$$

labels the rest particle. The collision matrix elements are written:

$$\mathcal{A}(S \to S') = \mathcal{A}\Big((s_*, s) \to (s'_*, s')\Big). \tag{4.2}$$

In the following, it will be useful to make use of the quantities:

$$\mathcal{A}\Big((0, s) \to (0, s')\Big) = A^{--}(s \to s'), \tag{4.3}$$

$$\mathcal{A}\big((1,s)\to(1,s')\big) = A^{++}(s\to s'), \tag{4.4}$$

$$\mathcal{A}\big((1,s)\to(0,s')\big) = A^{+-}(s\to s'), \tag{4.5}$$

$$\mathcal{A}\big((0,s)\to(1,s')\big) = A^{-+}(s\to s'), \tag{4.6}$$

and

$$A^{+}(s\to s') = A^{++}(s\to s') + A^{+-}(s\to s'), \tag{4.7}$$

$$A^{-}(s\to s') = A^{-+}(s\to s') + A^{--}(s\to s'). \tag{4.8}$$

Finally, we adopt the notation

$$\begin{aligned} I &= \star \text{ for the "rest" direction} \\ &= i \text{ for the } b \text{ "moving" directions,} \end{aligned}$$

in order to write, for a set of $b+1$ quantities

$$Q = \big\{Q_I; I = \star, 1\ldots b\big\}$$

and

$$\sum_I \equiv \sum_\star \sum_i .$$

We need now to generalize the concept of G-invariance. An isometry of a lattice is now a transformation which leaves the $\star$ coordinate invariant and is an isometry with respect to the b other coordinates. A set of $b+1$ tensors $T = \big\{T_I, I = \star, 1\ldots b\big\}$ is then G-invariant if any isometry mapping $\mathbf{c}_i$ in $\mathbf{c}_j$ maps T_i in T_j while leaving $T_\star$ unchanged. With the convention $\mathbf{c}_\star = \mathbf{0}$, we can then transform the properties **P1** and **P2** of the section 2 in:

P'1 Any G-invariant set of 1st-order tensors is given by

$$T_{I\alpha} = \lambda_I \mathbf{c}_{I\alpha} \quad \forall\alpha, \tag{4.9}$$

P'2 Any G-invariant set of 2nd-order tensors is given by

$$T_{I\alpha\beta} = \lambda \mathbf{c}_{I\alpha}\mathbf{c}_{I\beta} + \mu_t \delta_{\alpha\beta} \quad \forall(\alpha,\beta), \tag{4.10}$$

where respectively $\mu_t = \mu_r$ or μ_m for the rest or moving directions.

The properties of isotropy (2.6) to (2.9) are then easily generalized by replacing everywhere i by I. So are the properties of $\mathcal{A}$ and the equation governing the mean equilibrium populations which consists now of the system of $b+1$ equations

$$\begin{aligned} \Delta_I(N) &= \sum_s \sum_{s'} (S'_I - S_I)\mathcal{A}(S\to S')\prod_J N_J^{S_J}(1-N_J)^{(1-S_J)} \\ &= 0 \quad \forall I. \end{aligned} \tag{4.11}$$

Once more, the Δ_I are G-invariant which lead us to postulate the following expansion for the N_I:

$$N_i(\rho, \mathbf{u}) = d + \lambda \mathbf{c}_{i\alpha} u_\alpha + \mu Q_{i\alpha\beta} u_\alpha u_\beta + \eta \delta_{\alpha\beta} u_\alpha u_\beta + O(\mathbf{u}^3), \tag{4.12}$$

$$N_*(\rho, \mathbf{u}) = d_* + \eta_* \delta_{\alpha\beta} u_\alpha u_\beta + O(\mathbf{u}^3). \tag{4.13}$$

There are now six parameters to be determined. Because of the definition of ρ and $\mathbf{u}$

$$\rho = \sum_I N_I, \tag{4.14}$$

$$\rho \mathbf{u} = \sum_I N_I \mathbf{c}_I, \tag{4.15}$$

these parameters are constrained to satisfy three conditions:

$$\begin{aligned} bd &+ d_* = \rho, \\ \lambda &= \frac{\rho D}{bc^2}, \\ b\eta &+ \eta_* = 0. \end{aligned} \tag{4.16}$$

The problem becomes now to find the solutions of three systems of $b+1$ equations involving only three independent parameters. Fortunately, we can make use of the symmetries of the lattice and of the properties of A to simplify this problem. By G-invariance, the b Δ_i are equivalent. Because of the properties $P'1$ and $P'2$, we expect them to be zero with at most one condition at the first order and two at the second order. Moreover, the relation of mass conservation implies:

$$\Delta_* + \sum_i \Delta_i = 0. \tag{4.17}$$

We expect therefore the three systems to give at most four independent constraints on the parameters. This is still one too many but we will certainly be able to use the momentum conservation as in the section 3 to solve the difficulty. We now verify these remarks by computations.

We first need to expand the Δ_I up to the second order. Decomposing the summation on I according to the cases involving or not a rest particle, they can be written as

$$\Delta_* = \tag{4.18}$$

$$\sum_s \sum_{s'} (s'_* - s_*) \left[N_* A^+ + (1 - N_*) A^- \right] \prod_j N_j^{s_j} (1 - N_j)^{(1-s_j)},$$

$$\Delta_i = \tag{4.19}$$

$$\sum_s \sum_{s'} (s'_i - s_i) \left[N_* A^+ + (1 - N_*) A^- \right] \prod_j N_j^{s_j} (1 - N_j)^{(1-s_j)},$$

where we have set $A^+(s \to s') = A^+ \ldots$ etc to simplify the notations. Thanks to the results of the section 3, we already know the expansion of $\prod_j N_j^{s_j}(1-N_j)^{(1-s_j)}$. Using (4.13), we can also easily expand the term between braces. Note that it involves no first-order term, which simplifies the final computation.

