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Periodic Points and Entropies for Cellular Automata

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Abstract. For the class of permutive cellular automata the number of periodic points and the topological and metrical entropies are calculated.

1. Introduction

Cellular automata (CA) are infinite sequences of symbols of a finite alphabet that evolve in time according to a definite local rule. We can take, for example, a lattice where each vertex contains the value of a dynamical variable ranging over a finite set of numbers. At a certain time t_0 the values on the vertices of the lattice take a definite value, but at time $t_0 + 1$, the values of the dynamical variable can change according to a law defined locally. This kind of system, as shown by von Neumann [8] can simulate a Turing machine, and specific examples exist where the cellular automaton is capable of universal computation [3]. The simplest examples of CA rules are obtained when we restrict to one-dimensional lattices and the dynamic variable over each site of the lattice can take values on the finite field $\mathbb{Z}_2 = \{0, 1\}$ [10].

There is an extensive literature on the dynamics of one-dimensional CA and on the patterns generated by the evolution of the CA rules in the extended phase space $\mathbb{Z} \times \mathbb{Z}^+$ [12]. These patterns appear to be organized in several complexity classes according to the behavior of the iterates of random initial conditions [11]. In Wolfram's classification [11], we have four classes of CA maps. Class 1: Evolution leads to a homogeneous state. Class 2: Evolution leads to a set of separated simple or periodic structures. Class 3: Evolution leads to chaotic patterns. Class 4: Evolution leads to complex localized structures.

The concept of metrical entropy, introduced by Kolmogoroff [9], and its topological analog, the topological entropy [1] are invariants that quantify the intuitive notions of mean complexity and global complexity of a dynamical system. (For a discussion of this concepts see [2].) It has been shown by Brudno [4] that the complexity of a trajectory of a dynamical system is strictly related to the mean complexity, that is, if there exists an ergodic measure μ, the complexities of μ-almost all trajectories equals the metrical entropy. Under these conditions the classification of Wolfram can be made more precise by calculating the ergodic invariant measures and the metrical entropy for classes of CA maps.

The topological entropy gives an equal weight to the different kinds of complex behaviors of the trajectories of a dynamical system, and so, we can find systems with a positive topological entropy but with complex trajectories localized on a subset of phase space with zero Lebesgue measure. However, if the invariant measure associated to a compact dynamical system is unique, the topological entropy equals the metrical entropy.

We can define a third quantity to quantify the complexity of a dynamical system. Let f be a self map of a compact space X. Let $Per(f^n)$ denote the number of fixed points of f^n. We define the *periodic complexity* of the map f by

$$P(f) = \limsup_{n\to\infty} \frac{\log_2 Per(f^n)}{n}.$$

If f has the property of separability of trajectories, $P(f)$ is a lower bound for the topological entropy [5]. In topological Markov chains $P(f)$ equals the topological entropy [2].

This paper is organized as follows. In Section 2 we calculate the number of fixed points and the periodic complexity for the class of permutive CA maps (see the definitions below), and for the other classes an upper bound is given. It is also shown that orbits of points, under the evolution of a permutive CA map, are associated uniquely to orbits of points of a topological Markov chain. In Section 3 we calculate the entropies for permutive maps. Section 4 is devoted to the discussion of the results and some examples are presented. In the rest of the introduction we give some definitions that will be used in the subsequent sections.

Cellular automata are maps $\tau : \Sigma \to \Sigma$ where Σ is the space of doubly infinite sequences of 0 and 1, $\Sigma = \{0,1\}^{\mathbb{Z}}$. We restrict our study to the class of finite breadth CA, that is, those CA whose value at site number i, $i \in \mathbb{Z}$, at time $t+1$ depends on the values of site numbers $i+m, \ldots, i+n$, with $m < n$, at time t. The number $\gamma = n - m + 1$ is the *breadth* of the CA rule τ. Giving to $\mathbb{Z}_2$ the discrete topology and endowing Σ with the product topology, Σ is a compact topological space and a finite breadth CA map is a continuous function of Σ.

A simple representation of a CA map can be obtained through the shift map $\sigma : \Sigma \to \Sigma$ and a Boolean function of $\gamma \geq 2$ variables. Given $f : \{0,1\}^\gamma \to \{0,1\}$ we can define a cellular automaton map τ by

$$(\tau(x))_i = f(x_{i+m}, \ldots, x_{i+n}), \quad \text{for all } i \in \mathbb{Z}, \gamma = n - m + 1,$$

or,

$$\tau(x) = f(\sigma^m(x), \ldots, \sigma^n(x))$$

where $(\sigma(x))_i = x_{i+1}$ and $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \Sigma$. f is the *generating function* of the finite breadth map τ. An example of a CA map is given by $\tau(x) = \sigma^{-1}(x) \oplus \sigma^1(x)$, where $\oplus$ means addition modulo 2.

