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Spontaneously Activated Systems in Neurodynamics

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Abstract. Coupled activation and learning dynamical equations which spontaneously change locations of their attractors due to parametrical periodic excitations are introduced. The phenomenon is based upon two pathological characteristics of the system: failure of the Lipschitz condition at equilibrium points and zero Jacobian of the system.

1. Introduction

One of the fundamental limitations of artificial computational systems is that they behave too rigidly when compared with even the simplest biological systems. With regard to neurodynamical systems, this point has a simple phenomenological explanation: all such systems satisfy the Lipschitz condition that guarantees the uniqueness of the solutions subject to prescribed sets of initial conditions. Indeed, a dynamical system

$$\dot{x}_i = f_i(x_1, x_2, \dots x_n), \quad i = 1, 2 \dots n \tag{1.1}$$

subject to the initial conditions

$$x_i = \overset{0}{x}_i \ \text{ at } t = t_0 \tag{1.2}$$

has a unique solution:

$$x_i = f_i(t, x_0, t_0) \tag{1.3}$$

if all the derivatives $\partial f_i / \partial x_j$ exist and are bounded

$$\left| \frac{\partial f_i}{\partial x_j} \right| < \infty \tag{1.4}$$

The uniqueness of the solution (1.3) can be considered as a mathematical interpretation of rigid or predictable behavior of the corresponding dynamical system.

The concept of unpredictability in deterministic classical dynamics was first introduced in connection with the discovery of chaotic motions in nonlinear systems [1]. Such motions are caused by Lyapunov instability, which is characterized by a violation of a continuous dependence of solutions on the initial conditions during an unbounded time interval ($t \to \infty$). That is why the unpredictability in these systems develops gradually. Indeed, if two initially close trajectories diverge exponentially:

$$\epsilon = \epsilon_0 \exp \lambda t, 0 < \lambda < \infty \tag{1.5}$$

then for an infinitesimal initial distance $\epsilon \to 0$, the current distance ϵ becomes finite only at $t \to \infty$. For this reason, the Lyapunov exponents (the mean exponential rate of divergence) are defined in an unbounded time interval:

$$\sigma = \lim(\frac{1}{t})\ell n \frac{\epsilon}{\epsilon_0} \tag{1.6}$$

$$t \to \infty$$

However, it can be shown that the discovery of chaos has not shaken the rigid behavior of dynamical systems since all the chaotic motions are structurally stable. This means that although a particular trajectory of a chaotic dynamical system cannot be predicted, a certain set of average characteristics of the motion as well as the global structure of the limit sets are fully predictable.

In distributed dynamical systems, described by partial differential equations, there exists a stronger instability discovered by Hadamard. In the course of this instability, a continuous dependence of a solution on the initial conditions is violated during an arbitrary small time period. Such a blow-up instability is caused by a failure of hyperbolicity and transition to ellipticity [2,3]. In this work, we will show that a similar type of a blow-up instability leading to "discrete pulses" of unpredictability can occur in dynamical systems described by ordinary differential equations, if at some limit sets (for instance, equilibrium points) the Lipschitz condition is removed. We will also introduce a dynamical system which is characterized not only by unpredictable trajectories, but also by unpredictable structure, i.e., by unpredictably variable location of its attractors.

2. One-neuron dynamical system

We will start with the simplest differential equation in which the Lipschitz condition (1.4) is violated:

$$\dot{u} + \alpha u^k = 0, k < 1 \tag{2.1}$$

Indeed, here

$$\lim_{u \to 0} \frac{\partial \dot{u}}{\partial u} = -\alpha k u^{-(1-k)} = \begin{cases} -\infty & \text{if } \alpha > 0 \\ \infty & \text{if } \alpha < 0 \end{cases} \tag{2.2}$$

i.e., the Lipschitz condition fails at the equilibrium point $u = 0$.

Equation (2.1) can be considered as a one-neuron dynamical system where u is an activation potential.