The zero order gives the $b+1$ equations:

$$\Delta_\star{}^{(0)} = \sum_s \sum_{s'} (s'_\star - s_\star)\left[d_\star A^+ + (1-d_\star)A^-\right]\left(\frac{d}{1-d}\right)^p (1-d)^b, \quad (4.20)$$

$$\Delta_i{}^{(0)} = \quad (4.21)$$

$$\sum_s \sum_{s'} (s'_i - s_i)\left[d_\star A^+ + (1-d_\star)A^-\right]\left(\frac{d}{1-d}\right)^p (1-d)^b \quad \forall i.$$

In fact, as we have seen before, the last b equations are equivalent and they are linked to the first one by mass conservation. This system is then equivalent to only one of its equations, the first one for exemple, which involves only d and $d_\star$. Using the first equation of the system (4.16), these parameters are then determined by the system:

$$bd + d_\star = \rho, \quad (4.22)$$

$$\sum_s \sum_{s'} (s'_\star - s_\star)\left[d_\star A^+ + (1-d_\star)A^-\right]\left(\frac{d}{1-d}\right)^p (1-d)^b = 0.$$

Note that this is not a linear system in d so that it can have more than one solution. This was pointed out by Hénon who constructed an example where there are indeed three solutions [9].

The treatment of the first order is exactly the same as in the section 3 except for the i being replaced by I. This means that this order is identically zero without any further condition; this is good since we had no degree of freedom for the first-order parameter λ.

Expanding the quantity $\prod_j(s)$ and using the expansion of $N_\star$ (4.18), we obtain the system governing the second order:

$$\begin{aligned}\Delta_\star{}^{(2)} &= \sum_s \sum_{s'} \prod_j{}^{(2)}(s)\,(s'_\star - s_\star)\left[d_\star A^+ + (1-d_\star)A^-\right] \quad (4.23)\\ &+ \sum_s \sum_{s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_\star - s_\star)\left[A^+ - A^-\right]\eta_\star \delta_{\alpha\beta} u_\alpha u_\beta,\end{aligned}$$

$$\begin{aligned}\Delta_i{}^{(2)} &= \sum_s \sum_{s'} \prod_j{}^{(2)}(s)\,(s'_i - s_i)\left[d_\star A^+ + (1-d_\star)A^-\right] \quad (4.24)\\ &+ \sum_s \sum_{s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_i - s_i)\left[A^+ - A^-\right]\eta_\star \delta_{\alpha\beta} u_\alpha u_\beta \quad \forall i,\end{aligned}$$

where

$$\prod_j^{(2)}(s) = d^{p-1}(1-d)^{b-p-1} \tag{4.25}$$
$$\Big[\frac{\lambda^2}{d(1-d)}u_\alpha u_\beta \sum_k \sum_{j<k}(s_j-d)(s_k-d)\mathbf{c}_{j\alpha}\mathbf{c}_{k\beta}$$
$$+\mu u_\alpha u_\beta \sum_j (s_j-d)Q_{j\alpha\beta} + \eta u_\alpha u_\alpha \sum_j (s_j-d)\delta_{\alpha\beta}\Big].$$

Thanks to the G-invariance, we can write $\Delta_i = (\psi \mathbf{c}_{i\alpha}\mathbf{c}_{i\beta} + \phi_m \delta_{\alpha\beta})u_\alpha u_\beta$ and $\Delta_\star = \phi_r \delta_{\alpha\beta} u_\alpha u_\beta$. Moreover, mass conservation relates ψ, ϕ_m, and ϕ_r. The system of $b+1$ equations is then equivalent to only two equations (which is the number of free parameters):

$$\begin{aligned} \Delta_\star &= 0, \\ \Delta_i &= 0, \text{ for } i=1 \text{ for example.} \end{aligned} \tag{4.26}$$

Adding to them the the last equation of (4.16), we obtain the system of three equations linear in μ, η, and $\eta_\star$:

$$\begin{aligned} A^\star \eta_\star &+ B^\star \eta + C^\star \mu = D^\star, \\ A^i \eta_\star &+ B^i \eta + C^i \mu = D^i, \\ \eta_\star &+ b\eta = 0. \end{aligned} \tag{4.27}$$

The expression for the coefficients is given in the next page. To make the most of the G-invariance, we divide the first equation by b and we add the second one; the final equation involves now some quantities we can write schematically as: $X^i + \frac{1}{b}X^\star$. Thanks to the G-invariance and to the mass conservation, these quantities can be written as:

$$X^i + \frac{1}{b}X_\star = \psi_X Q_{i\alpha\beta}u_\alpha u_\beta. \tag{4.28}$$

Because of the mass conservation, A^I and B^I, which are composed of zero-order tensors, must satisfy $A^i = -\frac{1}{b}A^\star$ and a similar equation for B. This means that $\psi_A = \psi_B = 0$. The system becomes

$$\begin{aligned} A^\star \eta_\star + B^\star \eta + C^\star \mu &= D^\star, \\ \psi_C \mu &= \psi_D, \\ \eta_\star + b\eta &= 0. \end{aligned} \tag{4.29}$$