There are 2^{2^γ} Boolean functions $f(x_1,\ldots,x_\gamma)$ of γ variables and some of them do not depend of x_1 or x_γ. An easy calculation shows that the number of Boolean functions that depends of x_1 and x_γ is

$$N(\gamma) = 2^{2^\gamma} - 2\cdot 2^{2^{\gamma-1}} + 2^{2^{\gamma-2}}, \quad \text{for any } \gamma \geq 2.$$

In the following, $\mathcal{C}(\gamma; m,n)$, with $m < n$, is the class of CA maps whose generating functions depend of x_1 and x_γ and $\gamma = n-m+1$. For each fixed pair of integers m and n, $\mathcal{C}(2;m,n)$ and $\mathcal{C}(3;m,n)$ have respectively 10 and 228 elements.

Following Milnor [7], we say that τ is a *right-permutive* CA map (RP), iff, its generating function f verifies to the condition

$$f(x_1,\ldots,x_\gamma) = \overline{f(x_1,\ldots,\overline{x_\gamma})}, \quad \text{for all } (x_1,\ldots,x_\gamma) \in \mathbb{Z}_2^\gamma.$$

τ is *left-permutive* (LP), iff,

$$f(x_1,\ldots,x_\gamma) = \overline{f(\overline{x_1},\ldots,x_\gamma)}, \quad \text{for all } (x_1,\ldots,x_\gamma) \in \mathbb{Z}_2^\gamma,$$

where $\bar{0} = 1$ and $\bar{1} = 0$. τ is *permutive* (P) iff both the above conditions are verified. Denoting by $P(\gamma;m,n)$ the number of permutive CA maps and by $Q(\gamma;m,n)$ the number of right- or left-permutive CA maps in $\mathcal{C}(\gamma;m,n)$, it can be found easily that

$$\begin{aligned} P(\gamma;m,n) &= 2^{2^{\gamma-2}} \\ Q(\gamma;m,n) &= \left(2\sum_{i=1}^{\gamma-1} N(i)\right) - P(\gamma;m,n) \end{aligned}$$

where $N(1) = 2$.

2. Periodic points

Let $\{A_j\}$ with $0 \leq j \leq 2^\gamma - 1$ represent the elements of $\mathbb{Z}_2^\gamma$, where $j = i_\gamma + 2\cdot i_{\gamma-1} + 4\cdot i_{\gamma-2} + \cdots + 2^{\gamma-1}\cdot i_1$ and $(i_1,\ldots,i_\gamma) \in \mathbb{Z}_2^\gamma$. This defines an invertible map $\pi_\gamma : \mathbb{Z}_2^\gamma \to \{A_j\}$. For example, if $\gamma = 2$, we have the symbolic representation,

$$A_0 \leftrightarrow (0,0), A_1 \leftrightarrow (0,1), A_2 \leftrightarrow (1,0), A_3 \leftrightarrow (1,1).$$

We can now identify the set $\Sigma = \mathbb{Z}_2^{\mathbb{Z}}$ with a subset of $\tilde{\Sigma}_\gamma = \{A_j\}^{\mathbb{Z}}$ by taking $x = (\ldots,x_{-1},x_0,x_1,\ldots) \in \Sigma$ and identifying consecutive and overlapping γ-blocks of elements of $\mathbb{Z}_2$ with the elements of the set $\{A_j\}$. For $\gamma = 2$, we have, for example,

$$\ldots 01101001110\ldots \Leftrightarrow \ldots A_1A_3A_2A_1A_2A_0A_1A_3A_3A_2\ldots$$

Hence, we have constructed a bijection $h_\gamma : \Sigma \to S_\gamma \subset \tilde{\Sigma}_\gamma$ such that $\sigma_\gamma(h_\gamma(x)) = h_\gamma(\sigma(x))$, where σ_γ is the shift map over $\tilde{\Sigma}_\gamma$. The map $h_\gamma : \Sigma \to S_\gamma$ is defined by

$$(h_\gamma(x))_i := \pi_\gamma(x_{i+m},\ldots,x_{i+n}) = b_i, \quad \text{for all } i \in \mathbb{Z} \text{ and } \gamma = n-m+1,$$

where $b_i = A_j$ and $j = x_{i+n} + 2 \cdot x_{i+n-1} + \ldots + 2^{\gamma-1} \cdot x_{i+m}$.

