Complex Systems **3** (1989) 457–470

Stochastic Stability of Nonsymmetric Threshold Networks

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Abstract. For a nonsymmetric threshold network equipped with an asynchronous dynamics, we show that if the product of weights in any cycle of units is nonnegative, then each trajectory converges to a stable state with probability one. We also show that such networks have a natural feed-forward layer structure and stability is achieved in a hierarchical order due to this structure. It follows then that this new class of networks can perform similar tasks as symmetric networks as well as new tasks due to its relation to directed graphs.

1. Introduction

Discrete dynamical systems of threshold type received a lot of attention recently. The main assumption which is usually made is that of symmetry of the system, i.e. the interactions between units ("neurons" or "spins") are assumed symmetric. This assumption is natural for physical interpretations of these models (spin glasses); however, it is totally unnatural as far as their interpretations for physiological neural networks are concerned. There is no evidence of any symmetry of interactions of real neurons. Also, the symmetry assumption implies indirected graph of interactions and makes such networks a natural tool for studying combinatorial optimization problems (cf. e.g. Hopfield and Tank [7] for the traveling salesman problem and Peterson and Anderson [9] for the graph bisection problem). However, it seems that combinatorial problems which lead to directed graphs are not natural in this setting.

The main feature of the symmetric Hopfield model [6] which makes it a useful tool for combinatorial optimization, as well as a model for content addressable memory, is that the system converges to equilibrium after a finite number of switches. Nonsymmetric asynchronous systems may oscillate, in general, and general conditions for their stability seem not to be known so far.

In this paper we study a class of *nonsymmetric* threshold networks which have similar stability properties. More precisely, we consider systems with the property that any trajectory of the system converges to an equilibrium with probability one. We show that this class includes systems characterized by the condition that the product of weights of interactions of any directed cycle of units (neurons) is nonnegative. We call such systems nonfrustrated, or balanced. This class of systems has many nice properties and has a far richer structure than the class of symmetric systems.

Stability analysis of symmetric synchronous networks is well summarized in Bruck and Goodman [2] and leads to cycles of length at most two (cf. Goles and Olivos [5], Poljak [10], Odlyzko and Randall [8]). The main tool used there as well as for asynchronous symmetric systems (cf. Hopfield [6], Goles-Chacc, Fogelman-Soulie, and Pellegrin [3]) is Lyapunov functions. The same technique cannot be applied directly in our context as our systems converge in stochastic sense only and they may not posess any Lyapunov function. On the other hand, stability properties of balanced systems are more robust with respect to perturbations of the weights (and thresholds) of the system, comparing to stability of symmetric systems under arbitrary perturbations

In section 2 we state our main results. In section 3 we show that any balanced system has a natural layer, feed-forward structure. More precisely, by a change of coordinates the system can be brought to a cascade of subsystems (with a directed acyclic graph of interconnections), each of them having non-negative weights of interactions but with arbitrary interconnections between subsystems in the cascade.

In section 4 we prove the main results showing that stability is achieved hierarchically according to the layer feed-forward structure.

2. Statement of the results

Let $G = (V, E)$ be a directed graph, where $V = V_1, \ldots, V_n$ is a collection of vertices of the digraph, called units, and $E \subset V \times V$ is a collection of directed edges. For simplicity, we will often denote the ith edge V_i by i. The edges and the vertices of the graph will be assigned labels. Namely, $W = \{W_{ij}\}_{(i,j)\in E}$ will be a collection of real numbers called weights, and $T = \{T_1, \ldots, T_n\}$ will be another collection of real numbers called thresholds. Any triple $N = (G, W, T)$ will be called a *threshold network* or simply a network.

For most of this paper, without loss of generality one may assume that G is the full digraph. We will always consider the full matrix of weights $W = \{W_{ij}\}_{i=0,j=0}^{i=n,j=n}$, where we define $W_{ij} = 0$ if $(i,j) \notin E$. The *effective digraph* of the network N will be the digraph $G_{eff} = (V, E_{eff})$, where $(i,j) \in E_{eff}$ if and only if $W_{ij} \neq 0$.

We shall introduce the following asynchronous dynamics in our threshold network N, which can be considered as a generalization of the dynamics used by Hopfield [6]. We assume that time is discrete, $t = 0, 1, 2, \ldots$. Each unit V_i, $i = 1, \ldots, n$, is supposed to be in one of the finite number of possible states which are real numbers: $S_i^1 < \cdots < S_i^r$ and the number of possible states $r = r_i \geq 2$ may depend on the unit. The state of the jth unit at time t

is denoted by $x_j(t)$. If $r_i = 2$ for all i then we call the network the two-state network.

