# Local Graph Transformations Driven by Lyapunov Functionals 

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#### Abstract

We study the dynamical behavior of automata networks defined by $x(t+1)=x(t)+f(A x(t)+b)$; where $A$ is a symmetric $n \times n$ matrix, $b$ is a real $n$-vector and $f$ is the subgradient of a convex function. More precisely we prove, by using Lyapunov operators associated to the network, that the steady state behavior of these automata is simple: fixed points or two-cycles. We also give bounds for the transient time needed to reach the steady state. These networks appear in applications such as image restauration or phase unwrapping [6]. For this last application, we give bounds for the transient length.


## 1. Introduction

In this paper we characterize the dynamics of automata networks defined by

$$
\begin{equation*}
x(t+1)=x(t)+f(A x(t)+b) ; \quad x(t) \in \mathbb{R}^{n} \tag{1.1}
\end{equation*}
$$

where $A$ is a $n \times n$ symmetric matrix, $b$ a real $n$-vector, and $f\left(u_{1}, \ldots, u_{n}\right)=$ $\left(f_{i}\left(u_{i}\right)\right)$ from $\mathbb{R}^{n}$ into itself is the subgradient of a convex function from $\mathbb{R}^{n}$ into $\mathbb{R}$ called the potential associated to $f$. Transformation (1.1) may be seen as a network where each site is updated synchronously according to the following local rule:

$$
\begin{equation*}
x_{i}(t+1)=x_{i}(t)+f_{i}\left(\sum_{j=1}^{n} a_{i j} x_{j}(t)+b_{i}\right) \quad i=1, \ldots, n \tag{1.2}
\end{equation*}
$$

The characterization of the dynamics of automata (i.e., periodic behavior, transient length to reach the steady state, etc.) is a hard problem and few general results are known. For some particular classes there exist short cut theorems that permit us to determine the dynamic only by knowing the cellular space structure and the local rule.

[^0]The class defined in (1.1) was introduced first in [6] to modelize phase unwrapping problems, i.e., to compute the phase of an analytic function from a set of values of the function in a discrete grid. The automaton that accomplishes this task consists of a finite undirected graph $G$ where each site, labeled $1, \ldots, n$, initially contains a sample coded as an integer $x_{i}(0)$ of the principal value. All sites are updated synchronously according to a local average rule

$$
\begin{equation*}
x_{i}(t+1)=x_{i}(t)+f\left(S_{i}(t)\right) \quad \text { for } i=1, \ldots, n \tag{1.3}
\end{equation*}
$$

where

$$
S_{i}(t)=\sum_{j \in V_{i}}^{n} x_{j}(t)-d_{i} x_{i}(t)
$$

$V_{i}$ is the neighborhood of site $i$, defined by $V_{i}=\{j /$ site $j$ is connected to $i\}$, $d_{i}=\left|V_{i}\right|$ and $f$ is the local transition function:

$$
f\left(S_{i}(t)\right)=\left\{\begin{array}{cl}
-1 & \text { if } S_{i}(t)<0  \tag{1.4}\\
0 & \text { if } x_{i}(t)=x_{j}(t) \text { for any } j \in V_{i} \\
+1 & \text { otherwise }
\end{array}\right.
$$

or

$$
f\left(S_{i}(t)\right)=\left\{\begin{array}{cl}
-1 & \text { if } S_{i}(t)<0  \tag{1.5}\\
0 & \text { if } S_{i}(t)=0 \\
+1 & \text { otherwise }
\end{array}\right.
$$

The only difference between functions (1.4) and (1.5) is the tie-case (i.e., $\left.S_{i}(t)=0\right)$. In the former, the current state remains unchanged iff all the neighbors and the central cell have the same value; in the latter it remains unchanged for all configurations in the average local value.

As an example of the phase-unwrapping procedure, let a finite one-dimensional array of sites $\{1,, 2 .,,, n\}$ with neighborhood $V_{i}=\{i-1, i, i+1\}$ for $2 \leq i \leq n-1$ and $V_{1}=\{1,2\}, V_{n}=\{n-1, n\}$ and local transition function as in (1.3):

$$
\begin{aligned}
x_{i}(t+1)=x_{i}(t) & +f\left(x_{i-1}(t)+x_{i+1}(t)-2 x_{i}(t)\right) \text { for } 2 \leq i \leq n-1 \\
x_{1}\left(t+1^{\prime}\right)=x_{1}(t) & +f\left(x_{2}(t)-x_{1}(t)\right) \text { and } x_{n}(t+1)=x_{n}(t) \\
& +f\left(x_{n-1}(t)-x_{n}(t)\right)
\end{aligned}
$$