Let $M_\gamma = [M_{ij}]$, $0 \leq i,j \leq 2^\gamma - 1$, be a $2^\gamma \times 2^\gamma$ matrix. With $i = i_\gamma + 2 \cdot i_{\gamma-1} + \ldots + 2^{\gamma-1} \cdot i_1$ we define M_γ by $M_{ij} = 1$ if $j = 2 \cdot i_\gamma + \ldots + 2^{\gamma-1} \cdot i_2$ or $j = 1 + 2 \cdot i_\gamma + \ldots + 2^{\gamma-1} \cdot i_2$ and $M_{ij} = 0$ otherwise. We say that $A_i \longrightarrow A_j$ is an *allowed* or *compatible transition* for M_γ iff $M_{ij} = 1$. The set S_γ is completely characterized by the *Markov transition matrix* M_γ, i.e., $b = (\ldots, b_{-1}, b_0, b_1, \ldots) \in S_\gamma$ *iff* $b_i \longrightarrow b_{i+1}$ *is an allowed transition of* M_γ, *for all* $i \in \mathbb{Z}$. For $\gamma = 2$, we have

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial of M_2 is $\lambda^3(\lambda - 2)$ and the number of periodic configurations of S_γ with period n is 2^n. For each class $\mathcal{C}(\gamma; m, n)$ we call M_γ the *space transition matrix to the right* (STMR). The *space transition matrix to the left* (STML) is the transposed of the STMR, M_γ^t.

We can now associate to a CA rule $\tau \in \mathcal{C}(\gamma; m, n)$, with $\gamma \geq 2$, and $m \leq 0 \leq n$, a time transition matrix. Let $(b_m, \ldots, b_n)$ be a γ-block of elements of $\{A_j\}$, $0 \leq j \leq 2^\gamma - 1$. We define the map $f^* : \{A_j\} \to \{A_j\}$ by

$$f^*(b_1, \ldots, b_\gamma) := \pi_\gamma(f(\pi_\gamma^{-1}(b_1)), \ldots, f(\pi_\gamma^{-1}(b_\gamma))) = c$$

where $c \in \{A_j\}$ and f is the generating function of τ. The map f^* induces a CA map $\tau^* : S_\gamma \to S_\gamma$, by

$$\tau^*(b) := f^*(\sigma_\gamma^m(b), \ldots, \sigma_\gamma^n(b))$$

The time transition matrix $T_\tau = [T_{ij}]$, $0 \leq i,j \leq 2^\gamma - 1$, associated to τ is defined by

$$T_{ij} = \begin{cases} 1, & \text{if } f^*(b_m, \ldots, b_0, \ldots, b_n) = c, \text{ with } b_0 = A_i \text{ and } c = A_j \\ 0, & \text{otherwise} \end{cases}$$

where $m \leq 0 \leq n$. We will call T_τ the *time transition matrix* (TTM) associated to τ.

Proposition 2.1. *Let $\tau \in \mathcal{C}(\gamma; m, n)$ with $\gamma \geq 2$, $h_\gamma : \Sigma \to S_\gamma$ and $\tau^* : S_\gamma \to S_\gamma$ as defined above. Then, τ and τ^* are topologically conjugate, $\tau^* \circ h_\gamma = h_\gamma \circ \tau$.*

Proof. With $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \Sigma$ and by the definitions of the maps h_γ and π_γ, we have

$$\begin{aligned} (h_\gamma(\tau(x)))_i &= A_{2^{\gamma-1} \cdot \tau(x)_{i+m} + \ldots + \tau(x)_{i+n}} \\ &= A_{2^{\gamma-1} \cdot f(x_{i+m+m}, \ldots, x_{i+m+n}) + \ldots + f(x_{i+n+m}, \ldots, x_{i+n+n})} \end{aligned}$$

and

$$\begin{aligned}(\tau^*(h_\gamma(x)))_i &= f^*((h_\gamma(x))_{i+m}, \ldots, (h_\gamma(x))_{i+n}) \\ &= f^*(A_{2^{\gamma-1} \cdot x_{i+m+m}+\ldots+x_{i+m+n}}, \ldots, A_{2^{\gamma-1} \cdot x_{i+n+m}+\ldots+x_{i+n+n}}) \\ &= \pi_\gamma(f(x_{i+m+m}, \ldots, x_{i+m+n}), \ldots, f(x_{i+n+m}, \ldots, x_{i+n+n})) \\ &= A_{2^{\gamma-1} \cdot f(x_{i+m+m}, \ldots, x_{i+m+n})+\ldots+f(x_{i+n+m}, \ldots, x_{i+n+n})}\end{aligned}$$

for all $i \in \mathbb{Z}$. Comparing the above expressions, we have the desired result. ∎

