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Global Dynamics in Neural Networks

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Abstract. The Hedlund-Richardsod Theorem states that a global mapping from configuration space to itself can be realized by a Euclidean cellular automaton if and only if it takes the quiescent configuration to itself, commutes with shifts, and is continuous in the product topology. An analogous theorem characterizing the realizability of self-mappings of finite or infinite configuration space via neural networks is established. It follows that, under natural hypotheses, a uniform limit of global dynamics is a global dynamics. We also give sufficient conditions for the global dynamics of a neural network to be realized by a cellular automaton.

1. Introduction

Deterministic cellular automata [7,8] can be viewed as discrete dynamical systems with local dynamics defined by a finite-state machine. This local dynamics induces a global dynamics, i.e., a self-mapping of configuration (i.e., global-state) space $\mathcal{C}$. Since not all self-maps of $\mathcal{C}$ arise this way, it is natural to ask what self-mappings of the configuration space can be realized as the global dynamics associated with some cellular automaton. This question is answered satisfactorily by Richardson's Theorem [5]. (A one-dimensional version of this result follows from earlier work by Hedlund [3].) Since the set of local states of a cellular automaton is finite and the number of cells countable, the configuration space in its natural topology is homeomorphic to the Cantor set. Synchronous, discrete neural networks (or connectionist models) can also be thought of as discrete dynamical systems with their local dynamics defined by local, non-uniform activation functions together with a local non-uniform input function consisting of the weighted sum of input values. Hence, local dynamics give rise to global dynamics on both the net-input space and the activation space of the network. It is natural to ask again what global properties of a self-map of the net-input space arise from such local non-uniform dynamics.

We provide necessary and sufficient conditions to characterize such self-maps for any infinite ring R with unity and family of activation functions

$f_i : \mathcal{R} \to \mathcal{R}$ satisfying $f_i(0) = 0$ (in order to avoid spontaneous generation of activation) and $f_i(1)$ a unit in $\mathcal{R}$ (to avoid trivial networks). Recall that the *support* of a configuration $x : \mathcal{C} \to \mathcal{C}$ is the set of vertices having non-zero value. For each cell k, define the k^{th} *pixel* to be the configuration that is non-zero only at k by $e_i^k = 1$ if $k = i$, and 0 otherwise.

Theorem 1. *A self map $T : \mathcal{C} \to \mathcal{C}$ is realizable by a neural network with activation functions $\{f_i : \mathcal{R} \to \mathcal{R}\}$ if and only if*

1. *$T(0) = 0$;*
2. *T is continuous;*
3. *$T(e^k)$ has finite support for each pixel configuration e^k; and*
4. *T and $\{f_i\}$ are related by*

$$T(\sum_j x_j e^j)_i = \sum_j \frac{f_i(x_j)}{f_i(1)} T(e^j)_i$$

Condition 1 disallows spontaneous generation within the network. Condition 2 allows the recovery of the underlying network structure. Conditions 2 and 3 reflect the local infiniteness of the network. Condition 4 mirrors the local dynamics of the network. (See Section 2 for their precise meanings. It will be shown in section 4 that conditions 1 and 2 imply that the sum in condition 4 is finite.) Thus, as one would expect, the first two conditions are two of those from Richardson's Theorem, while the other two result from the less regular type of architecture of the network, and from the characteristic form of its local dynamics.

Theorem 1 has natural applications. In the important case of a *finite* neural network, conditions 2 and 3 are automatically satisfied. It also implies that, under natural conditions, the uniform limit [1] of neural network global dynamics is also a global dynamics. A direct proof of this result may be somewhat involved.