For the initial configuration (892357) we have for rule (1.4) and (1.5) respectively:

| Rule (1.4): | 892357 | Rule (1.5): | 892357 |  |
| :--- | :--- | :--- | :--- | :--- |
| 983466 |  |  | 983456 |  |
| 874556 |  | 874455 |  |  |
| 765465 |  | 765545 |  |  |
| 676556 |  | 666454 |  |  |
| 767665 | two-cycle |  | 656454 |  |
| 676756 |  | 565545 | two-cycle |  |
|  |  |  | 656454 |  |

Transformation (1.3) is a particular case of (1.1). Its periodic behavior was first studied by Odlyzko and Randall [7] by using a nondecreasing operator introduced in [1] in the context of neural networks:

$$
\begin{equation*}
E(x(t))=-\sum_{i, j=1}^{n} a_{i j} x_{i}(t) x_{j}(t-1) \tag{1.6}
\end{equation*}
$$

where the weights $a_{i j}$ correspond to the incidence matrix of graph $G$ :

$$
a_{i j}=\left\{\begin{array}{cl}
1 & \text { if }(i, j) \text { is an edge of } G  \tag{1.7}\\
-d_{i} & \text { if } i=j \\
0 & \text { otherwise }
\end{array}\right.
$$

The authors proved that expression (1.6) is nonincreasing for any trajectory of the phase unwrapping algorithm and, in doing so, that in the steady state there exists only fixed points or two-cycles. Unfortunately, $E(x(t))$ may remain constant in the transient phase which makes it impossible to use for bounding the transient length, i.e., the maximum number of steps to reach the steady state. Later, we have introduced in [5] a strictly decreasing operator for transformation (1.3) that roughly corresponds to expression (1.6), plus nonlinear terms. In this context we give bounds for the transient in $O(M e)$, where $M=\max _{i}\left|x_{i}(O)\right|$ and $e$ is the number of edges in the graph $G$. Our bounds are large and probably the optimal bound is $O(M n)$ but it seems to be a difficult problem. Finally, in [9] the author gives also bounds $O(\mathrm{Me})$ for the phase unwrapping transformation with a better constant term than in [5]. In order to do that he uses cyclically monotone functions in the framework of convex analysis.

In this paper we determine Lyapunov functionals for transformation (1.1) when the rule $f$ is the odd subgradient of a convex function and matrix $A$ is symmetric. Lyapunov operators permit us to characterize the periodic behavior of the network (only fixed points or two-cycles in the steady state) and to give a bound $O\left(M e^{2}\right)$ for the transient time. In the particular transformation (1.3), i.e., the phase unwrapping algorithm, with tie-rule (1.5), we determine a better constant for the $O(\mathrm{Me})$ bound of the transient time.

We also study the dynamic behavior of (1.1) when the connection matrix $A$ is positive-definite. In this context we determine a Lyapunov functional which drives any trajectory to fixed points. This strong hypothesis is interesting in applications of automata networks to modelize associative memories [?], i.e., given a finite set of patterns, to determine a network whose dynamics have these patterns as fixed points, and given an initial condition, the evolution converges to the nearest memorized pattern. In this case, $A$ is the correlation matrix between the discrete pattern to be memorized and satisfies our assumption.

## 2. Preliminaries

Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be the subgradient of a convex function $g: \mathbb{R}^{n} \rightarrow \mathbb{R}$, i.e.,

$$
g(u) \geq g(v)+\langle f(v), u-v\rangle \quad \forall u, v \in \mathbb{R}^{n}
$$

where $\langle$,$\rangle is the usual scalar product in \mathbb{R}^{n}$. In this context, $g$ is called the potential associated to $f$. We shall say that $f$ is strict if in (2.1) equality holds iff $f(u)=f(v)$.

It is not difficult to see that a sufficient (but not necessary) condition for strictness is for $g$ to be strictly convex. In the applications, the previous condition is too strong and we shall see that in the model of local graph transformation (1.2) we only need a weaker condition to ensure $f$ strict.

Some elementary properties of subgradients that we shall use in this paper are the following:

## Lemma 1.

1. $g(0)=0 \Longrightarrow\langle f(v), v\rangle \geq g(v) \quad \forall v \in \mathbb{R}^{n}$
2. $f(0)=0, g(0)=0 \Longrightarrow g(v) \geq 0 \quad \forall v \in \mathbb{R}^{n}$

Proof. Directly from inequality (2.1) applied to the couple of vectors $(0, v)$, $(v, 0)$ respectively.