The network is supposed to evolve in time in one of the following two possible modes. In the *serial mode*, at each particular instant t one of the units $V_1, \dots, V_n$ is chosen at random, with probability $p_1 > 0, \dots, p_n > 0$, for updating his state (we assume that $p_1 + \cdots + p_n \leq 1$). In the *parallel mode*, in each step every unit updates independently of the others with probability $0 < p_1 < 1, \dots, 0 < p_n < 1$. In each mode, the updating of the jth unit takes place according to the following rule based on the sign of the input to this unit, where the input to the jth unit is defined as

$$I_j = \sum_{i=1}^{n} W_{ij} x_i(t) - T_j. \tag{2.1}$$

If this input is positive then the updating unit changes its state to its next larger possible state, and if it is negative, the unit takes its next possible smaller state. If such next larger (smaller) states do not exist or if the input is zero, the updating unit remains in its present state. (We will think of the thresholds as "external inputs" which are fixed during the evolution of the network; that is why we include them in the above sum.)

A threshold network with the above dynamics will be called a *dynamic threshold network*. We will often say that a unit V_j "wants to switch" or "is unhappy" if, once chosen for updating, it can change its state. Otherwise, we will say that the unit is "happy."

If there is no unit which wants to switch, then we say that the network is in an equilibrium state. If at a particular instant t a unit changes its state, we will say that this unit switches or that the network switches. Note that a given initial state of the network does not determine its further behavior (trajectory) uniquely and the trajectory may depend on the order in which the units are chosen for updates.

In order to analyze stability of dynamic threshold networks, we should choose a proper definition of stability. The following example is helpful.

Consider a network where the effective graph is a directed cycle of k units with zero thresholds and all weights equal to one. Assume that all units may take only two possible values $+1$ and -1 as their states and consider the serial mode. It is easy to see that if $k \geq 3$, this network can oscillate indefinitely. In fact, assume that at the initial state all units but one take value $+1$. In the first step, the unit next to the one with value -1 can switch to -1, while the one previously in the state -1 can switch to $+1$ in the following step. This process can repeat again and again so that the "negative disturbance" propagates indefinitely around the cycle and we have infinite oscillations. However, it can be easily seen that the probability that the units will be chosen for updating in the required order (for infinite number of steps) is zero. Therefore, oscillations are possible but one should expect that they are of probability zero. Similar phenomena can occur in the parallel mode for $k \geq 2$. We would like to ignore such inessential oscillations in our considerations.

We will call a dynamic threshold network *almost surely* stable if, with probability one, each trajectory converges to an equilibrium state after a finite number of switches. More precisely, this means that the oscillating trajectories are of measure zero, if we consider the natural measure on the set of trajectories which is constructed via the Kolmogorov's theorem. (Note that our dynamic threshold network defines a Markov chain.)

The problem of characterizing almost sure stability of the network in terms of its weights and thresholds is difficult and will not be discussed here. However, often it is more reasonable to ask whether the network is stable independent of the values of the thresholds. For example, in the context of neural networks our thresholds incorporate external inputs which are assumed constant during the considered evolution of the network. Their values may change after long intervals of time, or they may be unknown, but we would like to have stability of the network independent of their values. Therefore, the following definition is natural.

We will call a dynamic threshold network *internally stable* if it is almost surely stable for any values of the thresholds.

We will say that a cycle of units $V_{i_1}, V_{i_2}, \ldots, V_{i_k}, V_1$, $k \geq 1$, is *balanced* or *nonfrustrated* if the following condition holds:

$$W_{i_1 i_2} W_{i_2 i_3} \cdots W_{i_{k-1} i_k} W_{i_k i_1} \geq 0. \tag{2.2}$$

If the above product is negative, we call the above cycle *frustrated*. We will say that our threshold network is *balanced* if any cycle of units in this network is balanced. Any network with nonnegative weights is balanced, and so is any acyclic network (a network with acyclic effective graph).

For cycles of length $k = 1$ (self-loops) our condition reads $W_{ii} \geq 0$, and for cycles of length two it says that

$$W_{ij} W_{ji} \geq 0, \quad i, j = 1, \ldots, n. \tag{2.3}$$

Our main result is the following.