In case $a > 0$, this point is a terminal attractor [4] which is characterized by an infinite local stability. It is approached by a transient solution $u(t)$ with the initial condition $u_0 = u(0)$ during a finite time t_0:

$$t_0 = \int_{u_0}^{u \to 0} \frac{du}{\alpha u^k} = \frac{u_0^{1-k}}{\alpha(1-k)}, k < 1 \tag{2.3}$$

i.e., the transient solution $u(t)$ intersects the constant solution $u = 0$; that leads to the loss of the uniqueness of the solution at $u = 0$. It is easily verifiable that for $k > 1$ the Lipschitz condition holds:

$$\lim \frac{\partial \dot{u}}{\partial u} \to 0 \tag{2.4}$$

$u \to 0$

and the integral in (2.3) diverges. This means that the solution $u(t)$ asymptotically tends to $u = 0$, but never approaches it, i.e., the uniqueness of the solution is preserved.

Let us turn to the case $\alpha < 0$. Now, the point $u = 0$ represents an infinitely unstable (terminal) repeller. Indeed, linearizing equation (2.1) with respect to the point $u = 0$ one obtains:

$$\dot{u} + \lambda u = 0, \lim_{u_* \to 0} (-\alpha k u_*^{-(1-k)}) = \infty \tag{2.5}$$

If the initial condition u_0 is infinitely close to the repeller, then the transient solution will escape it during a finite period t_0:

$$t_0 = \int_{u_0 \to 0}^{\tilde{u}} \frac{du}{\alpha u^k} = \frac{\tilde{u}^{1-k}}{(1-k)\alpha} \tag{2.6}$$

Indeed, t_0 is finite if $\tilde{u}$ is bounded despite the fact that $u_0 \to 0$ (obviously, $t_0 \to \infty$ if $k > 1$).

Let us analyze the transient escape from the terminal repeller at $k = 1/3$, assuming that $|u_0| \to 0$. The solution to equation (2.1) reduces to the following:

$$u = \pm t^{3/2}, \quad u \neq 0 \tag{2.7}$$

Hence, two different solutions are possible for "almost the same" initial conditions (figure 1). The most essential property of this result is that the

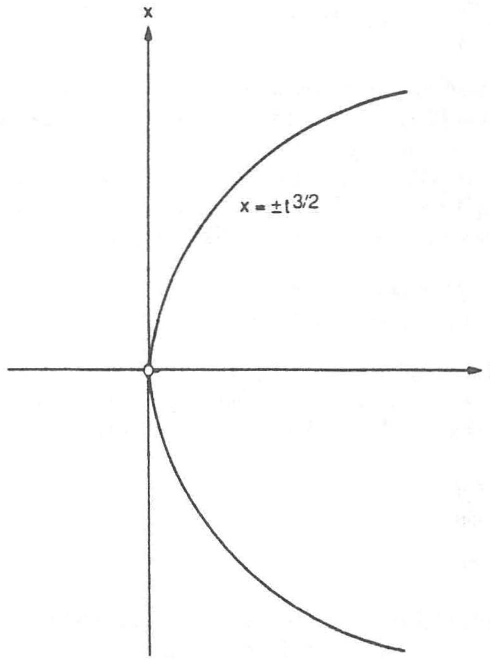

Figure 1: Escape from terminal repeller.

divergence of the solutions (2.7) is characterized by an unbounded Lyapunov exponent:

$$\sigma = \lim_{t \to t_0} \left(\frac{1}{t} \ln \frac{2t^{3/2}}{2|u_0|} \right) = \infty \tag{2.8}$$

$$|u_0| \to 0$$

in which t_0 is an arbitrarily small (but finite) positive quantity. In contrast to equation (1.6), here the Lyapunov exponent can be defined in an arbitrarily small time interval, since during this interval the initial infinitesimal distance between the solutions becomes finite. Thus, a terminal repeller represents a vanishingly short but infinitely powerful "pulse of unpredictability" which is "pumped" into the dynamical system.