Corollary 1. *A self-map $T : \mathcal{C}^V \to \mathcal{C}^V$ is realizable by a neural network with given activation functions $\{f_i : \mathcal{R} \to \mathcal{R}\}$ if it is the uniform limit of a sequence of neural network global dynamics $T_n : \mathcal{C}^V \to \mathcal{C}^V$ with the same activation functions and uniformly bounded support of pixel images $T_n(e^k)$.*

Proof. The first two conditions of Theorem 1 are easily verified for T. The third follows since uniform convergence implies pointwise convergence, which on each coordinate $\mathcal{R}$ emans that each sequence $\{T_n(e^k)_j\}_{n \geq 0}$ is eventually constant; thus $T(e^k)$ can contain as many nonzero cells as the uniform bound on the support of $T_n(e^k)$. Likewise, a given $T(\sum_j x_j e^j)_i$ in condition 4 is eventually equal to the corresponding sum for T_n, so it remains to check that the sets $\{j : T_n(e^j)_i \neq 0\}$ are eventually equal to $\{j : T(e^j) \neq 0\}$. This follows since that sum has only finitely many nonzero terms. ∎

If the underlying diagraph of a neural network happens to be a cellular space, it is reasonable to ask under what conditions the global dynamics might be that of a cellular automaton.

Theorem 2. *Let $T : \mathcal{C} \rightarrow \mathcal{C}$ be a self-map arising from a neural network whose activation functions are $\{f_i : \mathcal{R} \rightarrow \mathcal{R}\}$ and let F be the product of the f_i. If*

1. *the underlying diagraph D of the network is an Euclidean cellular space;*
2. *the activation functions of the network are identical and onto;*
3. *T commutes with shifts of pixels.*

then there is a mapping $\tau : \mathcal{C} \rightarrow \mathcal{C}$ that is realized by a cellular automaton on D so that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{C} & \xrightarrow[\tau]{} & \mathcal{C} \end{array}$$

In Section 2 we formally define the brand of neural network considered here. In section 3 the characterizing conditions for the global dynamics of such a network are established. Section 4 is devoted to recovering the underlying network structure from a suitable self-map of the Cantor set. The results of section 3 and 4 constitute the proof of Theorem 1. Section 5 is devoted to the proof of Theorem 2.

2. Neural Networks

A neural network consists of *cells* or processors capable of some arithmetic, connected by links bearing *weights* [6]. Cells sum their weighted inputs and apply an activation function to calculate their new activation state. This means that a neural network is built on a *directed graph* with *vertices* V, and various *arcs* from vertex j to vertex i representing links and labeled with a weight w_{ij} from the ring $\mathcal{R}$. In order for the network to be physically realizable, each vertex must have only finitely many incoming and outgoing arcs, i.e., the diagraph must be *locally finite.* In order to have full computational ability (i.e., that of Turing machines [2]) neural networks must allow for an arbitrarily large number of cells. We will simply assume a countably infinite number.

Formally, a *neural network* is a triple $\mathcal{N} = \langle \mathcal{R}, D, \{f_i\} \rangle$ consisting of a finite ring $\mathcal{R}$, a countably infinite, locally finite, arc-weighted, directed graph D, and a family of activation functions, f_i, one for each vertex i in D. The *local dynamics* of $\mathcal{N}$ is defined by equations

$$net_i(t+1) = \sum_j w_{ij} a_j(t), \tag{2.1}$$

where

$$a_j(t) = f_j(net_j(t)) \tag{2.2}$$

is the activation of the cell j at time t, and the sum is taken over all cells j supporting links into i.

At a fixed time t, the vector of net-inputs assigns to each cell i of D a new value net_i from the ring $\mathcal{R}$, and is thus a member of $\mathcal{R}^V$. (Recall that $\mathcal{R}^V$ is the set of all functions from V to $\mathcal{R}$.) Similarly the vector of activations is also a member of $\mathcal{R}^V$. In the following we shall use $\mathbf{C}$ to refer to activation space and call it the *configuration* (activation) space of the network, and use $\mathcal{R}^V$ to refer to the *net-input* space. At each tick of the time clock, the current net-input vector changes. This change reflects the global dynamics of the network $\mathcal{N}$.

For each cell i, the activation function f_i maps $\mathcal{R}$ onto $\mathcal{R}$. This family of functions gives rise to a product function $F : \mathcal{R}^V \to \mathbf{C}$ so that the following diagram commutes under composition of functions:

$$\begin{array}{ccc} \mathcal{R}^V & \xrightarrow{F} & \mathbf{C} \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ \mathcal{R} & \xrightarrow[f_i]{} & \mathcal{R} \end{array}$$

where π_i is the projection from $\mathcal{R}^V$ onto the i^{th} component in $\mathcal{R}$. This means that for each cell i and each net-input vector $x \in \mathcal{R}^V$, $F(x)_i = f_i(x_i)$.