Theorem 1. *Any balanced dynamic threshold network in serial or parallel mode is internally stable.*

The following corollaries follow immediately from theorem 1.

Corollary 1. *If either all the weights of a dynamic threshold network are nonnegative or they are all nonpositive and all the cycles in the effective graph are of even length, then the network is internally stable.*

Corollary 2. *Consider a network given by a dynamic threshold network with the graph G being a subgraph of the regular n-dimensional Euclidean lattice. If all the nonzero weights of this network are of the same sign, the network is internally stable.*

The class of balanced networks is insensitive with respect to some changes of its parameters. Suppose that the parameters of the network, W_{ij} and T_i, are subject to change but the effective digraph of the network remains constant. In other words, we assume a constant digraph $G = (V, E)$ of connections with nonzero weights W_{ij}, $(i, j) \in E$. In such a natural class of networks, the condition (2.2) is stable under small time-independent perturbations of the weights and any time-independent perturbations of the thresholds. Therefore, the almost sure stability of the network is preserved under such perturbations. This property does not hold for symmetric dynamic threshold networks, as shown by the following example.

Consider a network which consists of three units with the weights $W_{12} = W_{23} = -W_{31} = a$, $W_{21} = W_{32} = -W_{13} = b$, and $W_{11} = W_{22} = W_{33} = 0$. Assume that all the thresholds are zero. If $a = b > 0$, then this is a symmetric dynamic threshold network and so it is stable. However, if $a = b+\epsilon$ with $\epsilon > 0$ small but nonzero, then the network will start to oscillate as the connections in one direction of the 3-cycle will dominate over the connections in the other direction (note that both cycles are frustrated).

The following example analyzes stability of a single directed cycle. We will say that the threshold T_i is *blocking* if there is a state of the ith unit in which this unit is happy, independent of the states of the other units.

Suppose that the effective graph of our network is a single directed cycle. This network is almost surely stable if and only if the cycle is balanced or at least one of its thresholds is blocking. To see this, let us consider two consecutive units i and j in our directed cycle and consider the corresponding nonzero weight W_{ij}. It is clear that the jth unit is not blocking if and only if the number T_j lies strictly between the numbers $W_{ij}S_i^{\min}$ and $W_{ij}S_i^{\max}$, where $S_i^{\min}$ and $S_i^{\max}$ are the minimal and maximal values of the state of the ith unit. If we update the units consecutively around the cycle until they reach their happy states, during the second round it is necessary for a switch of a particular unit that the previous unit switched in order that the input to the given one changed. Evidently, if one of the thresholds is blocking, then its unit can only switch in one direction (upward or downward) and the switchings cannot propagate indefinitely through this unit. If there is no blocking threshold, and switchings propagate around the whole cycle, then the change ΔS_i of the ith state depends on the change of the previous unit so that $\operatorname{sgn}(\Delta S_i) = \operatorname{sgn}(W_{i-1i}\Delta S_{i-1})$. Writing such equalities for each edge in the cycle and substituting one into another, we find after the complete round that the sign of the change of the ith unit after the round should be equal to the sign of its change at the beginning of the round times the sign of the product of the weights around the cycle. If this product is nonnegative, there cannot be any change of the first unit after the second round. This means that we have reached an equilibrium. It is not difficult to see (and it will follow from our next results) the implication that oscillating trajectories are of probability zero for such a balanced cycle. However, if our cycle is frustrated, then the sign of the change of the first unit after the round is opposite to the sign at the beginning of the round and the change can propagate again and again, leading to oscillations.

To formulate a converse result, we need more definitions. Suppose a sequence $i_1, \ldots, i_k$ defines a directed cycle in the effective digraph of the network (by definition, such a cycle consists of the sequence of different units $V_1, \ldots, V_k$ and the sequence of directed edges joining consecutive units and the edge joining the last unit to the first one). An *essentially frustrated* cycle will be any frustrated cycle $i_1, \ldots, i_k$ with the following property:

$$\forall 1 \le p \le k \sum_{q \ne p-1} |W_{i_q i_p}| \Delta S_{i_q} < |W_{i_{p-1} i_p}| \Delta S_{i_{p-1}}, \tag{2.4}$$

where the indices p and q in the sum are taken modulo k (i.e., k is identified with 0) and $\Delta S_i = S_i^{\max} - S_i^{\min}$. Equivalently, an essentially frustrated cycle is a directed frustrated cycle of units such that for any unit in the cycle, the possible values of the input to the unit have the following separation property. For any fixed state of the units outside the cycle, the values of the input to this unit given by all the possible extreme states of units in the frustrated cycle can be strictly separated by a real number into two groups so that in each of these groups the unit preceding the given unit in the cycle takes only one of its extreme values (in other words, the extreme state of the preceding unit decides to which of the two separated groups the value of the input belongs).