In order to illustrate the unpredictability in such a non-Lipschitzian dynamics, we will turn from equation (2.1) to the following equation:

$$\dot{u} - vu^{1/3} = 0, \text{ while} \tag{2.9}$$

$$v = \cos \omega t \tag{2.10}$$

Assuming that $u \to 0$ at $t \to 0$, one obtains regular solutions

$$u = \pm \left(\frac{2}{3\omega} \sin \omega t \right)^{3/2}, \quad u \neq 0 \tag{2.11}$$

and a singular solution (an equilibrium point)

$$u = 0 \tag{2.12}$$

During the first time period

$$0 < t < \frac{\pi}{2\omega} \tag{2.13}$$

the equilibrium point (2.12) is a terminal repeller (since $v > 0$). Therefore, within this period the solutions (2.11) have the same property as the solutions (2.7): their divergence is characterized by an unbounded Lyapunov exponent.

During the next time period

$$\frac{\pi}{2\omega} < t < \frac{3\pi}{2\omega} \tag{2.14}$$

the equilibrium point (2.12) becomes a terminal attractor (since $v < 0$), and the system which approaches this attractor at $t = \pi\omega$ remains motionless until $t > 3\pi/2\omega$. After that, the terminal attractor converts into the terminal repeller, and the system escapes again, and so on.

It is important to notice that each time when the system escapes the terminal repeller, the solution splits into two symmetric branches, so that the total trajectory can be combined from 2^n pieces, where n is the number of cycles, i.e., it is the integer part of the quantity $(t/2\pi\omega)$, figure 2. As one can see, here the nature of the unpredictability is significantly different from the unpredictability in chaotic systems. This difference will be emphasized even more by the next example; let us replace equations (2.9) and (2.10) by the following:

$$\dot{u} - v(\sin u)^{1/3} = 0, v = -1 + 2e^{-t}\cos \omega t, \tag{2.15}$$

assuming again that $u \to 0$ at $t \to 0$. Since $v > 0$ at $t = 0$, the equilibrium point $u = 0$ initially is a terminal repeller. Hence, the regular solution will consist of two possible (positive and negative) escaping branches which will approach the neighboring terminal attractors at $u = \pi$ or $u = -\pi$, respectively. The system will be at rest in one of these two attractors until v becomes negative, i.e., until these terminal attractors become terminal repellers. After that, the solution will split again into two possible escaping branches, while the system can continue to escape the equilibrium point $u = 0$

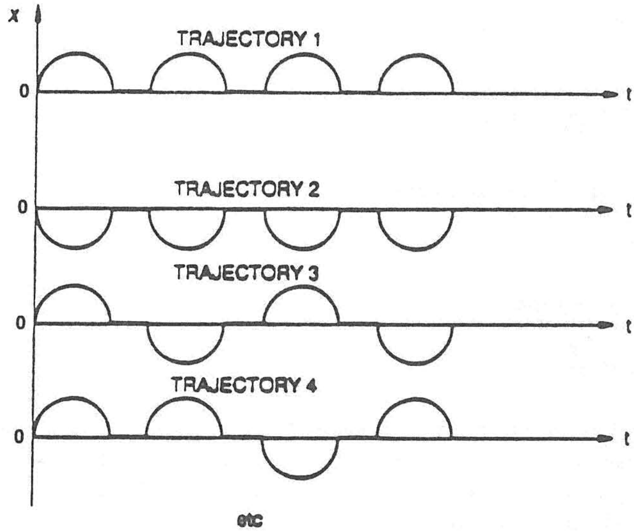

Figure 2: Unpredictability caused by alternating terminal attractors and repellers at an equilibrium point.

or return to it. However, because $v \to -1$ at $t \to \infty$, all the equilibrium points

$$u = \pm 2\pi k, \quad k = 0, 1, 2 \ldots, \tag{2.16}$$

will eventually become permanent terminal attractors, and the system will relax at one of them. Because of branching of the solution, however, it is impossible to predict which one of the competing attractors (2.16) will be finally approached by the system. Hence, here the unpredictability is represented not by a chaotic attractor, but rather by a set of competing static attractors, figure 3.