We have constructed a CA map $\tau^* : S_\gamma \to S_\gamma$, over a "larger" set of symbols ($\{A_j\}, 0 \leq j \leq 2^\gamma - 1$), that is topologically equivalent to τ. We shall see below the conditions for the orbit of a point $b \in S_\gamma$ by the iteration of τ^*, $\mathcal{O}(b) := \{x \in S_\gamma : x = \tau^{*^n}(b), n \geq 0\}$, to be completely "reconstructed," given a semi-infinite sequence $(b_0, b_1, \ldots)$ of elements of $\{A_j\}$ whose transitions $b_i \xrightarrow{t} b_{i+1}$ are all compatible with T_τ. In this way the number of periodic points of the dynamical system defined by τ^* equals the number of periodic points of the subshift of the finite type defined by T_τ and the shift map σ_γ over S_γ. In the following we look for conditions on τ such that a "temporal transition" $b \xrightarrow{t} c$ compatible with T_τ generates uniquely the "spatial transitions" $b \xrightarrow{s} d(b,c)$ and $e(b,c) \xleftarrow{s} b$ compatible with M_γ and M_γ^t, respectively. In a diagrammatic form we have

$$\begin{array}{ccccc} e(b,c) & \underset{M_\gamma^t}{\xleftarrow{s}} & b & \underset{M_\gamma}{\xrightarrow{s}} & d(b,c) \\ & & T_\tau \downarrow t & & \\ & & c & & \end{array}$$

Lemma 2.2. *Let $\tau \in \mathcal{C}(\gamma; m, n)$ with $\gamma \geq 2$ and $m \leq 0 \leq n$. Let T_τ and M_γ [resp. M_γ^t] be, respectively, the TTM associated to τ and the STMR [resp. STML]. If $b \xrightarrow{t} c$ is an allowed transition of T_τ, τ is RP [resp. LP] and $n > 0$ [resp. $m < 0$] then, there exists one and only one allowed transition $b \xrightarrow{s} d$ [resp. $e \xleftarrow{s} b$] compatible with M_γ [resp. M_γ^t].*

Proof. Suppose that τ is RP. Let $b \xrightarrow{t} c$ be an allowed transition of T_τ. With $\pi_\gamma^{-1}(b) = (x_m, \ldots, x_0, \ldots, x_n)$ and $\pi_\gamma^{-1}(c) = (y_m, \ldots, y_0, \ldots, y_n)$, we have $f(x_m, \ldots, x_n) = y_0$. As y_1 is fixed by $\pi_\gamma^{-1}(c)$ because $n > 0$ and, by the RP condition, there exists one, and only one, $x \in \mathbb{Z}_2$ such that $f(x_{m+1}, \ldots, x_n, x) = y_1$. So, $d = \pi(x_{m+1}, \ldots, x_n, x)$. A similar proof is obtained when τ is LP. ∎

Theorem 2.3. *Let $\tau \in \mathcal{C}(\gamma; m, n)$ with $\gamma \geq 2$. If τ is RP [resp. LP] and $m \leq 0 < n$ [resp. $m < 0 \leq n$] then, for any allowed sequence $B = (b^0, b^1, \ldots, b^k, \ldots)$ compatible with T_τ, there exist sequences $D^i = (d_1^i, \ldots, d_k^i, \ldots)$ [resp. $E^i = (\ldots, e_{-k}^i, \ldots, e_{-1}^i)$] with $i \geq 0$, such that, for every $i \geq 0$ and $k \geq 1$, $d_k^i \xrightarrow{t} d_k^{i+1}$ [resp. $e_{-k}^i \xrightarrow{t} e_{-k}^{i+1}$] and $d_k^i \xrightarrow{s} d_{k+1}^i$ [resp. $e_{-k-1}^i \xleftarrow{s} e_{-k}^i$] are compatible with T_τ and M_γ [resp. M_γ^t], respectively.*

Moreover, $b^i \xrightarrow{s} d_1^i$ [resp. $e_{-1}^i \xleftarrow{s} b^i$] is compatible with M_γ [resp. M_γ^t] for every $i \geq 0$, and the sequences D^i [resp. E^i] are completely determined by B.

Proof. Suppose that τ is RP and that B is an allowed T_τ sequence. By lemma 2.2, there exist uniquely determined M_γ-compatible transitions $b^0 \xrightarrow{s} d_1^0$ and $b^1 \xrightarrow{s} d_1^1$, determined, respectively by the T_τ transitions $b^0 \xrightarrow{t} b^1$ and $b^1 \xrightarrow{t} b^2$. Now we want to prove that $d_1^0 \xrightarrow{t} d_1^1$ is T_τ-admissible. With $\pi_\gamma^{-1}(d_1^0) = (x_{m+1}, \ldots, x_{n+1})$ and $\pi_\gamma^{-1}(d_1^1) = (y_{m+1}, \ldots, y_{n+1})$, as we have seen in the proof of lemma 2.2, x_{n+1} is completely determined by y_1, i.e., $f(x_{m+1}, \ldots, x_{n+1}) = y_1$. So, as $m \leq 0$ and $n > 0$, the transition $d_1^0 \xrightarrow{t} d_1^1$ is always admissible, independently of y_{n+1}. As $d_1^0 \xrightarrow{t} d_1^1$ is admissible, by lemma 2.2 there exists, say, d_2^0, such that $d_1^0 \xrightarrow{s} d_2^0$ is M_γ-admissible. Using induction with the same arguments we construct semi-infinite sequences $D^i = (d_1^i, \ldots, d_k^i, \ldots)$ all compatible with M_γ and completely determined by B. When τ is LP the proof is similar. ∎