Theorem 2. *If a dynamic threshold network is internally stable, then it does not contain any essentially frustrated cycle.*

The following is an example of a minimal frustrated cycle which is not essentially frustrated (we will assume that units have only two possible states $+1$ and -1). The network N_1 consists of three units 1, 2, 3, and the only nonzero weights are $W_{12} = 1/2$, $W_{23} = W_{32} = -W_{31} = 1$. It is easy to see that this network is internally stable and, unless there is a blocking threshold, the couple (both ways interacting units 2 and 3) will be in the same state in equilibrium.

It is also possible that a network may oscillate even if it does not contain any essentially frustrated cycle. Consider the network which is constructed from three distinct copies of N_1 (described above) connected together into a cycle so that the second unit of the first copy of N_1 is identified with the first unit of the second copy of N_1, the second unit of the second N_1 is identified with the first unit of the third N_1, and the second unit of the third N_1 is identical with the first unit of the first N_1. It is easy to see that this network (with zero thresholds) has two stable equilibria: all units in the state $+1$, and all units in the state -1. On the other hand, if in the initial state one of the three couples (two units joined by edges both ways) is in one state (say, $+1$) and the other two couples are in the other state (-1, respectively), then the network will oscillate indefinitely.

The above example can be generalized to k copies of the network N_1 connected into a cycle, analogously as above. Then, a straightforward analysis of such a network shows that there are two or four stable equilibria: all units in the same state (two possible), all couples in consecutively changing states $+1$, -1, $+1$, -1, and so on (the number of copies of N_1 must be even and then there are two such equilibria). However, if we start from a state in which two consequtive couples are in the same state, then the network will oscillate indefinitely. Additionally, such a network has an interesting memory-like property. Namely, even if it oscillates indefinitely, it preserves the number of groups of $+1$'s and -1's in consecutive couples.

In the following two sections we show how balanced networks achieve equilibrium.

3. Feed-forward layer structure

Any threshold network has an inherent hierarchical layer structure which depends on its matrix of weights $W = \{W_{ij}\}$, more precisely on the distribution of zeros in this matrix (i.e., on the effective digraph G_{eff} of the network.) In networks in which the matrix of weights changes in time and therefore the effective digraph changes, one can define an analogous hierarchical structure for the digraph of the network G.

Consider a threshold network N and its effective digraph G_{eff}. Define the following quasi-order in the set of vertices. We will say that a unit i *influences* a unit j (or i precedes j) if there is a path from i to j in the effective digraph. If the unit i influences j, and vice versa, then we shall say that both are in the same *influence class*. The relation of belonging to the same influence class is an equivalence relation, so the set of units splits into disjoint influence classes. The influence classes of a threshold network N will sometimes be called *causal layers* and will be denoted by $L_1, \ldots, L_m$.

There is a natural partial order (induced by the quasi-order of "influence") in this set of influence classes $L_1, \ldots, L_m$. Namely, we say that L_p influences L_q if there a unit in L_p which influences a unit in L_q. Note that we obtain in this way a new acyclic digraph $G_c = (V_c, E_c)$, where the set of vertices V_c consists of influence classes of N and an edge (L_p, L_q) is in E_c if and only if L_p influences L_q.

We call a threshold network (or its matrix of weights W) *strongly connected*, if each unit in it influences any other unit. Influence classes are strongly connected.

It is an elementary fact in graph theory that in any acyclic digraph one can reorder the vertices so that if an edge (p, q) belongs to the graph, then $p < q$. In our case, this means that, after reordering the influence classes $L_1, \ldots, L_m$, we obtain that if L_p influences L_q then $p < q$. It is easy to define this reordering. Namely, take as the first class any class L_p which is not influenced by any other class. Then, remove this class from the set of influence classes. Take as the second class any of the remaining influence classes which is not influenced by any one of them. Then, remove this class from the considered set of classes. Continue this procedure until all the classes are removed. This argument leads to the following elementary fact which is stated for further reference.