Expressing the D^I in term of the C^I, F^I, and B^I and solving the system, we obtain finally μ, η and $\eta_\star$ which are

$$\begin{aligned} \mu &= d\Big(\frac{\rho}{bd}\Big)^2 G(\rho)\left[1 - \frac{1}{(1-2d)}\frac{\psi_F}{\psi_C}\right], \\ \eta &= d\Big(\frac{\rho}{bd}\Big)^2 \frac{G(\rho)}{B^\star - bA^\star}\left[\frac{1}{(1-2d)}\Big(C^\star\frac{\psi_F}{\psi_C} - F^\star\Big) + \frac{c^2}{D}B^\star\right], \\ \eta^\star &= -bd\Big(\frac{\rho}{bd}\Big)^2 \frac{G(\rho)}{B^\star - bA^\star}\left[\frac{1}{(1-2d)}\Big(C^\star\frac{\psi_F}{\psi_C} - F^\star\Big) + \frac{c^2}{D}B^\star\right], \end{aligned} \tag{4.30}$$

where

$$G(\rho) = \frac{D^2}{2c^4}\frac{1-2d}{1-d}. \tag{4.31}$$

We can note that the μ obtained above is very similar to the one obtained in the monospeed case. Thanks to (4.28), we can indeed express ψ_C and ψ_F in terms of the collision laws as

$$\begin{aligned} N\psi_F &= \sum_s \sum_{s'} \left[d_* A^+ + (1-d_*)A^-\right] \\ &\left(\frac{d}{1-d}\right)^p (1-d)^b \frac{1}{d(1-d)} P_\alpha(s) P_\beta(s) [Y_{\alpha\beta}(s') - Y_{\alpha\beta}(s)], \end{aligned} \tag{4.32}$$

$$\begin{aligned} N\psi_C &= \sum_s \sum_{s'} \left[d_* A^+ + (1-d_*)A^-\right] \\ &\left(\frac{d}{1-d}\right)^p (1-d)^b \frac{1}{d(1-d)} Y_{\alpha\beta}(s) [Y_{\alpha\beta}(s') - Y_{\alpha\beta}(s)], \end{aligned} \tag{4.33}$$

where

$$N = \frac{bc^4(D-1)}{D}. \tag{4.34}$$

In this case, the collision matrix $A(s \to s')$ of the monospeed model has been replaced by an effective collision matrix $\left[d_* A^+ + (1-d_*)A^-\right]$. From its expression and the properties of the A^+, it can be shown that this matrix satisfies all the properties of A: it is positive, it satisfies the relation of normalization, it conserves mass and momentum and it is invariant under any isometry in G. But it depends now on the density of the rest particles which means that we have a new degree of freedom to vary. This will be important in the maximization of the Reynolds number.

Expression of the coefficients

$$A^\star = \sum_{s,s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_\star - s_\star) \left[A^+ - A^-\right] \delta_{\alpha\beta} u_\alpha u_\beta$$

$$A^i = \sum_{s,s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_i - s_i) \left[A^+ - A^-\right] \delta_{\alpha\beta} u_\alpha u_\beta$$

$$B^\star = \sum_{s,s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_\star - s_\star) \left[d_* A^+ + (1-d_*)A^-\right] \sum_j s_j \frac{\delta_{\alpha\beta} u_\alpha u_\beta}{d(1-d)}$$

$$B^i = \sum_{s,s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_i - s_i)$$

$$\Big[d_\star A^+ + (1-d_\star)A^-\Big]\sum_j s_j \frac{\delta_{\alpha\beta}u_\alpha u_\beta}{d(1-d)}$$

$$C^\star = \sum_{s,s'}(s'_\star - s_\star)\Big[d_\star A^+ + (1-d_\star)A^-\Big]\Big(\frac{d}{1-d}\Big)^p(1-d)^b$$

$$\frac{1}{d(1-d)}\sum_j s_j Q_{j\alpha\beta}u_\alpha u_\beta$$

$$C^i = \sum_{s,s'}(s'_i - s_i)\Big[d_\star A^+ + (1-d_\star)A^-\Big]\Big(\frac{d}{1-d}\Big)^p(1-d)^b$$

$$\frac{1}{d(1-d)}\sum_j s_j Q_{j\alpha\beta}u_\alpha u_\beta$$

$$D^\star = -\sum_{s,s'}(s'_\star - s_\star)\Big[d_\star A^+ + (1-d_\star)A^-\Big]\Big(\frac{d}{1-d}\Big)^p(1-d)^b$$

$$\frac{\lambda^2}{d^2(1-d)^2}\sum_{j<k}(s_j-d)(s_k-d)\mathbf{c}_{j\alpha}\mathbf{c}_{k\beta}u_\alpha u_\beta$$

$$D^i = -\sum_{s,s'}(s'_i - s_i)\Big[d_\star A^+ + (1-d_\star)A^-\Big]\Big(\frac{d}{1-d}\Big)^p(1-d)^b$$

$$\frac{\lambda^2}{d^2(1-d)^2}\sum_{j<k}(s_j-d)(s_k-d)\mathbf{c}_{j\alpha}\mathbf{c}_{k\beta}u_\alpha u_\beta$$

$$F^\star = \sum_{s,s'}(s'_\star - s_\star)\Big[d_\star A^+ + (1-d_\star)A^-\Big]\Big(\frac{d}{1-d}\Big)^p(1-d)^b$$

$$\frac{1}{d(1-d)}\sum_{j,k} s_j s_k \mathbf{c}_{j\alpha}\mathbf{c}_{k\beta}u_\alpha u_\beta$$

$$F^i = \sum_{s,s'}(s'_i - s_i)\Big[d_\star A^+ + (1-d_\star)A^-\Big]\Big(\frac{d}{1-d}\Big)^p(1-d)^b$$

$$\frac{1}{d(1-d)}\sum_{j,k} s_j s_k \mathbf{c}_{j\alpha}\mathbf{c}_{k\beta}u_\alpha u_\beta$$