Thus, in this item we have introduced a new type of unpredictability in dynamical systems caused by failure of the Lipschitz condition at equilibrium points. It has been demonstrated that, unlike the chaotic systems, the non-Lipschitzian dynamics may exhibit an unpredictability characterized by unbounded Lyapunov exponents. The sources of these unbounded exponents are terminal repellers which "pump the unpredictability" in the form of vanishingly short, but infinitely powerful "pulses." That is why a set of possible trajectories in phase space is not a Cantor set (as in chaotic system), but rather a countable set of a combinatorial nature. (Because of that, the global unpredictability in the non-Lipschitzian dynamics is associated with an exponential complexity.) Hence, in this respect the non-Lipschitzian dynamics has some connections with the "digital world."

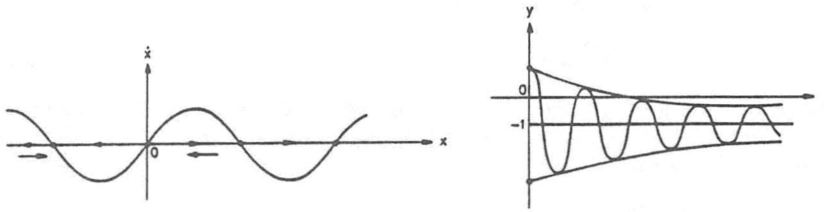

Figure 3: Unpredictability caused by competing terminal attractors.

It is important to emphasize that in chaotic systems the unpredictability is caused by a supersensitivity to the initial conditions, while the uniqueness of the solution for fixed initial conditions is guaranteed by the Lipschitz condition. In contrast, in the non-Lipschitzian dynamics presented here, the unpredictability is caused by the failure of the uniqueness of the solution at some of the equilibrium points.

The non-Lipschitzian dynamics introduced above may have some practical applications. Indeed, it represents dynamical systems with a multiple-choice response to an initial deterministic input. Such models can become an underlying idealized framework for dynamical systems with "creativity," whose response is based upon a "hidden logic." This logic might be incorporated into the system in the form of an appropriate dynamical microstructure of terminal repellers or by additional external inputs. As will be shown below, such an approach can be useful in dynamical modeling of neural networks. Indeed, a neural network with n terminal repellers would be able to make 2^n totally different decisions under slightly different external inputs performing thereby an "intrinsic" logic. The most significant property of such neural networks would be their ability to be activated not by external inputs, but rather by internal rhythms (see equation (2.9), or equation (2.15)). Indeed, as soon as terminal attractor is converted into terminal repeller, it activates the system. Such a behavior can be compared with higher-level cognitive processes since it is based upon interactions between attractors (i.e., upon the "knowledge" in the system) in contradistinction to perception and recognition performances which are based upon external inputs.

3. One-neuron-one-synapsis dynamical system

In this section, based upon the non-Lipschitzian approach to dynamical systems, we will introduce a self-developing dynamical system which spontaneously changes the locations of its attractors. The simplest version of such a system consists of a one-neuron activation dynamics

$$\dot{u} = -(u - Tu^2)^{1/3} \sin \omega t, \quad \omega = \text{const} \tag{3.1}$$

and one-synapsis learning dynamics

$$\dot{T} = (u - Tu^2)^{2/3} \sin^2 \omega t \tag{3.2}$$

while an external input is represented by periodic parametrical excitation.