3. Global dynamics of neural networks

Formally, the *global dynamics* $T : \mathcal{R}^V \to \mathcal{R}^V$ of $\mathcal{N}$ is given for any configuration $x \in \mathcal{R}^V$, by

$$T(x)_i = net_i(F(x)) \tag{3.1}$$

T gives rise to the following commutative diagram:

$$\begin{array}{ccc} \mathcal{R}^V & \xrightarrow{T} & \mathcal{R}^V \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ \mathbf{C} & \xrightarrow[net_i]{} & \mathcal{R} \end{array}$$

This global dynamics T describes the evolution of the entire network from all possible global states as a dynamical system. The four properties of Theorem 1 are satisfied.

1. $T(0) = 0$. This now follows from equations (2.1) and (2.2). ∎

2. $T(e^k)$ *has finite support*, since our graph D is locally finite. ∎

For our next property of T we must make a small excursion into topology. (See [1] for any unfamiliar definitions or results used below.) Since $\mathcal{R}$ is a finite ring, give it the discrete topology, i.e., let every subset of $\mathcal{R}$ be open. Given $\mathcal{R}^V$ the product topology, i.e., basic open sets are those that restrict to specified values at a fixed finite number of cells. With these topologies,

3. *T is a continuous map.*

This proof is most easily followed by referring to the diagram below equation (3.1). A function into a product space is continuous if and only if its composition with each projection function π_i is continuous. Thus F is continuous since each f_i is (every function on a discrete space is continuous). The continuity of net_i requires a little argument. Take any $x \in \mathcal{R}^V$ so that $net_i(x) = r \in \mathcal{R}$. $\{r\}$ is an open neighborhood of r in $\mathcal{R}$. Only finitely many cells, $i_1, \ldots, i_n$ contribute to $net_i(x)$. Let U be the subset of $\mathcal{R}^V$ consisting of all those configurations that agree with x on each of $i_1, \ldots, \subset_n$. The set U is open in $\mathcal{R}^V$ and $net_i(U) \subseteq \{r\}$. Thus net_i is continuous, and so is T. ∎

Finally we want to establish a global analogue of equation (2.1) in terms of T.

4. *For each $x \in \mathcal{R}^V$,*

$$T(x)_i = \sum_j \frac{f_j(x_j)}{f_j(1)} T(e^j)_i$$

Proof. We first observe that, if k has a link into i then

$$T(e^k)_i = net_i(F(e^k)) = net_i(f_k(1)e^k) = \sum_j w_{ij}a_j = w_{ik}(1),$$

otherwise $T(e^k)_i = 0$. Since $f_k(1)$ has an inverse, we can express the weight w_{ik} as

$$w_{ik} = \frac{T(e^k)_i}{f_k(1)} \tag{3.2}$$

We can now calculate $T(x)_i$ as follows:

$$T(x)_i = net_i(F(x)) = \sum_j w_{ij}a_j = \sum_j \frac{T(e^j)_i}{f_j(1)} f_j(x_j),$$

the second sum now being taken over all j's, since for cells j with no links into i, $T(e^j)_i = 0$. ∎

Thus we have shown that the conditions of Theorem 1 are necessary. In the next section we prove their sufficiency.

4. Realization of self-maps via a neural network

Given a map $T : \mathcal{C} \to \mathcal{C}$ and an activation function $F : \mathcal{C} \to \mathcal{C}$ satisfying the conditions of the Theorem 1, we want to recover T as the global dynamics of some neural network based on F. We will first construct the underlying directed graph, and then show that, with the weights defined by equation (3.2), T is the global dynamics of the resulting network.

The finite ring $\mathcal{R}$ is given. Choose a countable set of vertices V. Give $\mathcal{R}$ the discrete topology and $\mathcal{R}^V$ the product topology. $\mathcal{R}^V$ is a perfect, totally disconnected, compact metric space, and as such is homeomorphic to the Cantor set [4, p. 97]. Thus, consider both T and F as mappings from $\mathcal{R}^V$ to $\mathcal{R}^V$.