We show now that the results obtained above are consistent with previous results. We write explicitly the expansion of the N_I as

$$N_i = d\left[1 + \frac{D}{c^2}\frac{\rho}{\rho_m} + \Big(\frac{\rho}{\rho_m}\Big)^2 G(\rho)\right. \tag{4.35}$$

$$\left.\Big([1-\omega_\star]Q_{i\alpha\beta} + \frac{c^2}{D}\frac{E_\star}{E_\star+1}[1+\omega_{\star\star}]\delta_{\alpha\beta}\Big)u_\alpha u_\beta\right],$$

$$N_\star = d_\star\left[1 - \Big(\frac{\rho}{\rho_m}\Big)^2 G(\rho)\frac{c^2}{D}\Big(\frac{\rho}{\rho_r} - 1\Big)\frac{E_\star}{E_\star+1}[1+\omega_{\star\star}]\delta_{\alpha\beta}u_\alpha u_\beta\right].$$

where

$\rho_m = bd$ density of the moving particles,

$\rho_r = d_\star$ density of the rest particle,

$$\omega_\star = \frac{1}{(1-2d)}\frac{\psi_F}{\psi_C}, \tag{4.36}$$

$$\omega_{\star\star} = \frac{D}{c^2}\frac{1}{B^\star(1-2d)}\left[C^\star\frac{\psi_F}{\psi_C} - F^\star\right], \tag{4.37}$$

$$E_\star = -\frac{1}{b}\frac{B^\star}{A^\star}. \tag{4.38}$$

With this expansion, we recover two limiting cases: the case with the semi-detailed assumption and the monospeed case.

If we make the *semi-detailed assumption*, all the configurations must be equivalent which implies

$$\begin{aligned} d_\star &= d, \\ A^\star &+ B^\star = 0, \\ \psi_F &= F^\star = 0. \end{aligned} \tag{4.39}$$

Using these results, we get therefore

$$\omega_\star = \omega_{\star\star} = 0,$$

$$E_\star = \frac{1}{b} \equiv E,$$

and then,

$$\begin{aligned} N_i &= d\left[1 + \frac{D}{c^2}\frac{\rho}{\rho_m}\mathbf{c}_i.\mathbf{u} + \left(\frac{\rho}{\rho_m}\right)^2 G(\rho)\right. \\ &\quad \left.\left(Q_{i\alpha\beta} + \frac{c^2}{D}\frac{E}{E+1}\delta_{\alpha\beta}\right)u_\alpha u_\beta\right] \\ N_\star &= d_\star\left[1 - \left(\frac{\rho}{\rho_m}\right)^2 G(\rho)\frac{c^2}{D}\frac{1}{E+1}\delta_{\alpha\beta}u_\alpha u_\beta\right] \end{aligned} \tag{4.40}$$

This is indeed the expansion obtained by D'Humières et al. [10].

The *monospeed case* corresponds to $d_\star = 0$ and to

$$\begin{aligned} A^{+-}(s \to s') &= A^{++}(s \to s') = A^{-+}(s \to s') = 0, \\ A^{--}(s \to s') &= A(s \to s'). \end{aligned}$$

This implies:

$$\begin{aligned} \frac{E^\star}{E^\star + 1} &= 0, \\ \left[d_\star A^+ + (1-d_\star)A^-\right] &= A(s \to s'), \end{aligned} \tag{4.41}$$

which enables us to recover the monospeed case described in section 3.

We need now to compute the viscosity in this model. This is done in a way very similar to the monospeed model [8] in appendix C. The result is, as expected, a generalization of the formula (3.42)

$$\begin{aligned} \nu &= \frac{\tau c^2}{D+2}(\chi - \frac{1}{2}) \\ &= \frac{\tau c^2}{D+2}\left[-\frac{1}{\psi_C} - \frac{1}{2}\right]. \end{aligned} \tag{4.42}$$

Note that the recipes given in (4.41) ensure that this expression is indeed consistent with the monospeed viscosity.

We can now use the similarity between the monospeed case and the case involving a rest particle to give an expression of the Reynolds number in term of quantities dependent on the collision laws. It is, using the same notation as in section 3:

$$R = M l_0 R_\star(\rho_0), \tag{4.43}$$

where, in this case,

$$R_\star(\rho_0) = c_s\Big(\frac{\rho_0}{\rho_m}\Big)\frac{4(D+2)}{\tau c^2}\frac{g_{sdb}(\rho_0)}{1-2d_0}\Big(\frac{\mu_6^{eff} - (1-2d)\mu_7^{eff}}{2\mu_7^{eff}+1}\Big). \tag{4.44}$$

The subscript eff in μ_7 and μ_6 means that they have been computed with the effective collision matrix $\left[d_\star A^+ + (1-d_\star)A^-\right]$ instead of $A(s \to s')$.

To maximize R, we need then to do the same thing as in the monospeed case but we have now another degree of freedom which is the density of the rest particles. It appears in two different ways: through the ratio $\frac{\rho}{\rho_m}$ and in the effective collision matrix. We can therefore expect the maximization of the Reynolds number to be more efficient. This is not surprising since it is just a consequence of the fact that more collisions are allowed when a rest particle is present.