The system (3.1), (3.2) possesses two pathological properties. First, it has zero Jacobian:

$$J = \begin{vmatrix} \partial \dot{u}/\partial u & \partial \dot{u}/\partial T \\ \partial \dot{T}/\partial u & \partial \dot{T}/\partial T \end{vmatrix} \equiv 0 \tag{3.3}$$

Because of that, the system has infinite number of equilibrium points which occupy two curves in the configuration space u, T:

$$u_0 = 0 \text{ and } u_0 = \frac{1}{T_0} \tag{3.4}$$

Second, at all the equilibrium points, the Lipschitz condition fails since

$$\left|\frac{\partial \dot{u}}{\partial u}\right| = \left|\frac{(2Tu - 1)\sin \omega t}{3(u - Tu^2)^{2/3}}\right| \to \infty \text{ if } u \to 0 \text{ or } u \to \frac{1}{T} \tag{3.5}$$

$$\left|\frac{\partial \dot{T}}{\partial T}\right| = \left|\frac{2(2Tu - 1)\sin^2 \omega t}{3(u - Tu^2)^{1/3}}\right| \to \infty \text{ if } u \to 0 \text{ or } u \to \frac{1}{T} \tag{3.6}$$

As a result of equations (3.5) and (3.6), the characteristic roots λ_1 and λ_2 of the Jacobian (3.3) at the equilibrium points (3.4) must be

$$\lambda_1 = 0, \quad |\lambda_2| \to \infty \tag{3.7}$$

Indeed, linearizing equations (3.1) and (3.2) with respect to the points (3.4) one finds

$$\lambda_1 = 0, \quad \lambda_2 = \frac{(2T_0 u_0 - 1)\sin \omega t}{3(T_0 u_0^2 - u_0)^{2/3}} - \frac{2u_0 \sin^2 \omega t}{3(T_0 u_0^2 - u_0)^{1/3}} \tag{3.8}$$

It is easy to verify that

$$\lambda_2 \to \begin{cases} \infty & \text{if } u_0 = \frac{1}{T_0} \\ -\infty & \text{if } u_0 = 0 \end{cases} \quad \text{for } \sin \omega t > 0 \tag{3.9}$$

$$\lambda_2 \to \begin{cases} -\infty & \text{if } u_0 = \frac{1}{T_0} \\ \infty & \text{if } u_0 = 0 \end{cases} \quad \text{for } \sin \omega t < 0 \tag{3.10}$$

Hence, when the equilibrium points $u_0 = 0$ are stable (they become terminal attractors [4]), the equilibrium points $u_0 = 1/T_0$ are unstable (they become terminal repellers) and vice versa.

One should note that, strictly speaking, the formula for λ_2 in equation (3.8) can be applied only if the explicit time t in equations (3.1,3.2) is considered as a slow changing parameter, i.e., if

$$\omega \ll \lambda_2 \tag{3.11}$$

However, since $|\lambda_2| \to \infty$ (see equation (3.7)) the inequality (3.11) holds for all bounded ω.

The Lipschitz condition fails not only in actual space, but in configuration space u, T. Indeed, as follows from equations (3.1) and (3.2), the differential equation for trajectories is

$$\frac{dT}{du} = -(u - Tu^2)^{1/3} \sin \omega t \tag{3.12}$$

and

$$\lambda_3 = \left| \frac{\partial \frac{dT}{du}}{\partial T} \right| = \left| \frac{u^2 \sin \omega t}{3(u - Tu^2)^{2/3}} \right| \to \begin{cases} \infty & \text{if } u = \frac{1}{T} \\ 0 & \text{if } u = 0 \end{cases} \tag{3.13}$$

Thus, the Lipschitz condition fails only at the curve $u_0 = \frac{1}{T_0}$ of the configuration space u, T. All the equilibrium points of this curve are terminal attractors for $\sin \omega t < 0$ and terminal repellers for $\sin \omega t > 0$.

As follows from equation (3.13), the Lipschitz condition holds at the curve $u_0 = 0$, while the equilibrium points of this curve possess a neutral stability.

Before analyzing the global behavior of the solutions to equations (3.1,3.2), we will first investigate the local properties of the escape from terminal repellers.

The solutions in an infinitesimal neighborhood of a terminal repeller have the following structure:

$$u = \overset{*}{u}\, e^{\lambda_2 t}, \quad T = \overset{*}{T}\, e^{\lambda_2 t}, \quad \lambda_2 \to \infty \tag{3.14}$$

in which $\overset{*}{u}$ and $\overset{*}{T}$ are initial disturbances.