Proposition 1. *For any matrix W there exists a reordering of the variables (units) such that this matrix takes a block lower triangular form, with strongly connected blocks on the diagonal.*

The above hierarchy of a network is clearly essential in studing stability of the network. Basically, the problem reduces to studying stability of the causal layers as the hierarchical connection between them does not produce instability (there are no cycles in between the causal layers).

More can be said about the structure of balanced threshold networks, more precisely, about matrices W which satisfy the condition (2.2).

Theorem 3. *A matrix W is balanced (i.e., satisfies the condition 2.2) if and only if there is a reordering of variables (units) and a change of their signs $x_i \longrightarrow \epsilon_i x_i$, $\epsilon_i = \pm 1$, $i = 1, \ldots, n$ which brings this matrix to a block lower triangular form, with the blocks on the diagonal strongly connected and with nonnegative coefficients.*

Proof. It is clear that condition (2.2) is satisfied for a block lower triangular matrix with nonnegative coefficients in the blocks on the diagonal. Namely, each product (2.2) which contains only coefficients in a block on the diagonal satisfies (2.2). If the product (2.2) has a coefficient which is not in any block on the diagonal, it must also have a coefficient above the diagonal blocks and so is equal to zero.

To show the converse fact, we assume that condition (2.2) is satisfied and we use proposition 1 to get W in a block lower triangular form. It is enough to show additionally that if the matrix W is balanced then after a possible change of signs of the variables we obtain nonnegative coefficients in diagonal blocks. This follows from lemma 1.

Lemma 1. *If the matrix W is balanced and strongly connected, then there is a "change of coordinates"*

$$x_i \to \epsilon_i x_i, \quad i = 1, \ldots, n, \quad \epsilon_i = \pm 1, \tag{3.1}$$

which makes all the weights nonnegative: $W_{ij}\epsilon_i\epsilon_j \geq 0$.

Proof. Let us take the effective graph of the network $G_{\mathit{eff}} = (V, E_{\mathit{eff}})$ and define the following numbers

$$S_{ij} = \operatorname{sgn} W_{ij}\ , \quad (i,j) \in E_{\mathit{eff}}. \tag{3.2}$$

As each cycle of units in G_{eff} is balanced, it follows that the following cocycle condition is satisfied for any cycle of units

$$S_{i_1 i_2} S_{i_2 i_3} \cdots S_{i_{k-1} i_k} S_{i_k i_1} = 1. \tag{3.3}$$

From an elementary cohomology argument it follows that there are numbers $\epsilon_1 = \pm 1, \ldots, \epsilon_n = \pm 1$ such that

$$S_{ij} = \frac{\epsilon_i}{\epsilon_j} = \epsilon_i \epsilon_j, \quad (i,j) \in E_{\mathit{eff}}. \tag{3.4}$$

Explicitly, the construction goes as follows. We pick up a unit, say 1, and define $\epsilon_1 = 1$. For any unit i there is a path $1 = i_1, \ldots, i_k = i$ from 1 to i in the effective digraph (as the network is strongly connected). We define ϵ_i as equal to the sign of the product of weights corresponding to this path

$$\epsilon_i = \operatorname{sgn}(W_{i_1 i_2} \ldots W_{i_{k-1} i_k}). \tag{3.5}$$

Since our network is balanced and strongly connected, it follows that this definition is independent of the path joining 1 to i. (If these signs were different for two such paths, we would pick a path from i to 1 which together with the previous two paths would give two cycles with different signs of the corresponding products of weights. This would contradict the fact that the network is balanced.)

It is evident from the definition of S_{ij} that the change of coordinates obtained with defined numbers $\epsilon_1, \ldots, \epsilon_n$ transforms the weights of the network into nonnegative numbers. ∎

4. Stability of balanced networks

We will show how a balanced dynamic threshold network converges to equilibrium by stating several facts and their justifications in the form of lemmas and their proofs.

Lemma 2. *If the network has nonnegative weights, then there is at least one stable equilibrium and there is a passage of nonzero probability from any initial state to an equilibrium. In the case of two-state network and serial mode, this passage can be achieved in n steps with probability not less then the product $\alpha = p_1 \cdots p_n$.*

Proof. We use a procedure analogous to that used by Noga Alon [1] to let the system go to a stable equilibrium. Suppose we start from an initial state. In the first stage of the algorithm, we let the units which are unhappy and can increase their states switch in some order. We close this stage when no unit can increase its state. Reaching such a state is of nonzero probability, as $p_1 > 0, \ldots, p_n > 0$.