5. Conclusion

We have shown that, within the (lattice) Boltzmann approximation, it is possible to extend the general theory developed by Frisch et al. [7] to cases involving violation of semi-detailed balance and that it seems a very promising way of increasing the Reynolds number. Moreover, whereas it has been shown by Hénon in [8] that the viscosity is always positive when the semi-detailed balance is assumed, this need not be true anymore with the violation of semi-detailed balance. It is not clear whether our model can already achieve arbitrarily small or even negative viscosities (which would imply the existence of a phase transition) and only numerical simulations and computations will enable us to clarify this point. However, as we have seen, it is very difficult to foresee how to chose the collision rules to reach the smallest possible viscosity. This may require a somewhat empirical exploration which is still in progress.

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Appendix A.

We give here the detailed computation of the expansion of the Δ_i given in the section 3.

The first step is to expand up to the second order the quantity:

$$\prod_j(s) \equiv \prod_j N_j^{s_j}(1-N_j)^{(1-s_j)}. \tag{A.1}$$

Using $s_i = 0$ or 1 and (3.4), we have

$$\begin{aligned} N_j^{s_j}(1-N_j)^{(1-s_j)} &= s_jN_j + (1-s_j)(1-N_j) \\ &= s_jd + (1-s_j)(1-d) + \lambda(2s_j-1)\mathbf{c}_j\mathbf{u} \\ &\quad +\mu(2s_j-1)Q_{i\alpha\beta}u_\alpha u_\beta + O(\mathbf{u}^3) \\ &= d^{s_j}(1-d)^{(1-s_j)} \\ &\quad \Big[1+\frac{(s_j-d)}{d(1-d)}(\lambda\mathbf{c}_j\mathbf{u}+\mu Q_{i\alpha\beta}u_\alpha u_\beta)\Big] \\ &\quad +O(\mathbf{u}^3) \end{aligned} \tag{A.2}$$

We can now put this result in (A.1) to get the three leading orders in the expansion of $\prod_j(s)$ which are respectively:

0-order term

$$\prod_j^{(0)}(s) = \Big(\frac{d}{1-d}\Big)^p(1-d)^b \tag{A.3}$$

with $p = \sum_i s_i$.

1st-order term

$$\prod_j^{(1)}(s) = \Big(\frac{d}{1-d}\Big)^p(1-d)^b\frac{\lambda}{d(1-d)}\mathbf{u}.\Big(\sum_j(s_j-d)\mathbf{c}_j\Big). \tag{A.4}$$

2nd-order term

$$\begin{aligned}\prod_j^{(2)}(s) &= \Big(\frac{d}{1-d}\Big)^p (1-d)^b \Big[\frac{\lambda^2}{d^2(1-d)^2} u_\alpha u_\beta \\ &\quad \sum_k \sum_{j<k} (s_j - d)(s_k - d)\mathbf{c}_{j\alpha}\mathbf{c}_{k\beta} \\ &+ \frac{\mu}{d(1-d)} u_\alpha u_\beta \sum_j (s_j - d) Q_{j\alpha\beta}.\Big]\end{aligned} \tag{A.5}$$

Thanks to the relations

$$\sum_i \mathbf{c}_i = 0, \tag{A.6}$$

$$\sum_i Q_{i\alpha\beta} = 0, \tag{A.7}$$

the terms independent of s can be eliminated in (A.4) and in (A.5).

Finally, we make use of the relation

$$\begin{aligned}&\sum_k \sum_{j<k} (s_j - d)(s_k - d)\mathbf{c}_{j\alpha}\mathbf{c}_{k\beta} \\ &= \frac{1}{2}\sum_k \sum_j (s_j - d)(s_k - d)\mathbf{c}_{j\alpha}\mathbf{c}_{k\beta} - \frac{1}{2}\sum_k (s_k - d)^2 \mathbf{c}_{k\alpha}\mathbf{c}_{k\beta} \\ &= \frac{1}{2}\sum_k \sum_j s_j s_k \mathbf{c}_{j\alpha}\mathbf{c}_{k\beta} \\ &\quad -\frac{1}{2}\sum_k (d^2 + (1-2d)s_k)(Q_{k\alpha\beta} + \frac{c^2}{D}\delta_{\alpha\beta}),\end{aligned} \tag{A.8}$$

to write the 2nd-order term as:

$$\begin{aligned}\prod_j^{(2)}(s) &= \Big(\frac{d}{1-d}\Big)^p (1-d)^b \\ &\quad \Big[\frac{\lambda^2}{2d^2(1-d)^2} u_\alpha u_\beta \sum_k \Big((2d-1)s_k - d^2\Big)\frac{c^2}{D}\delta_{\alpha\beta} \\ &+ \frac{\lambda^2}{2d^2(1-d)^2} u_\alpha u_\beta \sum_k \sum_j s_j s_k \mathbf{c}_{j\alpha}\mathbf{c}_{k\beta} \\ &+ \Big(\frac{\mu}{d(1-d)} + \frac{(2d-1)}{2d^2(1-d)^2}\lambda^2\Big) u_\alpha u_\beta \sum_k s_k Q_{k\alpha\beta}\Big].\end{aligned} \tag{A.9}$$

In order to make the notation more compact, we introduce the following quantities [8]:

$$P_\alpha(s) = \sum_i s_i \mathbf{c}_{i\alpha}, \tag{A.10}$$

$$Y_{\alpha\beta}(s) = \sum_i s_i Q_{i\alpha\beta}, \tag{A.11}$$

which represent respectively the first- and second-order moment in the state s.