As follows from equation (3.14), the transient solution may escape the repeller and approach some values $\tilde{u}$ and $\tilde{T}$ during a finite time period t_0 even if the initial disturbances are infinitesimal:

$$t_0 = \frac{1}{\lambda_2} \ln \frac{\tilde{u}}{\overset{*}{u}} = \frac{1}{\lambda_2} \ln \frac{\tilde{T}}{\overset{*}{T}}, \lambda_2 \to \infty, \overset{*}{u}, \overset{*}{T} \to 0, \tag{3.15}$$

while $\tilde{u}$ and $\tilde{T}$ are sufficiently small, but finite.

One should recall (see equation (1.5)) that for bounded λ the Lyapunov instability develops gradually: two initially-close trajectories diverge such:

$$\varepsilon = \varepsilon_0 \exp \lambda t, \; |\lambda| < \infty$$

that for an infinitesimal initial distance $\varepsilon_0 \to 0$, the current distance becomes finite only at $t \to \infty$. For this reason, the Lyapunov exponents are defined in an unbounded time interval (see equation (1.6)).

In this contradistinction, the escape from the terminal repeller (3.14) is similar to Hadamard instability in continuous systems [2] where the instability can be defined within a finite time interval. That is why the Lyapunov exponents for the instability (3.14) can be also defined in a finite time interval (compare to equation (2.8))

$$\sigma = \lim_{t \to t_0} \left(\frac{1}{t} \ln \frac{u}{\overset{*}{u}} \right) \to \infty \text{ if } \overset{*}{u} \to 0 \ldots \tag{3.16}$$

Thus, the divergence of the solutions (3.14) describing the escape from the terminal repeller are characterized by the unbounded Lyapunov exponent (3.16). This means that here as in the previous section, a terminal repeller represents a vanishingly short, but infinitely powerful "pulse of unpredictability" which is "pumped" into the dynamical system.

The solutions in an infinitesimal neighborhood of a terminal attractor have the following structure:

$$u = \overset{*}{u} e^{-\lambda_2 t}, \; T = \overset{*}{T} e^{-\lambda_2 t}, \; \lambda_2 \to \infty, \tag{3.17}$$

As follows from (3.17), a solution with finite initial condition $u = \overset{*}{u}$ at $t = 0$ may approach the terminal attractor in a finite time interval t_0:

$$t_0 = \frac{1}{\lambda_2} \ln \frac{\overset{*}{u}}{u} < \infty, \lambda_2 \to \infty, u \to 0, \tag{3.18}$$

while for a regular attractor this time is infinite.

The structure of the solutions around terminal repellers and attractors in the configuration space u, T is similar to equations (3.14) and (3.17) with the only difference being the role of the argument is played by u instead of t:

$$T = \overset{*}{T} e^{\lambda_3 u} \tag{3.19}$$

where

$$\lambda_2 \to \begin{cases} \infty & \text{if } \sin \omega t > 0 \\ -\infty & \text{if } \sin \omega t < 0 \end{cases} \tag{3.20}$$

Let us turn now to the global behavior of the solutions to equations (3.1,3.2) and start with the following initial conditions:

$$u = 0.5, \quad T = 2 \text{ at } t = 0 \tag{3.21}$$

According to equation (3.8) for $0 < t < \pi/w$, the point (3.21) is a terminal repeller. In the case of precisely zero disturbances, the system would rest