We shall say that $f$ is odd iff $f(-u)=-f(u) \forall u \in \mathbb{R}^{n}$ (hence $f(0)=0$ ). Clearly if $g$ is a differentiable even potential then $f=\nabla g$ is its subgradient and it is odd, but in general a subgradient of an even potential is not necessarily odd.

Applications that are the subgradient of a convex function are characterized as cyclically monotone functions [10]. A particular class of cyclically monotone functions are the positive ones $[2,3]$, i.e., those which satisfy $\forall u, v \in \mathbb{R}^{n}\langle f(u)-f(v), u\rangle \geq 0$. This positivity property allows us to associate a quadratic Lyapunov operator to symmetric automata networks $[2,3]$. Other results of convex analysis can be seen in $[10]$ and applications to discrete iterations are given in $[2-4,8]$.

Finally, we define a cycle of transformation (1.1) as a finite sequence of vectors $(x(t))_{t=0}^{p-1}$ such that $x(t+1)=x(t)+f(A x(t)+b)$ for $0 \leq t \leq p-1$, where the indexes are taken module $p$, and $x(t) \neq x\left(t^{\prime}\right) \quad \forall t \neq t^{\prime}$, with $t, t^{\prime} \in\{0, \ldots, p-1\}$. In this context $p$ is called the period of the cycle. Also we define the transient length of transformation (1.1) as $T=\max _{x(0) \in \mathbb{R}^{n}}\{t: x(t)$ does not belong to a cycle\}. Obviously $T$ may be not bounded and we write $T=+\infty$, when iteration (1.1) is not bounded. In other cases, for instance in local graph transformations, initial conditions are taken in $\mathbb{Z}^{n}$ and (1.2) evolves in a finite set, so $T<\infty$.

## 3. Lyapunov functions

In this paragraph we associate to transformation (1.1) a strictly decreasing operator (called Lyapunov function) driving the dynamic of the network. We study two cases: when $f$ is the subgradient of a convex function $g$ and when $f$ is a positive function. Even though positive applications are also the subgradient of a convex functions (i.e. $g(x)=\langle x, f(x)\rangle$ ), the associated Lyapunov operator is simpler than in the general case.

### 3.1 Subgradient-potential analysis

Let $(f, g)$ be a couple subgradient-potential, we prove:
Theorem 1. Let $A$ be a symmetric $n \times n$ matrix, $b$ a real $n$-vector, and $f$ a strict odd subgradient of an even potential $g$. Then $E(u)=-\langle u, A u\rangle-$ $g(A u+b)-\langle 2 b, u\rangle, u \in \mathbb{R}^{n}$, is a Lyapunov function for transformation (1.1), i.e., for any trajectory $(x(t))_{t \geq 0}$ of (1.1) the sequence $(E(x(t)))_{t \geq 1}$ is decreasing and also satisfies $E(x(t))<E(x(t-1))$ iff $x(t-1) \neq x(t+1)$.

Proof. From expression (1.1) we get $E(x(t))=-\langle x(t-1)+f(A x(t-1)+$ b), $A x(t)\rangle-g(A x(t)+b)-\langle 2 b, x(t)\rangle$. Since $A$ is symmetric and $x(t)-x(t-2)=$ $f(A x(t-1)+b)+f(A x(t-2)+b)$ :

$$
\begin{aligned}
\Delta_{t} E= & E(x(t))-E(x(t-1)) \\
= & -\langle f(A x(t-1)+b), A x(t-1)+A x(t)\rangle \\
& -g(A x(t)+b)+g(A x(t-1)+b)-\langle 2 b, x(t)-x(t-1)\rangle
\end{aligned}
$$

but, since $x(t)-x(t-1)=f(A x(t-1)+b)$, we have
$\Delta_{t} E=-\langle f(A x(t-1)+b), A x(t-1)+b+A x(t)+b\rangle-g(A x(t)+b)+g(A x(t-1)+b)$
Let $u(t)=A x(t)+b$; hence,

$$
\Delta_{t} E=-\langle f(u(t-1)), u(t-1)+u(t)\rangle-g(u(t))+g(u(t-1))
$$

Since $f$ is a subgradient of $g, f$ is odd, $g$ is even: $\Delta_{t} E=-g(u(t))+$ $g(-u(t-1))+\langle f(-u(t-1)), u(t)-(-u(t-1))\rangle \leq 0$. Since $f$ is strict $\Delta_{t} E=0$ iff $f(u(t))=f(-u(t-1))$ and $f$ odd implies $f(u(t))+f(u(t-1))=0$. Finally, from definition of (1.1): $\Delta_{t} E=0$ iff $x(t+1)-x(t-1)=f(u(t))+f(u(t-1))=$ 0 and we conclude that $\Delta_{t} E<0$ iff $x(t+1) \neq x(t-1)$.