In the second stage, we do a converse transition. Namely, we let the units which are unhappy and can decrease their states switch in some order, until no other decreasing switch is possible.

The state reached in this way is an equilibrium. In fact, by the description of the second stage, no more units want to decrease its state. The same is true for increasing switches. In fact, at the end of the first phase there were no units that wanted to increase their state and, during the second phase, this situation was maintained as possibly more units decreased their state so the sum $I_j = \sum W_{ij}x_i - T_j$ decreased or, at least, did not increase (here we need that W_{ij} are nonnegative).

Of course, the transition to an equilibrium described above is of nonzero probability. In the case of two-state network it follows from a result of Alon [1] that the above procedure can be refined so that the equilibrium is achieved with at most n switches, each unit switching at most once. Therefore, the probability of achieving an equilibrium in n steps is not less then $\alpha = p_1 \cdots p_n$ and so the lemma is proved. ∎

The result of the above lemma can be extended to balanced networks. It is in the following lemma where the hierarchical feed-forward layer structure intervenes and a hierarchical order of achieving stability is revealed.

Lemma 3. *If we replace the assumption on nonnegative weights in lemma 2 with the assumption that the network is balanced, then the same assertions hold.*

Proof. It is enough to use theorem 3, which says that we can reorder the units and change the signs of the coordinates so that any causal layer does not influence the causal layers with lower indices, and the weights within each layer are nonnegative. The lemma then follows from the fact that the assertion of lemma 2 applies to any of our causal layers.

Applying this assertion to consecutive causal layers, we see that the network reaches an equilibrium in stages corresponding to reaching an equilibrium by each consecutive layer. Namely, given a layer, if all the previous layers reached an equilibrium then their influence on this layer reduces to changing the thresholds only. Then, the given layer comes to an equilibrium with nonzero probability. In the next stage, the next layer reaches an equilibrium and so on, until all the network stabilizes. In the case of two-state networks, the probability that a given layer reaches an equilibrium (in the serial mode) is not less than the product of the probabilities p_i corresponding to this given layer in as many steps as the number of units in it. Therefore, the assertion of lemma 2 follows for the whole network. ∎

To conclude proving internal stability of balanced networks (theorem 1) it is enough to use the following elementary fact for Markov chains.

Lemma 4. *If from any state of a network there is a passage (of nonzero probability) to a stable equilibrium, then all the trajectories end in stable equilibria, with probability one.*

Proof. For any state $x \in X$ denote by $p_x > o$ the probability of passing to an equilibrium in k_x steps. Let $p > 0$ be the minimum of p_x, $x \in X$, and let k be the maximum of k_x. Then, the probability that the network, starting from a given initial state, does not fall into an equilibrium after k steps is not greater that $q = 1 - p < 1$. The same probability after $2k$ steps is not greater then q^2 and, after mk steps, it is not greater then q^m. As the number of steps approaches infinity, this probability approaches zero. This means that each trajectory falls eventually to an equilibrium, with probability one. ∎

One easily concludes from the above proof and lemmas 2 and 3 that the two-state network in serial mode comes to an equilibrium in nm steps with the probability at least $1-(1-\alpha)^m$.

In this way, the proof of theorem 1 is completed as its assertion follows immediately from lemmas 3 and 4.

Proof of Theorem 2. Assuming that the network contains an essentially frustrated cycle, we will prove that the network is not internally stable. Let us mentally split the network into the essentially frustrated cycle of units and the remaining units. Given an initial state of the network, there are two possible ways it can evolve. Either the remaining units do not reach an equilibrium after a finite number of switches, or they reach an equilibrium (they reach this equilibrium with nonzero probability). In the first case, we have indefinite oscillations in the network.

In the second case, we can change the thresholds of the units in the frustrated cycle in such a way that, together with the influence of the remaining units (which stabilized their states, already), the effective thresholds of the units in the frustrated cycle are zero. This means that the units of the frustrated cycle behave like they had zero thresholds and did not see the rest of the network. The result follows then from the following lemma.