First, we find suitable links among the vertices in V in order to determine the underlying digraph D. Observe that the sequence $\{e^k\}$ of pixels (in an arbitrary order) converges to the all-quiescent configuration O. Since $T(O) = O$ and T is continuous, $\{T(e^k)\}$ also converges to O. This means that for a fixed node i, $T(e^j)_i$ is nonzero for at most finitely many j's. Therefore, the sum in condition 4 of Theorem 1 must be finite. Define a digraph D on the vertex set V by putting an arc from j to i just in case $T(e^j)_i \neq 0$. This shows that the underlying directed graph D of the network is completely determined by T independently of the f_i's. Together with condition 3, it also shows that D is locally finite.

Next, we need to find appropriate weights so that the resulting neural network defined by D and F induces T as global dynamics. Apportion weights according to equation (3.2), so that, by condition 4 of Theorem 1 and equation (2.1),

$$T(x)_i = \sum_j \frac{T(e^j)_i}{f_j(1)} f_j(x_j) = \sum_j w_{ij} f_j(x_j) = net_i(f(x)),$$

and this satisfies equation (3.1). ∎

5. Application to cellular automaton

In this section we will use the deterministic version [5, Corollary 2] of Richardson's Theorem to prove Theorem 2. Assume the conditions of Theorem 2 and refer to the diagram contained in its statement. We first define τ and then prove it to be the global dynamics of a cellular automaton by verifying that the three conditions of Richardson's Theorem are satisfied.

Note that for any two points $x, y \in \mathcal{R}^V$, if $F(x) = F(y)$ then $T(x) = T(y)$ by condition 4 in Theorem 1. Hence, if $z \in \mathcal{C}$ and $x,y \in F^{-1}(z)$, then $F(T(x)) = F(T(y))$. Therefore, we may define $\tau : \mathcal{C} \to \mathcal{C}$ as follows: since F is onto, for $z \in \mathcal{C}$ choose some $x \in F^{-1}(z)$ and let $\tau(z) = F(T(x))$. Thus τ is well defined and the diagram commutes. Further, τ satisfies Richardson's conditions:

A. $\tau(O) = O$, since $T(O) = O$ and $F(O) = O$.

B. Since $\mathcal{C}$ is compact and F is continuous, F is a closed map. Since F is also onto, it is a quotient map. Thus τ is *continuous* since $F \circ T$ is continuous.

C. *τ commutes with shift operators*: given any configuration $x \in \mathcal{C}$ and a lattice point k, define the shift operator S_k by $S_k(x)_i = x_{i-k}$. By hypothesis, T commutes with each shift operator at pixels, i.e.,

$$T(e^{j+k})_i = T(e^j)_{i-k}$$

for all nodes i and j. Since the network is homogenous, $j \in N_i$ if and only if $j-k \in N_{i-k}$, where N_i is the cellular neighborhood of the cell i. Thus, since $T(e^j)_i = 0$ for cell with no links into i,

$$\begin{aligned} T(S_k(x))_i &= \sum_{j-k \in N_{i-k}} \frac{f(x_{j-k})}{f(1)} T(e^j)_i \\ &= \sum_{j-k \in N_{i-k}} \frac{f(x_{j-k})}{f(1)} T(e^{j-k})_{i-k} \\ &= \sum_{j \in N_{i-k}} \frac{f(x_j)}{f(1)} T(e^j)_{i-k} = T(x)_{i-k} = T(x)_{i-k} \\ &= S_k(T(x))_i, \end{aligned}$$

i.e., T commutes with all shifts. Since F is the product of identical maps $f : \mathcal{R} \to \mathcal{R}$, it also commutes with all shifts. Using these facts and the definition of τ it follows that

$$\begin{aligned} S_k \circ \tau \circ F &= S_k \circ F \circ T = F \circ S_k \circ T \\ &= F \circ T \circ S_k = \tau \circ F \circ S_k \\ &= \tau \circ S_k \circ F. \end{aligned}$$

But F is an epimorphism (i.e., is onto) and thus right-cancelable. Therefore $S_k \circ \tau = \tau \circ S_k$. ■

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