Corollary 1. Under the previous hypothesis, if a trajectory $(x(t))_{t \geq 0}$ of transformation (1.1) is ultimately periodic, then the period is either one or two.

Proof. Since $\Delta_{t} E<0$ iff $x(t-1) \neq x(t-1)$, in the cycle we have $x(t+1)=$ $x(t-1)$, that is, a fixed point or a two-cycle.

Remark. When $A$ is a nonsymmetric matrix, or $f$ not necessarily odd, we may have large periods. For instance take the following symmetric matrix:

$$
A=\left(\begin{array}{cc}
-1 & 1 \\
1 & -2
\end{array}\right)
$$

and $f\left(u_{1}, u_{2}\right)=\left(f_{1}\left(u_{1}\right), f_{2}\left(u_{2}\right)\right)$ defined by:

$$
f_{1}(x)=\left\{\begin{array}{cl}
-1 & \text { if } x<0 \\
0 & \text { if } x=0 \\
+1 & \text { if } x>0
\end{array} \text { and } f_{2}(x)=\left\{\begin{array}{cl}
-1 & \text { if } x<0 \\
0 & \text { if } x=0 \\
2 & \text { if } x>0
\end{array}\right.\right.
$$

Clearly, $f_{1}$ is the odd subgradient of $g_{1}(x)=|x|$ and $f_{2}$, which is not odd, is the subgradient of

$$
g_{2}(x)=\left\{\begin{array}{cl}
2 x & \text { if } x \geq 0 \\
-x & \text { if } x<0
\end{array}\right.
$$

It is easy to show that the transformation (1.1) admits the three-cycle: $(1,1) \rightarrow(1,0) \rightarrow(0,2)$.

### 3.2 Positive functions analysis

When the function $f$ is strictly positive (i.e. $\langle f(u)-f(v), u\rangle \geq 0 \forall u, v \in \mathbb{R}^{n}$ and equality holds iff $f(u)=f(v)$ or $u=0$ ) we may obtain a more compact Lyapunov function. In fact we have:

Theorem 2. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be a strictly positive odd function, $A$ a symmetric $n \times n$ matrix, and $b$ a real $n$-vector such that $b \notin\langle A \cdot j\rangle$ (the subspace generated by the columns of $A$ ).

Then $E(x(t))=-\langle x(t), A x(t-1)\rangle-\langle b, x(t)+x(t-1)\rangle$ is a Lyapunov function associated to (1.1).

Proof. Let $(x(t))_{t \geq 0}$ be a trajectory of (1.1). Since $A$ is symmetric:

$$
\Delta_{t} E=-\langle x(t)-x(t-2), A x(t-1)+b\rangle
$$

From definition of (1.1) and since $f$ is odd:

$$
\Delta_{t} E=-\langle f(A x(t-1)+b)-f(-(A x(t-2)+b)), A x(t-1)+b\rangle
$$

since $b \notin\left\langle A_{\cdot j}\right\rangle, A x(t-1)+b \neq 0$ and $f$ strict implies, similarly to the proof of theorem 1, $\Delta_{t} E \leq 0$ and $\Delta_{t} E<0$ iff $x(t) \neq x(t-2)$.

We shall see in the next section the application of the previous Lyapunov operators to characterize the periodic behavior and to obtain bounds for the graph transformation dynamics.

## 4. Application to the graph transformation dynamic

Let $G$ be a graph with the following incidence matrix:

$$
a_{i j}=\left\{\begin{array}{cl}
1 & \text { if } i \neq j \text { and }(i, j) \text { is an edge of } G  \tag{4.1}\\
-d_{i} & \text { if } i=j \text { where } d_{i} \text { is the degree of vertex } i \\
0 & \text { otherwise }
\end{array}\right.
$$

and the transformation:

$$
\begin{equation*}
x_{i}(t+1)=x_{i}(t)+f_{i}\left(\sum_{j=1}^{n} a_{i j} x_{j}(t)\right) \quad 1 \leq i \leq n \quad x(0) \in \mathbb{Z}^{n} \tag{4.2}
\end{equation*}
$$

where $f_{i}: \mathbb{R} \rightarrow \mathscr{Z}$ is an odd nondecreasing function such that $\left|f_{i}(x)\right| \leq\left\lceil\frac{|x|}{d_{i}}\right\rceil$ for $1 \leq i \leq n$.

Transformation (4.2) with the matrix $A$ defined in (4.1) is a generalization proposed in $[10]$ for the model of phase unwrapping $[6,8]$ and is a particular case of (1.1).