In the conditions of the last theorem, when τ is RP and given $B = (b^0, b^1, \ldots)$, we have the following diagram:

$$\begin{array}{ccccccc}
b^0 & \xrightarrow{s} & d_1^0 & \xrightarrow{s} & d_2^0 & \xrightarrow{s} & d_3^0 \ldots \\
t\downarrow & & t\downarrow & & t\downarrow & & \vdots \\
b^1 & \xrightarrow{s} & d_1^1 & \xrightarrow{s} & d_2^1 \ldots & & \\
t\downarrow & & t\downarrow & & \vdots & & \\
b^2 & \xrightarrow{s} & d_1^2 \ldots & & & & \\
t\downarrow & & \vdots & & & & \\
b^3 \ldots & & & & & & \\
\vdots & & & & & &
\end{array}$$

where all vertical and horizontal transitions are compatible with T_τ and M_γ, respectively, and the D^i are determined by B. If τ is P we can prolong the diagram to the left in an analogous way and write $\tau^*(E^i, b^i, D^i) = (E^{i+1}, b^{i+1}, D^{i+1})$. Hence, we have:

Corollary 2.4. *If $\tau \in \mathcal{C}(\gamma; m, n)$ is a P CA map, $\gamma \geq 3$ and $m < 0 < n$, then $Per(\tau^k) = Trace(T_\tau^k)$ for every $k \geq 1$. If τ is nonpermutive, $\gamma \geq 2$ and $m \leq 0 \leq n$, then $Per(\tau^k) \leq Trace(T_\tau^k)$.*

Proof. If τ is permutive and $B = (b^0, \ldots, b^{k-1}, b^0, \ldots, b^{k-1}, \ldots)$ is periodic and T_τ-admissible, by theorem 2.3 we have that $\tau^*(E^i, b^i, D^i) = (E^{i+1}, b^{i+1}, D^{i+1})$, for every $i \geq 0$. By lemma 2.2, $d_i^0 = d_i^k$ and $e_{-i}^0 = e_{-i}^k$, for every $i \geq 1$. So, (E^0, b^0, D^0) is a periodic point of τ^*. As D^i and E^i are uniquely determined by B, the first assertion of the corollary follows by proposition 2.1 and by the fact that the number of periodic transitions of a Markov matrix equals its trace. If τ is nonpermutive , by construction of T_τ and M_γ, to every periodic point $x \in \Sigma$ of τ with period n, corresponds a closed loop of length n in the graph defined by T_τ. So, $Per(\tau^k) = Per(\tau^{*^k}) \leq Trace(T_\tau^k)$. ∎

Lemma 2.5. *Let $\tau \in \mathcal{C}(\gamma; m, n)$ with $\gamma \geq 2$. If τ is P then τ is $2^{\gamma-1}$-to-one as well as any finite block of elements of $\mathbb{Z}_2$.*

Proof. Let $x \in \Sigma$ and $a = (a_1, \ldots, a_{\gamma-1}) \in \mathbb{Z}_2^{\gamma-1}$. By the RP condition, there exists one, and only one, $y_1 \in \mathbb{Z}_2$ such that $f(a_1, \ldots, a_{\gamma-1}, y_1) = x_0$. By the LP condition, there exists one, and only one, $y_{-1} \in \mathbb{Z}_2$ such that $f(y_{-1}, a_1, \ldots, a_{\gamma-1}) = x_{-1}$. A configuration

$$y = (\ldots, y_{-n}, \ldots, y_{-1}, a_1, \ldots, a_{\gamma-1}, y_1, \ldots, y_n, \ldots)$$

such that $\tau(y) = x$ is then obtained by induction. As a is an arbitrary element of $\mathbb{Z}_2^{\gamma-1}$, the proposition follows. ∎

Corollary 2.6. *If $\tau \in \mathcal{C}(\gamma; m, n)$ is permutive, $\gamma \geq 3$ and $m < 0 < n$, then $P(\tau) = \gamma - 1$.*

Proof. Let T_τ be the TMM associated to τ. By lemma 2.5 , for each A_i there exists $2^{\gamma-1}$ T_τ-allowed transitions $A_i \to A_j$. So,

$$\sum_{j=0}^{2^\gamma-1} T_{ij} = 2^{\gamma-1},$$

independently of i. By the Frobenius-Perron Theorem the spectral radius λ of T_τ is bounded by

$$\min_i(\sum_j T_{ij}) \leq \lambda \leq \max_i(\sum_j T_{ij})$$

and so, $\lambda = 2^{\gamma-1}$. As $Trace(T_\tau^k) \sim \lambda^k$, the result follows by corollary 2.4 and the definition of $P(\tau)$ (see *Introduction*). ∎

We have shown that all permutive CA maps within a class $\mathcal{C}(\gamma; m, n)$ have the same periodic complexity. In the next section we prove that the same conclusion holds for the entropies. Finally, we note that the Markov matrix T_τ associated to any CA map τ enables us to construct an algorithm to find the period n points of τ. In fact, if $x \in \Sigma$ is a periodic point of τ with period n then, associated to x, there exists a closed loop of length n in the graph defined by T_τ. So, from all the possible loops of length n we can test those that can be prolonged to the right and left with transitions compatible with M_γ and M_γ^t.