Substituting these quantities in (A.3), (A.4), and (9) and coming back to the definition of the Δ_i, we obtain the expansion in (3.8) of (3.32) of section 3.

Appendix B.

In this section, we compute the value of the coefficient μ_5 defined in (3.32) of section 3.

Using (3.28), it can be written as

$$\begin{aligned}\mu_5 &= \frac{D}{2(D-1)bc^4} \\ &\quad \sum_s \sum_{s'} A(s \to s') d^{p-1}(1-d)^{b-p-1} \sum_\alpha \sum_\beta P_\alpha P_\beta Y_{\alpha\beta} \\ &= \frac{1}{d(1-d)} \frac{D}{(D-1)2bc^4} \\ &\quad \sum_{\alpha\beta} \sum_{i,j,k} Q_{i\alpha\beta} \mathbf{c}_{j\alpha} \mathbf{c}_{k\beta} \sum_s \Big(\frac{d}{1-d}\Big)^p (1-d)^b s_i s_j s_k \sum_{s'} A(s \to s').\end{aligned} \tag{B.1}$$

The sum on s' is one because of (2.12). The sum on s can be written as

$$\sum_s s_i s_j s_k \Big(\frac{d}{1-d}\Big)^p (1-d)^b = \sum_{s_1=0}^{1} \cdots \sum_{s_n=0}^{1} s_i s_j s_k \Big(\frac{d}{1-d}\Big)^p (1-d)^b,$$

or, using $p = \sum_i s_i$,

$$\Big(\prod_{m \neq i,j,k} \sum_{s_m=0}^{1} d^{s_m}(1-d)^{1-s_m} \Big) \sum_{s_i=0}^{1} s_i d^{s_i}(1-d)^{1-s_i} \tag{B.2}$$

$$\sum_{s_j=0}^{1} s_j d^{s_j}(1-d)^{1-s_j} \sum_{s_k=0}^{1} s_k d^{s_k}(1-d)^{1-s_k}.$$

Three cases are to be distinguished:

if $i = j = k$, (B.2) is d (the sum on s_i is d, the sum on s_j for $j \neq i$ is 1).

if $i = j \neq k$ or $j = k \neq i$ or $k = i \neq j$, (B.2) is d^2.

if $i \neq j$ and $j \neq k$ and $k \neq i$, (B.2) is d^3.

The sum on i, j, k needs then to be decomposed in three kinds of terms:

$$S_2 \equiv \sum_i \sum_{j=i} \sum_{k=j}, \tag{B.3}$$

$$K(i) \equiv \sum_i \sum_{j\neq i} \sum_{k=j}, \tag{B.4}$$

$$L \equiv \sum_i \sum_{j\neq i} \sum_{k\neq i,j} . \tag{B.5}$$

Therefore, μ_5 becomes:

$$\begin{aligned} \mu_5 &= \frac{1}{2Nd(1-d)}\Big[dS_2 + d^2\big(K(i)+K(j)+K(k)\big) + d^3 L\Big] \\ &\quad \sum_{\alpha,\beta} Q_{i\alpha\beta}\mathbf{c}_{j\alpha}\mathbf{c}_{k\beta}. \end{aligned} \tag{B.6}$$

For convenience, we decompose L and the K thanks to S_2 and the quantities

$$S_1(i) \equiv \sum_i \sum_j \sum_{k=j}, \tag{B.7}$$

$$S_0 \equiv \sum_i \sum_j \sum_k . \tag{B.8}$$

This decomposition is

$$\begin{aligned} K(i) &= S_1(i) - S_2, \\ L &= S_0 + 2S_2 - \Big(S_1(i) + S_1(j) + S_1(k)\Big). \end{aligned} \tag{B.9}$$

The term between braces of (B.6) is then

$$\begin{aligned} d^3 S_0 &+ d^2(1-d)\big(S_1(i)+S_1(j)+S_1(k)\big) \\ &+ d(1-d)(1-2d)S_2. \end{aligned} \tag{B.10}$$

We can now use the relations

$$\sum_i Q_{i\alpha\beta} = 0 \text{ and } \sum_i \mathbf{c}_{i\alpha} = 0,$$

to set to zero the S_1 and S_0 terms. We obtain finally

$$\mu_5 = (1-2d)\frac{1}{2N} S_2 Q_{i\alpha\beta}\mathbf{c}_{j\alpha}\mathbf{c}_{k\beta}. \tag{B.11}$$

Coming back to the definition of S_2 and using (2.9), we can easily compute this quantity. The final result is

$$S_2 Q_{i\alpha\beta}\mathbf{c}_{j\alpha}\mathbf{c}_{k\beta} = N,$$

and therefore

$$\mu_5 = \frac{1}{2}(1-2d).$$

Appendix C.

In this appendix, we compute the viscosity in the case where a rest particle is present. To do that, we follow the method used in [8] which consists of five steps:

1. Define a steady, homogeneous state, with linear shear velocity field.
2. Write equations for collisions.
3. Write equations for propagation.
4. Solve these equations to determine the structure of the steady state.
5. Compute the momentum flux and the viscosity.