Transformation (4.2) is clearly integral and given an initial state $x(0) \in$ $\mathbb{Z}^{n}$ it evolves in the finite set $([-M, M] \cup \mathscr{Z})^{n}$ where $M=\max _{i}\left|x_{i}(0)\right|$.

Now, we shall give two useful lemmas for nondecreasing functions.

Lemma 2. Let $f: \mathbb{R} \rightarrow \mathbb{Z}$ be a nondecreasing odd function, $f \neq 0$, such that $|f(x)| \leq\left\lceil\frac{|x|}{d}\right\rceil, d \in \mathbb{Z}$. Then there exists an odd function $\tilde{f}: \mathbb{R} \rightarrow \mathbb{Z}$ verifying $\left.\tilde{f}\right|_{\mathbb{Z}}=\left.f\right|_{\mathbb{Z}},|\tilde{f}(x)| \leq\left\lceil\frac{|x|}{d}\right\rceil$. Furthermore, its discontinuity points are of the form $m \pm \frac{1}{2} ; m \in \mathbb{Z}$.

Proof. It suffices to prove this lemma in $\mathbb{R}_{+}$and define $\tilde{f}(x)=-f(-x)$ for $x<0$. Since $f \neq 0$ is a nondecreasing integral function, there exists an increasing sequence of natural numbers (eventually finite) $\left\{n_{i}\right\}_{i \geq 0}, n_{0}=0$, such that $f\left\{n_{i}, \ldots, n_{i+1}-1\right\}=\left\{f\left(n_{i}\right)\right\}$ and $f\left(n_{i}\right)<f\left(n_{i+1}\right)$. We define $\tilde{f}$ as follows: $\tilde{f}(x)=f(0)=0$ for $x \in\left[0, n_{1}-\frac{1}{2}\left[\right.\right.$ and, for $i \geq 1, \tilde{f}(x)=f\left(n_{i}\right)$ for $x \in\left[n_{i}-\frac{1}{2}, n_{i+1}-\frac{1}{2}\left[\right.\right.$. Clearly $\left.\tilde{f}\right|_{\mathbb{Z}}=\left.f\right|_{\mathbb{Z}}$ and $|\tilde{f}(x)| \leq\left\lceil\frac{|x|}{d}\right\rceil$. 트븐

Since the arguments of transformation (4.2) are integers the trajectories for functions $\left(\tilde{f}_{i}\right)$ are the same as those of $\left(f_{i}\right)$. Thus, without loss of generality, we shall suppose that the functions $\left(f_{i}\right)$ have their discontinuities at points $m \pm \frac{1}{2} ; m \in \mathbb{Z}$.

Also, it is not difficult to see that $g_{i}(u)=\int_{0}^{u} f_{i}(\xi) d \xi$ is an even potential associated to $f_{i}$ verifying $g_{i}(0)=0$. Furthermore, the functions $\left(f_{i}\right)$ defined in lemma 2 are strict on $\mathbb{Z}$; in fact we have the following result:

Lemma 3. Let $f: \mathbb{R} \rightarrow \mathbb{Z}$ be a nondecreasing odd function with discontinuity points $m \pm \frac{1}{2} ; m \in \mathbb{Z}$, then for any $u, v \in \mathbb{Z}: \theta(u, v)=g(u)-g(v)-$ $f(v)(u-v) \geq \frac{1}{2}$ iff $f(u) \neq f(v)$, where $g(x)=\int_{0}^{x} f(\xi) d \xi$ is the even potential associated to $f$.

Proof. Let us suppose $f(u)=f(v)$, since $(f, g)$ is a subgradient-potential couple $\theta(u, v) \geq 0$ and $\theta(v, u) \geq 0$. If one of the previous inequalities is strict then $\theta(u, v)+\theta(v, u)=(f(v)-f(u))(v-u)>0$. Since $f(u)=f(v)$ we conclude $\theta(u, v)=\theta(v, u)=0$ which is a contradiction with the fact that $\theta(u, v) \geq \frac{1}{2}>0$.