3. Entropies

Let $x, y \in \Sigma$ and define the distance function $d : \Sigma \times \Sigma \to \mathbb{R}^+$ by $d(x, y) = \sum_{n=-\infty}^{\infty} \frac{|y_n - x_n|}{2^{|n|}}$. The neighborhood base for the topology induced by the metric d consists of cylinder sets,

$$C_{r,s}^a := \{x \in \Sigma : (x_r, \ldots, x_s) = (a_1, \ldots, a_{s-r+1})\}$$

where $a = (a_1, \ldots, a_{s-r+1})$ is a fixed element of $\mathbb{Z}_2^{s-r+1}$. In this topology Σ is compact. Let $\mathcal{B}$ be the σ-algebra generated by all the cylinder sets $C^a_{r,s}$ and define the product measure $\mu : \mathcal{B} \to \mathbb{R}^+$ through

$$\mu\left(C^{a_1}_{m_1,m_1} \cap C^{a_2}_{m_2,m_2} \cap \ldots \cap C^{a_k}_{m_k,m_k}\right) := \frac{1}{2^k}$$

for any $a_i \in \mathbb{Z}_2$, $m_i \in \mathbb{Z}$ and $k \geq 1$. Bakenship and Rothaus [7] have shown that the τ invariance of μ is a necessary and sufficient condition for the surjectivity of τ. By lemma 2.5 we have that *every permutive cellular automaton map is invariant with respect to the measure* μ.

Let $\alpha = \{A_1, \ldots, A_n\}$ and $\beta = \{B_1, \ldots, B_m\}$ be either covers or partitions of Σ and denote their *join* by $\alpha \vee \beta := \{A_i \cap B_j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$. The cover β is a *refinement* of the cover α, $\alpha \prec \beta$, iff every element of β is a subset of some element of α. A finite partition α of Σ is called *generating* for τ iff $\bigvee_{k=-\infty}^{\infty} \tau^k \alpha = \epsilon$, where ϵ is the partition of Σ into points.

Let α be an open cover of Σ and $N(\alpha)$ the number of sets in a subcover of minimal cardinality. The topological entropy of τ with respect to α is

$$h(\tau, \alpha) := \lim_{k \to \infty} \frac{\log_2 N(\alpha \vee \tau^{-1}\alpha \vee \ldots \vee \tau^{-k+1}\alpha)}{k}$$

and the topological entropy of τ is $\sup h(\tau, \alpha)$, where the sup is taken over all finite covers of Σ. If $\{\alpha_i\}$ is a sequence of refining covers of Σ and $diameter(\alpha_i) \to 0$ as $i \to \infty$ then, $h(\tau) = \sup_i h(\tau, \alpha_i)$ [1].

If τ leaves invariant a measure μ, the metrical entropy of τ relative to the partition α and the measure μ is

$$H(\tau, \mu, \alpha) := \lim_{k \to \infty} \frac{H_\mu(\alpha \vee \tau^{-1}\alpha \vee \ldots \vee \tau^{-k+1}\alpha)}{k}$$

where $H_\mu(\alpha) = -\sum_{i=1}^{n} \mu(A_i) \log_2 \mu(A_i)$. The metrical entropy of τ relative to the measure μ is $H(\tau, \mu) = \sup H(\tau, \mu, \alpha)$ and, if α is a generating partition, $H(\tau, \mu) = H(\tau, \mu, \alpha)$ [9].

The main result of this section is the following theorem:

Theorem 3.1. *Let $\tau \in \mathcal{C}(\gamma; m, n)$ with $\gamma \geq 3$ and $m < 0 < n$. If τ is P then, the topological and metrical entropies of τ are*

$$h(\tau) = H(\tau, \mu) = \gamma - 1$$

where μ is the product measure on Σ.

We now give some preparatory lemmata for the proof of theorem 3.1.

Lemma 3.2. *Let $\tau \in \mathcal{C}(\gamma; m, n)$ with $\gamma \geq 3$ and $m < 0 < n$. If τ is P then, for every $r > k + m$ and $s < p + n$, we have*

$$(\tau^{-1} C^a_{k,p}) \bigcap C^b_{r,s} = \emptyset$$

or,

$$(\tau^{-1} C^a_{k,p}) \bigcap C^b_{r,s} = C^d_{k+m,p+n}$$

where d depends of a and b.