Appendix C.1 Steady state

We choose the steady state which is obtained by perturbation of the equilibrium state at null speed. The mean population of the node x is then given by

$$N_I = d_I + \zeta_I(x) \quad \text{with } \zeta_I \ll 1. \tag{C.1}$$

Because of the definition of the density, ζ satisfies:

$$\sum_I \zeta_I = 0. \tag{C.2}$$

We assume that the mean velocity, defined as

$$\rho \mathbf{u} = \sum_I c_I N_I, \tag{C.3}$$

takes the form:

$$u_\alpha = \sum_\beta T_{\alpha\beta} x_\beta \quad (\alpha = 1 \ldots D). \tag{C.4}$$

The $T_{\alpha\beta}$ are the components of the velocity gradient:

$$T_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta}. \tag{C.5}$$

It is then natural to suppose that the ζ_I are also a linear function of the x_α (see [8]):

$$\zeta_I = k_{I\beta} x_\beta + \epsilon_I. \tag{C.6}$$

We insert this expression in (C.3) and (C.2) to get the relations

$$\begin{aligned}
\sum_I \epsilon_I &= 0, \qquad (C.7)\\
\sum_I \epsilon_I \mathbf{c}_{I\alpha} &= 0 \quad \forall\alpha,\\
\sum_I k_{I\beta} &= 0 \quad \forall\beta,\\
\sum_I \mathbf{c}_{I\alpha} k_{I\beta} &= \rho T_{\alpha\beta} \quad \forall\alpha,\beta.
\end{aligned}$$

We get therefore $(D+1)^2$ equations. Since we have $(D+1)(b+1)$ unknowns in the problem, we need to find some more equations to solve it. This is done requiring that the populations N_I must be invariant under the evolution operator, that is, under the operations collision plus propagation.

Appendix C.2 Collision

As in the section 4, the collision equation is

$$\begin{aligned}
N'_I - N_I &= \sum_s \sum_{s'} (S'_I - S_I)\mathcal{A}(S \to S') \prod_J N_J{}^{S_J}(1-N_J)^{(1-S_J)} \qquad (C.8)\\
&= 0 \quad \forall I.
\end{aligned}$$

We put in this equation the expression of the N_I and we use the fact that they are the solution of (C.8) at zero speed and the G-invariance to eliminate a constant term. We get to first order:

$$\begin{aligned}
\zeta'_\star - \zeta_\star &= \sum_s \sum_{s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_\star - s_\star)\left[A^+ - A^-\right]\zeta_\star \qquad (C.9)\\
&+ \sum_s \sum_{s'} (s'_\star - s_\star)\left[d_\star A^+ + (1-d_\star)A^-\right]\\
&\quad d^{p-1}(1-d)^{b-p-1}\sum_j (s_j - d)\zeta_j,
\end{aligned}$$

$$\begin{aligned}
\zeta'_i - \zeta_i &= \sum_s \sum_{s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_i - s_i)\left[A^+ - A^-\right]\zeta_\star \qquad (C.10)\\
&+ \sum_s \sum_{s'} (s'_i - s_i)\left[d_\star A^+ + (1-d_\star)A^-\right]\\
&\quad d^{p-1}(1-d)^{b-p-1}\sum_j (s_j - d)\zeta_j.
\end{aligned}$$

Appendix C.3 Propagation

For a steady state, the probability of arrival of a particle at a node equals the probability of leaving the previous node:

$$N_i(x + \tau \mathbf{c}_i) = N'_i(x). \qquad (C.11)$$

We substitute the expression of the N_I in this equation to get:

$$\zeta_i(x) - \zeta'_i(x) = -\tau \sum_\alpha k_{i\alpha} \mathbf{c}_{i\alpha}. \tag{C.12}$$

Since the rest particle does not propagate, we must have

$$N_\star(x) = N'_\star(x). \tag{C.13}$$

This means that the $\zeta_\star$ must obey the relation:

$$\zeta_\star(x) - \zeta'_\star(x) = 0. \tag{C.14}$$

From the mass conservation, we deduce that the ζ_i verify:

$$\sum_i [\zeta_i(x) - \zeta'_i(x)] = 0. \tag{C.15}$$

Appendix C.4 Computation of the solutions

We now combine the collision equations with the propagation equations to get $b+1$ complementary equations, namely

$$\begin{aligned} 0 &= -\sum_s \sum_{s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_\star - s_\star) \left[A^+ - A^-\right] \zeta_\star \\ &+ \sum_s \sum_{s'} (s'_\star - s_\star) \left[d_\star A^+ + (1-d_\star) A^-\right] \\ &\quad d^{p-1} (1-d)^{b-p-1} \sum_j (s_j - d) \zeta_j, \end{aligned} \tag{C.16}$$

$$\begin{aligned} \tau k_{i\alpha} \mathbf{c}_{i\alpha} &= \sum_s \sum_{s'} \left(\frac{d}{1-d}\right)^p (1-d)^b (s'_i - s_i) \left[A^+ - A^-\right] \zeta_\star \\ &+ \sum_s \sum_{s'} (s'_i - s_i) \left[d_\star A^+ + (1-d_\star) A^-\right] \\ &\quad d^{p-1} (1-d)^{b-p-1} \sum_j (s_j - d) \zeta_j. \end{aligned} \tag{C.17}$$

It would be hard to solve this problem without guessing the form of the solutions. Fortunately, we can use the results in the monospeed case to search for solutions of the form

$$\mathrm{k}_{I\alpha} = K_\alpha + L_{\alpha\beta} \mathbf{c}_{I\beta}. \tag{C.18}$$

We replace this definition (C.18) in (7) to get:

$$\begin{aligned} K_\alpha &= 0 \quad \forall \alpha, \\ L_{\alpha\beta} &= \frac{\rho D}{bc^2} T_{\alpha\beta} = d \frac{\rho}{\rho_m} \frac{D}{c^2} T_{\alpha\beta} \quad \forall \alpha, \beta, \end{aligned} \tag{C.19}$$

where $\rho_m = bd$. Note that if we combine now the expression obtained for the $L_{\alpha\beta}$ and the relation (C.15), we obtain

$$\sum_{\alpha} T_{\alpha\alpha} = 0$$

which is not surprising since the flow is incompressible.