Let us suppose now $f(u) \neq f(v)$. Since $f$ is odd and $g$ is even, it suffices to analyze the case $u>v \geq 0, u, v \in \mathbb{Z}$. Since $f$ is nondecreasing, $f(u)>$ $f(v) \geq 0$, then

$$
\theta(u, v)=\int_{v}^{u} f(\xi) d \xi-f(v)(u-v)
$$

Since $u, v \in \mathbb{Z}$ and $f(u)>f(v)$ there exists at least a discontinuity point in $] v, u\left[\right.$. Let us take the bigger one, $\left.q-\frac{1}{2} \in\right] v, u[; v+1 \leq q \leq u ; q \in \mathbb{Z}$ then

$$
\int_{v}^{u} f(\xi) d \xi \geq f(v)\left(q-\frac{1}{2}-v\right)+f(u)\left(u-\left(q-\frac{1}{2}\right)\right)
$$

hence

$$
\theta(u, v) \geq(f(u)-f(v))\left(u-\left(q-\frac{1}{2}\right)\right) \geq \frac{1}{2}
$$

Clearly, the previous lemma ensures that the functions $\left(f_{i}\right)$ are strict. We may now characterize the dynamic behavior of the transformation (4.2):

Theorem 3. Let $A$ be the symmetric matrix defined in (4.1) and $\left(f_{i}\right)$ a collection of nondecreasing integral odd functions. Then the transformation (4.2) admits only fixed points or two-cycles.

Proof. Since for a given $x(0) \in \mathbb{Z}^{n}$, transformation (4.2) takes values in a finite set, any trajectory is ultimately periodic. Furthermore, as $f_{i}$ is a nondecreasing strict odd function, it is the subgradient of $g_{i}(x)=\int_{0}^{x} f_{i}(\xi) d \xi$. By defining $f(u)=\left(f_{i}\left(u_{i}\right)\right), g(u)=\sum_{i} g_{i}\left(u_{i}\right)$ is the potential associated to $f$. We obtain the result directly from theorem 1 and corollary 1 .

Corollary 2. The phase unwrapping transformations (1.3) with rules (1.4) or (1.5) admits only fixed points or two-cycles.

Proof. For (1.3) with rule (1.4), we have

$$
x_{i}(t+1)=x_{i}(t)+f\left(\sum_{j \in V_{i}} x_{j}(t)-d_{i} x_{i}(t)\right)
$$

with $f(u)=-1$ if $u<0,0$ if $u=0$ and 1 if $u>0$. Since the argument of $f$ is integral, it is easy to see that

$$
f(u)=\left\{\begin{array}{cl}
-1 & \text { if } u<-\frac{1}{2} \\
0 & \text { if }-\frac{1}{2}<u<\frac{1}{2} \\
+1 & \text { otherwise }
\end{array}\right.
$$

hence $f$ is strict (see lemma 3). Thus, $E(x(t))$ defined in theorem 1 is a Lyapunov functional with threshold $b=0$.

Now, for (1.3) with rule (1.5) it was proved in [7] that for any step $t$ in the periodic phase and for any site $i: x_{i}(t+1) \neq x_{i}(t)$, which implies that the argument of $f$ never vanishes. Hence, rule (1.5) coincides with (1.4) in steady state. We then conclude that in the periodic phase $E(x(t))$ is also a Lyapunov functional for (1.3) with rule (1.5). Finally, the two periodic behavior holds directly from previous comments and corollary 1 .

Also we may study the transient length, $T$, of transformation (4.2):

Theorem 4. Let $A$ be the matrix defined in (4.1) and let $\left(f_{i}\right)$ be a set of odd nondecreasing strict integral functions. Then the Lyapunov function associated to the transformation (4.2)

$$
E(x(t))=-\langle x(t), A x(t)\rangle-g(A x(t))
$$

is bounded by

$$
|E(t)| \leq M^{2}\|A\|_{1}=4 M^{2} e, \text { for } t \geq 1
$$

where $M=\max _{i}\left|x_{i}(0)\right|,\|A\|_{1}=\sum_{i, j}\left|a_{i j}\right|$ and $e$ is the number of edges in the graph $G$.

Proof. As in the proof of theorem 1, we have

$$
E(x(t))=-\langle x(t-1), A x(t)\rangle-\langle f(A x(t-1)), A x(t)\rangle-g(A x(t))
$$

where $g\left(u_{1}, \ldots, u_{n}\right)=\sum_{i=1}^{n} g_{i}\left(u_{i}\right)$ is the potential of $f(u)=\left(f_{1}\left(u_{1}\right), \ldots, f_{n}\left(u_{n}\right)\right)$. Since $g_{i}(x)=\int_{0}^{x} f_{i}(\xi) d \xi, g_{i}(0)=0 \forall i=1, \ldots n$; from lemma 1, property 1 , we have $g(A x(t)) \leq\langle f(A x(t)), A x(t)\rangle$; and from the definition of the transformation (4.2), $x(t+1)-x(t-1)=f(A x(t))+f(A x(t-1))$ we get