Lemma 5. *A dynamic threshold network which consists of an essentially frustrated cycle is not internally stable.*

Proof. Without losing generality, we may assume that our cycle is $V_1, \ldots, V_n, V_1$. We set the thresholds in our cycle so that zero is a separating number for all possible values of the input, for each unit in the cycle (this is possible as the cycle is essentially frustrated). We see that the sign of the input to the ith unit is determined by the state of the unit preceding in the cycle, provided this unit is in one of its extreme states. By a small perturbation of the thresholds, we can maintain this property and achieve additionally that for any possible state of the network (there is only a finite number of them) the inputs to all units are nonzero. It follows then that in any equilibrium state, each unit must be in one of its extreme states.

We will show that the network has no equilibrium state. In fact, in an equilibrium state we would have that two consecutive units in the cycle are in the same extreme state (both in maximal or both in minimal) if and only if the corresponding weight is positive. Otherwise, they would be in opposite extreme states. As we go around the cycle, we get an even number of jumps between maximal and minimal states of the consecutive units in an equilibrium. This means that there must be a even number of negative weights in the cycle, which contradicts our assumption that the cycle is frustrated. It follows then that the essentially frustrated cycle with the thresholds defined above has no equilibria, and so it is not almost surely stable. ∎

5. Concluding remarks

We will conclude with several remarks on possible applications of our results and comparing the class of balanced threshold networks with the class of symmetric dynamic threshold networks.

First, it seems that the class of balanced threshold networks has a much richer structure than the traditional class of symmetric networks. Therefore, this class may have larger computational abilities. We will not discuss the latter problem here but will only comment on the structure of balanced networks.

As shown in theorem 3, each balanced network has a natural layer structure (as opposed to any symmetric network which splits into a finite number of strongly connected networks not influencing each other). There is a natural order of layers and there are no cycles inbetween layers (in general, the order is not linear). However, layers are strongly connected and so there are cycles within layers. The layers are like symmetric dynamic threshold networks; in particular, they could be symmetric. It seems that they can perform the same task of content addressable memories, or autoassociation or refinement of input strings. However, we have additionally interconnections between layers which can perform other types of computation. It follows from lemma 1 that, up to a change of "coordinates," the weights within each layer can be taken nonnegative. However, there are no restrictions on the signs of connections inbetween the layers.

Our class of networks is more robust with respect to possible perturbations of the parameters of the network. As we mentioned earlier, the class of balanced networks is preserved under all changes of weights which preserve their signs. In fact, our main result can be generalized to networks with time-dependent weights.

Theorem 4. *Each balanced dynamic threshold network with time dependent weights $W_{ij} = W_{ij}(t)$ such that $sgn(W_{ij}(t)) = const$, time-dependent probabilities $p_1(t) > \tilde{p}_1, \dots, p_n(t) > \tilde{p}_n$, and constant thresholds is almost surely stable.*

The proof of this theorem is completely analogous to the proof of theorem 1 and so is omitted.

In view of the above theorem and the fact that each perturbation of a balanced network which preserves the signs of the weights gives a balanced network, we can consider all learning rules which preserve the sign of the weights. In particular, in each step $t \to t+1$ the weights may change according to the rule

$$W_{ij}(t+1) - W_{ij}(t) = \begin{cases} \epsilon x_i(t)x_j(t), & \text{if } |W_{ij}| > \epsilon \\ 0, & \text{if } |Wij| \leq \epsilon, \end{cases} \tag{5.1}$$

where $0 < \epsilon < 1$ and we consider two-state networks with $x_i(t) = \pm 1$.

Another quite different rule is

$$W_{ij}(t+1) - W_{ij}(t) = \epsilon W_{ij}(t)x_i(t)x_j(t). \tag{5.2}$$

More generally, we may take

$$W_{ij}(t+1) - W_{ij}(t) = f(W_{ij}(t))x_i(t)x_j(t), \tag{5.3}$$

where f is a real function such that $f(0) = 0$, and $0 < f'(W) < 1$. Once we start with a balanced network and we use one of the above learning rules, our network remains balanced. In fact, it will preserve the feed-forward layer structure in theorem 3, as this structure depends only on the effective graph of the network and on the signs of the weights.

Acknowledgments

This work was done while the author was visiting Rutgers Center for Systems and Control, Rutgers University, and was supported by National Science Foundation DMS-8803396. The author is grateful to Adam Idzik, Eduardo Sontag, and Hector Sussmann for helpful discussions concerning the subject of the paper.

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