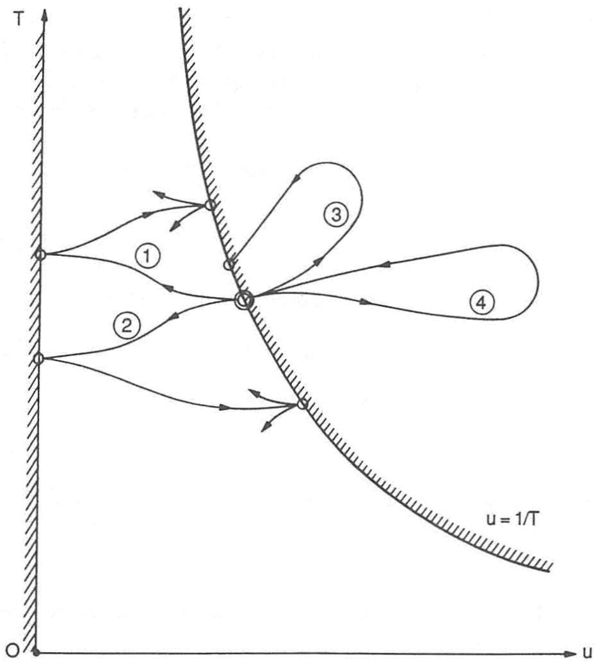

Figure 4: Spontaneous changes of point attractors.

forever at this point. In the presence of infinitesimal disturbances $\overset{*}{u}$ and $\overset{*}{T}$ the system can "choose" an escape scenario from the four combinations:

$$u = \pm \overset{*}{u}\, e^{\lambda_2 t},\ T = \pm \overset{*}{T}\, e^{\lambda_2 t},\ \overset{*}{u},\ \overset{*}{T} \rightarrow 0,\ \lambda_2 \rightarrow \infty, \tag{3.22}$$

Although initially the differences between the positive and the negative solutions are infinitesimal, their transient divergence is characterized by unbounded Lyapunov exponents (3.16) in both actual and configuration spaces. The escaping solutions 1 and 2 (see figure 4) will approach the corresponding terminal attractors located on the line $u = 0$; they will remain there until $\sin \omega t > 0$, i.e., until all these attractors become repellers. Then, two from the four new branches of each of the solutions will return to the curve $u = 1/T$ giving the rise to another branches of the solutions, and so on.

It is important to notice that each time the system escapes the terminal repeller, the solution splits into four possible branches, so that the total trajectory can be combined from 4^n pieces, where n is the integer part of the quantity $t/2\pi\omega$. The number of different structures (i.e., different attractors) which the system can attain is less than the number of different trajectories for two reasons. First, some of the solutions (see solution 4 in figure 4) can return to the old attractors. Second, the solutions 1 and 2 do not branch on the curve $u = 0$, because in the configuration space the points of this curve are not terminal—they have neural stability (see equation (3.13)). That is why the number of structural changes in the system (3.1,3.2) has the order of $2^{n/2}$.

The system (3.1,3.2) can be represented in the following autonomous form:

$$\dot{u} = (u - Tu^2)^{1/3} v_2, \quad \dot{T} = -\dot{u}^2 \tag{3.23}$$

if the new variable v_2 satisfies the following differential equations:

$$\dot{v}_1 = \omega v_2 + v_1(1 - v_1^2 - v_2^2), \quad \dot{v}_2 = -\omega v_1 + v_2(1 - v_1^2 - v_2^2) \tag{3.24}$$

Indeed, equations (3.24) have a stable limit cycle:

$$v_1 = \cos \omega t, \quad v_2 = -\sin \omega t, \quad \omega = \text{const} \tag{3.25}$$

and therefore equations (3.23,3.24) are equivalent to equations (3.1,3.2).

4. General case

Equations (3.1) and (3.2) can be generalized to the case of n neurons u_i and n^2 synaptic interconnections T_{ij}

$$\dot{u}_i = -\left[u_i + \sum_{j=1}^{n} T_{ij} V(u_j)\right]^{1/3} v_i, \quad \dot{T}_{ij} = -\dot{u}_i \dot{u}_j \tag{4.1}$$

$$\dot{v}_i = -\omega w_i + v_i(1 - v_i^2 - w_i^2), \quad \dot{w}_i = \omega v_i + w_i(1 - v_i^2 - w_i^2) \tag{4.2}$$

in which $v(u_j)$ is a sigmoid function.