$$
E(x(t)) \geq-\langle x(t+1), A x(t))\rangle
$$

On the other hand, since $E(x(t))=-\langle x(t), A x(t)\rangle-g(A x(t))$ and $f(0)=$ 0 , from lemma 1 , property 2 , we obtain $E(x(t)) \leq-\langle x(t), A x(t)\rangle$; hence,

$$
-\langle x(t+1), A x(t)\rangle \leq E(x(t)) \leq-\langle x(t), A x(t)\rangle \text { for } t \geq 1
$$

and thus $\mid E\left(x(t) \mid \leq M^{2}\|A\|_{1}=4 M^{2} e\right.$ for $t \geq 1$.
Corollary 3. By the previous hypothesis, the transient length of transformation (4.2) is bounded by

$$
T \leq 4 M^{2} \cdot\|A\|_{1}+1=16 M^{2} e+1
$$

Proof. It suffices to point out that $\forall t \geq 1$ such that $x(t+1) \neq x(t-1)$, $\left|\Delta_{t} E\right| \geq \frac{1}{2}$ (lemma 3). The result follows directly from theorem 4.

Corollary 4. Let $G$ be an undirect connected graph. Then, the transient length for the phase unwrapping transformation (1.3) with local rule (1.5) is bounded by $T \leq 16 M e+2 M-3 \leq 18 M e$, where $e$ is the number of edges in $G$ and $M=\max _{i}\left|x_{i}(0)\right|$.

Proof. As in the proof of corollary 2, we may consider the equivalent function $f$ with discontinuity points $\pm \frac{1}{2}$, which is strict. Thus $E(x(t))$ is a Lyapunov functional for (1.3) with $b=0$ and connection matrix (4.1).

Let $\Omega(t)=\max \{\langle x(t), A x(t)\rangle,\langle x(t+1), A x(t)\rangle\}$. Similar to the proof of theorem $4,|E(x(t))| \leq \Omega(t)$. On the other hand, it is easy to see that for any site $i$,
(a) $\left|S_{i}(t)\right| \leq 2 d_{i} \Rightarrow\left|S_{i}(t+1)\right| \leq 2 d_{i}$
(b) $\left|S_{i}(t)\right|>2 d_{i} \Rightarrow\left|x_{i}(t)\right|<M-2$

From (b) it is direct that for $t \geq 2 M-3,\left|S_{i}(t)\right| \leq 2 d_{i}$ and from (a) we conclude that this property remains in time. Then, for $t \geq 2 M-3$,

$$
|E(x(t))| \leq \Omega(t) \leq 2 M \sum_{j=1}^{n} d_{i}=4 M e
$$

On the other hand, since the function $f$ is strict with discontinuity points $\pm \frac{1}{2},\left|\Delta_{t} E\right| \geq \frac{1}{2}$, then we conclude $T \leq 16 \mathrm{Me}+2 \mathrm{M}-3 \leq 18 \mathrm{Me}$.

## 5. Positive-definite matrices

Let $A$ be a positive-definite symmetric matrix; hence, $A={ }^{t} R R$. Let $f$ be a subgradient of a convex function $g$ and $b$ a real $n$-vector. For any trajectory $(x(t))_{t \geq 0}$ of transformation (1.1), we define

$$
\begin{aligned}
E(x(t)) & =\langle x(t), A x(t-1)\rangle-g(A x(t)+b)-g(A x(t-1)+b)- \\
& -\sum_{i=1}^{n} \frac{1}{2}\left(\sum_{j=1}^{n} r_{i j} x_{j}(t)\right)^{2}-\sum_{i=1}^{n} \frac{1}{2}\left(\sum_{j=1}^{n} r_{i j} x_{j}(t-1)\right)^{2}
\end{aligned}
$$

where $R=\left(r_{i j}\right)$ such that $A={ }^{t} R R$.
Theorem 5. With the previous hypothesis $E(x(t))$ is a Lyapunov function associated to the transformation (1.1).

Proof. Clearly $\Delta_{t} E=\langle A x(t)-A x(t-2), x(t-1)\rangle-g(A x(t)+b)+g(A x(t-$ $2)+b)-\sum_{i=1}^{n} \frac{1}{2}\left(\sum_{j=1}^{n} r_{i j} x_{j}(t)\right)^{2}+\sum_{i=1}^{n} \frac{1}{2}\left(\sum_{j=1}^{n} r_{i j} x_{j}(t-2)\right)^{2}$.