Proof. Let $a = (a_k, \dots, a_p) \in \mathbb{Z}_2^{p-k+1}$. By the definition of τ,

$$\tau^{-1} C^a_{k,p} = \bigcup_{c \in G} C^c_{k+m,p+n}$$

where $G = \{(x_{k+m}, \dots, x_{p+n}) : f(x_{i+m}, \dots, x_{i+n}) = a_i, k \leq i \leq p\}$. By lemma 2.5, the set G has $2^{\gamma-1}$ elements and it can be constructed as in the proof of lemma 2.5, by "extending" to the right and left the set of all the $\gamma-1$-blocks of elements of $\mathbb{Z}_2$. With $r > k+m$ and $s < p+n$, if the block $b = (b_r, \dots, b_s)$ agrees with one of the blocks $(x_r, \dots, x_s)$ we can choose $d = (x_{k+m}, \dots, b_r, \dots, b_s, \dots, x_{p+n})$ and $(\tau^{-1} C^a_{k,p}) \cap C^b_{r,s} = C^d_{k+m,p+n}$. If b does not agree with either of the sub-blocks of the elements of G we have an empty intersection. ∎

Lemma 3.3. *Let $\tau \in \mathcal{C}(\gamma; m, n)$ with $\gamma \geq 3$ and $m < 0 < n$. Let $\alpha(im, in) = \{C^a_{im,in} : a \in \mathbb{Z}_2^{i(n-m)+1}\}$. If τ is P then, for any $i \geq 1$, we have,*

$$N(\alpha(im, in) \vee \tau^{-1}\alpha(im, in) \vee \ldots \vee \tau^{-k+1}\alpha(im, in)) \\ = 2^{i(n-m)+1+(k-1)(\gamma-1)}$$

and $\vee_{k=0}^{-\infty} \tau^k \alpha(im, in) = \epsilon$, where ϵ is the partition of Σ into points.

Proof. To simplify the notation let us put $\alpha := \alpha(im, in)$ and $\alpha_k = \alpha \vee \tau^{-1}\alpha \vee \ldots \vee \tau^{-k+1}\alpha$. By lemma 3.2 we have $\alpha_2 = \alpha \vee \tau^{-1}\alpha = \{C^a_{im+m,in+n}\}$. Continuity of τ implies that α_2 is a cover of Σ with minimal cardinality and so $N(\alpha \vee \tau^{-1}\alpha) = 2^{i(n-m)+1+(n-m)}$. From property 6 of [1], $\alpha_3 = \alpha \vee \tau^{-1}(\alpha \vee \tau^{-1}\alpha)$. As $im > im + m$ $(m < 0)$ and $in < in + n$ $(n > 0)$, by lemma 3.2, we have, $\alpha_3 = \{C^a_{im+2m,in+2n}\}$. With the same previous argument α_3 is a cover of Σ with minimal cardinality. By induction, we obtain $\alpha_k = \{C^a_{im+(k-1)m,in+(k-1)n} : a \in \mathbb{Z}_2^{(i+k-1)(n-m)+1}\}$. With $n - m = \gamma - 1$, the result for $N(\alpha_k)$ follows and, $lim_{k\to\infty} \alpha_k = \vee_{k=0}^{-\infty} \tau^k \alpha = \{x \in \Sigma\} = \epsilon$. ∎

Proof of theorem 3.1. Let $\{\alpha_i\} := \{\alpha(im, in) : i \geq 1\}$ be a sequence of covers of Σ. In the metric d, $diameter(\alpha_i) \to 0$ as $i \to \infty$. By lemma 3.3 and the definition of topological entropy, $h(\tau, \alpha_i) = \gamma - 1$ and so, $h(\tau) = \sup_i h(\tau, \alpha_i) = \gamma - 1$. For the metrical entropy, by definition of the measure μ, we have

$$\mu(C^a_{im+(k-1)m,in+(k-1)n}) = 1/2^{i(n-m)+1+(k-1)(\gamma-1)}$$

and so, $H_\mu(\alpha_i \vee \tau^{-1}\alpha_i \vee \ldots \vee \tau^{-k+1}\alpha_i) = i(n-m)+1+(k-1)(\gamma-1)$. As every partition α_i is generating (lemma 3.3) we have $H(\tau, \mu) = H(\tau, \mu, \alpha_i) = \gamma - 1$. ∎

4. Examples and conclusions

In general, any finite breadth cellular automaton map can be specified within a certain class $\mathcal{C}(\gamma; m, n)$, by a code number ranging in the interval $[0, 2^{2^\gamma}]$

[10]. For example, taking $\gamma = 2$, and letting $a_0 = f(0,0)$, $a_1 = f(0,1)$, $a_2 = f(1,0)$, and $a_3 = f(1,1)$, the CA map generated by $f(x_1, x_2)$ has code number $C = a_0 + 2 \cdot a_1 + 4 \cdot a_2 + 8 \cdot a_3$. When the a_i ($\in \mathbb{Z}_2$) range over all its values, there are 16 different Boolean functions of two variables. However, the class $\mathcal{C}(2; m, n)$, for fixed m and n, has only ten elements because some of the generating functions do not explicitly depend on x_1 or x_2. In this representation, reducible generating functions have code numbers 0, 3, 5, 10, 12, and 15.