We need now to check that the solutions we have guessed are solutions of the system (C.17) and (C.16). We plug (C.18) into it and we separate the constant terms from the terms which are linear in x. Thanks to the G-invariance and to the momentum conservation, it can be shown that the terms linear in x, which involve only $L_{\alpha\beta}$, are identically null. The constant terms give $b+1$ equations for the ϵ_I :

$$\begin{aligned} 0 &= -\sum_s \sum_{s'} \Big(\frac{d}{1-d}\Big)^p (1-d)^b (s'_\star - s_\star)\left[A^+ - A^-\right]\epsilon_\star \\ &+ \sum_s \sum_{s'} (s'_\star - s_\star)\left[d_\star A^+ + (1-d_\star)A^-\right] \\ &\quad d^{p-1}(1-d)^{b-p-1}\sum_j (s_j - d)\epsilon_j, \end{aligned} \tag{C.20}$$

$$\begin{aligned} \frac{\tau D}{c^2} d \frac{\rho}{\rho_m} T_{\alpha\beta}\mathbf{c}_{i\alpha}\mathbf{c}_{i\beta} &= \sum_s \sum_{s'} \Big(\frac{d}{1-d}\Big)^p (1-d)^b (s'_i - s_i) \\ &\quad \left[A^+ - A^-\right]\epsilon_\star \\ &+ \sum_s \sum_{s'} (s'_i - s_i)\left[d_\star A^+ + (1-d_\star)A^-\right] \\ &\quad d^{p-1}(1-d)^{b-p-1}\sum_j (s_j - d)\epsilon_j. \end{aligned} \tag{C.21}$$

As in the monospeed case, the form of this system suggests using a "theory of linear response" that is to write that the perturbation, represented by the ϵ, is proportional to the excitation term:

$$\begin{aligned} \epsilon_i &= -\frac{\tau D}{c^2} d \frac{\rho}{\rho_m} \chi T_{\alpha\beta} Q_{i\alpha\beta}, \\ \epsilon_\star &= 0. \end{aligned} \tag{C.22}$$

Note that these expressions are consistent with the system (C.7).

We put these relations in the system (C.21) and use the notations used for the computation of the equilibrium population; we obtain

$$0 = \chi C^{*}{}_{\alpha\beta} T_{\alpha\beta}, \tag{C.23}$$

$$-T_{\alpha\beta}\mathbf{c}_{i\alpha}\mathbf{c}_{i\beta} = \chi C^{i}{}_{\alpha\beta} T_{\alpha\beta}. \tag{C.24}$$

Using the G-invariance, we can write

$$\begin{aligned} C^i_{\alpha\beta} &= \psi Q_{i\alpha\beta} + \phi\delta_{\alpha\beta}, \\ C^\star_{\alpha\beta} &= \phi_\star \delta_{\alpha\beta}. \end{aligned} \tag{C.25}$$

Using the mass conservation, we get $b\phi + \phi_\star = 0$. This implies that ψ is in fact ψ_C with the notations of the section 4. If we use now the incompressibility relation, we find that the system is equivalent to only one equation:

$$\chi = -\frac{1}{\psi_C}. \tag{C.26}$$

We can now write the expansion of the N_I as

$$\begin{aligned} N_i &= d\left[1 + \frac{D}{c^2}\frac{\rho}{\rho_m}\left(\mathbf{c}_{i\alpha}x_\beta - \tau\chi Q_{i\alpha\beta}\right)T_{\alpha\beta}\right] \\ N_\star &= d_\star. \end{aligned} \tag{C.27}$$

Since the rest particle does not propagate, it is normal that its expansion does not differ from the steady state at null speed.

Appendix C.5 Computation of the viscosity

As in [8], the components of the tensor momentum flux are

$$F_{\gamma\delta} = \sum_i \mathbf{c}_{i\gamma}\mathbf{c}_{i\delta}\left[N_i(x) + \frac{1}{2}\tau\mathbf{k}_i.\mathbf{c}_i\right]. \tag{C.28}$$

We substitute the expansion of the N_i and the $\mathbf{k}_i$ given by (C.26) and (C.18) and we use the isotropy relations (2.6) to (2.9) to get

$$\begin{aligned} F_{\gamma\gamma} &= \frac{\rho_m c^2}{D} - \frac{2\rho\tau c^2}{D+2}T_{\gamma\gamma} \\ F_{\gamma\delta} &= -\frac{\rho\tau c^2}{D+2}(\chi - \frac{1}{2})(T_{\gamma\delta} - T_{\delta\gamma}) \qquad (\gamma \neq \delta). \end{aligned} \tag{C.29}$$

The second equation proves that the shear viscosity is

$$\begin{aligned} \eta &= \frac{\rho\tau c^2}{D+2}(\chi - \frac{1}{2}) \\ &= \frac{\rho\tau c^2}{D+2}\left[-\frac{1}{\psi_C} - \frac{1}{2}\right]. \end{aligned} \tag{C.30}$$

Since the kinetic viscosity is $\frac{\eta}{\rho}$, we find the result given in section 4.

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