From transformation (1.1) and by defining $u(t)=A x(t)+b$ and $v(t)=$ $R x(t)$ we have

$$
\begin{aligned}
\Delta_{t} E & =\langle u(t)-u(t-2), f(u(t-2))\rangle-g(u(t))+g(u(t-2)) \\
& +\langle v(t)-v(t-2), v(t-2)\rangle-\sum_{i=1}^{n} \frac{1}{2} v_{i}^{2}(t)+\sum_{j=1}^{n} \frac{1}{2} v_{i}^{2}(t-2)
\end{aligned}
$$

since $g$ is the potential of $f$ :

$$
\begin{aligned}
\Delta_{t} E & \leq\langle v(t)-v(t-2), v(t-2)\rangle-\sum_{i=1}^{n} \frac{1}{2} v_{i}^{2}(t)+\sum_{i=1}^{n} \frac{1}{2} v_{i}^{2}(t-2) \\
& \leq \sum_{i=1}^{n}\left\{\left(-\frac{1}{2} v_{i}^{2}(t)+\frac{1}{2} v_{i}^{2}(t-2)+v_{i}(t-2)\left(v_{i}(t)-v_{i}(t-2)\right)\right.\right.
\end{aligned}
$$

Since $x$ is the gradient of the strictly-convex function $\frac{1}{2} x^{2}$ and as $A$ is invertible, we conclude

$$
\Delta_{t} E \leq 0 \quad \text { and } \quad \Delta_{t} E<0 \text { iff } x(t) \neq x(t-2) .
$$

Corollary 5. Let $A$ be a positive-definite symmetric matrix. Then, if a trajectory of (1.1) is ultimately periodic, the period is one (a fixed point).

Proof. Clearly from the previous theorem, the cycles if they exist are of period one or two. Let $\{x(0), x(1)\}$ be a limit cycle. Since $A$ is a positivedefinite matrix, let

$$
\begin{aligned}
0 \leq \gamma= & \langle x(0)-x(1), A(x(0)-x(1))\rangle= \\
= & \langle x(0),(A x(0)+b)-(A x(1)+b)\rangle+\langle x(1),(A x(1)+b) \\
& -(A x(0)+b)\rangle
\end{aligned}
$$

Now, from the definition of (1.1),

$$
\begin{aligned}
\langle x(t), A x(t) & -(A x(t-1)\rangle=\langle R x(t-1), R x(t)-R x(t-1)\rangle+ \\
& +\langle f(A x(t-1)+b),(A x(t)+b)-(A x(t-1)+b)\rangle \\
& \leq \frac{1}{2} \sum_{i=1}^{n}\left(\sum_{j=1}^{n} r_{i j} x_{j}(t)\right)^{2}-\frac{1}{2} \sum_{i=1}^{n}\left(\sum_{j=1}^{n} r_{i j} x_{j}(t-1)\right)^{2} \\
& +g(A x(t)+b)-g(A x(t-1)+b)
\end{aligned}
$$

since $(x(0), x(1))$ is a limit cycle:

$$
\begin{aligned}
\gamma & \leq \frac{1}{2} \sum_{i=1}^{n}\left(\sum_{j=1}^{n} r_{i j} x_{j}(0)\right)^{2}-\frac{1}{2} \sum_{i=1}^{n}\left(\sum_{j=1}^{n} r_{i j} x_{j}(1)\right)^{2}+g(A x(0)+b)- \\
& -g(A x(1)+b)+\frac{1}{2} \sum_{i=1}^{n}\left(\sum_{j=1}^{n} r_{i j} x_{j}(1)\right)^{2}-\frac{1}{2} \sum_{i=1}^{n}\left(\sum_{j=1} r_{i j} x_{j}(0)\right)^{2}+ \\
& +g(A x(1)+b)-g(A x(0)+b)=0
\end{aligned}
$$

as $A$ is a positive-definite matrix we conclude $x(0)=x(1)$.

Remark. It is interesting to point out that, in this case, the results hold for any cyclically monotone function $f$ (not necessarily odd).

## 6. Conclusions

We have determined Lyapunov functionals for a class of automata networks. This fact is important because these functionals drive the network dynamics and allow us to characterize the steady state and the transient behavior: very short periods (one or two) and polynomial transient time. The application of our approach to the phase unwrapping gives better bounds for the convergence of the algorithm in case of tie (1.5). Unfortunately, this approach is not powerful enough to study the convergence time for the tie rule (1.4), but we conjecture that the two tie rules have a similar behavior.

It is also important to point out that the symmetry assumption for the connection graph $G$ is crucial to obtain Lyapunov operators. If not, it is easy to built networks with nonbounded periods in the size of graph $G$.

Finally, the class studied here may be seen as a generalization of neural networks, where only two states are possible and also the synchronous update admits a Lyapunov functional [1,2].

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