In class $\mathcal{C}(2; 0, 1)$ the permutive CA maps have code numbers 6 and 9 and there are no simply right- or left-permutive maps. In class $\mathcal{C}(3; -1, 1)$ there are four permutive maps, with code numbers 90, 105, 150, and 165; there are also eight simply RP and eight simply LP maps.

The results of the previous sections apply to the classes $\mathcal{C}(2; 0, 1)$ and $\mathcal{C}(3; -1, 1)$ in the following way:

(a) **Class $\mathcal{C}(2; 0, 1)$.** Within this class, by corollary 2.4, we have only an upper bound for $P(\tau)$. However, in this special case, $P(\tau)$ can be exactly calculated.

Let ν be the permutation

$$\nu = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix}$$

and define the map $k(b) : S_2 \to S_2$ by, $(k(b))_i = \nu(b_i)$, for all $i \in \mathbb{Z}$. The map k is a bijection over S_2. Let $\tau'^* = k^{-1} \circ \tau^* \circ k$. Obviously, τ'^* and τ^* are topologically conjugate. Denoting by λ and λ' the spectral radius of T_τ and $T_{\tau'}$, we have that $Per(\tau^{*n}) = Per(\tau'^{*n})$ and, with $\lambda^* = \min\{\lambda, \lambda'\}$, $P(\tau^*) = P(\tau'^*) \leq \lambda^*$. For nonpermutive CA maps, we have calculated explicitly the spectral radius of all the TTM and we obtained, using the previous argument, that $Per(\tau^{*n}) \sim c(n)$, where $c(n)$ is some polynomial in n with degree at most 3. Hence, we have: *If $\tau \in \mathcal{C}(2; 0, 1)$ is nonpermutive then $P(\tau) = 0$.*

When τ is permutive ($\gamma = 2$), corollary 2.6 does not apply but, in [6] (theorem 2.5) it was shown that for the map with code number 6, $Per(\tau^n) = 2^{n-2^{n'}}$, where n' is the largest natural number such that $2^{n'}$ divides n. So, for rule number 6, $P(\tau) = 1$. The permutive map with code number 9 is topologically equivalent to the map with code 6, via the bijection k, and the same result applies. According to the criterion of classification of complex patterns by $P(\tau)$, we have that rules in $\mathcal{C}(2; 0, 1)$ are divided in two classes, one corresponding to $P(\tau) = 1$ and the other to $P(\tau) = 0$. In fact, the observation of the patterns in the extended phase space, generated by the iteration of the CA maps in this class, confirms this conclusion.

(b) **Class $\mathcal{C}(3;-1,1)$.** In this class we have, if τ is permutive, by corollary 2.6 and theorem 3.1, $P(\tau) = h(\tau) = H(\tau,\mu) = 2$. For simply RP or LP maps we found $Per(\tau^n) \leq Trace(T_\tau^n) = 3^n$, independently of the map, implying, $P(\tau) \leq \log_2 3$. The estimate for the number of periodic points can be very crude for certain values of n. For example, in the LP map with code number 30, we have $Per(\tau) = 3$ and $Per(\tau^2) = 3$. For permutive maps the same calculations lead exactly to $Per(\tau^n) = 4^n$, improving theorem 2.5 of [6]. The patterns generated by the permutive rules in $\mathcal{C}(3;-1,1)$ show all the same qualitative behavior corresponding to Class 3 in Wolfram's classification. For other classes numerical evidence suggest that all permutive CA maps belong to Class 3.

As we have seen, for all classes of permutive CA maps the relation $P(\tau) = h(\tau)$ holds. So, we advance the conjecture that for any finite breadth cellular automaton the same equality is true. This would lead to good estimates for the topological entropy through the calculation of the number of periodic points of τ^n (see the comments at the end of Section 2).

In class $\mathcal{C}(3;-1,1)$ all permutive maps have the same number of periodic points of period n for all $n \geq 1$. This suggests that all permutive maps can be topologically equivalent. For classes $\mathcal{C}(2;m,m+1)$ this is indeed the case. For classes $\mathcal{C}(3;m,m+2)$, rules 90 and 165 and rules 105 and 150 are topologically equivalent via the map locally defined by the permutation $\nu : \{A_0,\ldots,A_7\} \to \{A_7,\ldots,A_0\}$.

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