It is easily verifiable that equations (4.1,4.2) possess the same self-developing properties as the original dynamical system. Indeed, in the configuration subspaces,

$$\frac{dT_{ij}}{du_i} = -\dot{u}_j \tag{4.3}$$

which is equivalent to equation (3.12).

In addition, they can perform some qualitatively new effects: they can spontaneously relocate periodic or chaotic attractors as well as static attractors. To illustrate, we start with the following three-neuron network:

$$\dot{u}_1 = [-u_1 + T_{11} V(u_1)]^{1/3} v, \quad \dot{T}_{11} = -\dot{u}_1^2 \tag{4.4}$$

$$\dot{v} = -\omega w + v(1 - v^2 - w^2), \quad \dot{w} = \omega v + w(1 - v^2 - w^2) \tag{4.5}$$

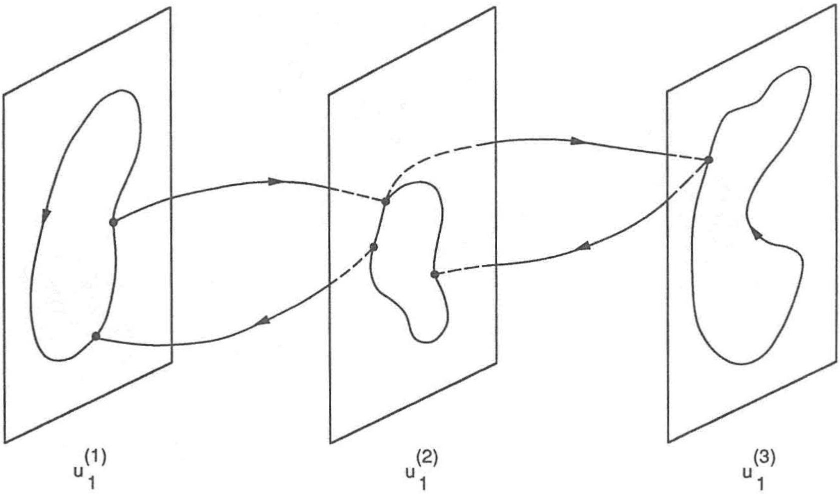

Figure 5: Spontaneous changes of limit cycles.

$$\dot{u}_2 = -u_2 + T_{22}V(u_2) + T_{23}V(u_3) + T_{21}V(u_1) \tag{4.6}$$

$$\dot{u}_3 = -u_3 + T_{33}V(u_3) + T_{32}V(u) + T_{31}V(u) \tag{4.7}$$

Clearly, equations (4.6,4.7) represent the conventional part of the neural network, while equations (4.4,4.5) describe its self-developing part. For simplicity, we assumed that equations (4.4,4.5) are decoupled from equations (4.6,4.7) (since $T_{12} = T_{13} = 0$), but equations (4.6,4.7) are still affected by equations (4.4,4.5). Let us set up the synaptic interconnections T_{22}, T_{23}, T_{32}, and T_{33} in equations (4.4,4.5) such that the solution has periodic attractors in the configuration planes $u_1 = \text{const}$ (figure 5). The spontaneous relocation of the static attractors for equations (4.4) will cause the corresponding relocations of limit cycles in the configuration planes $u_1 = \text{const}$ for the system (4.6,4.7) through the changes of their last terms $T_{21}V(u_1)$ and $T_{31}V(u_1)$, since the locations and the configurations of periodic attractors are parametrically dependent on u_1.

Spontaneous relocations of multiperiodic or chaotic attractors can be organized in the same way. For that purpose, the conventional part of the neural network must consist of at least three neurons, while the coefficients $T_{ij}(i, j = 2, 3, 4)$ should be set up such that the solution has multiperiodic (or chaotic) attractors in the three-dimensional configuration spaces $u_1 = \text{const}$ (figure 6). The self-developing part of the neural network can be represented

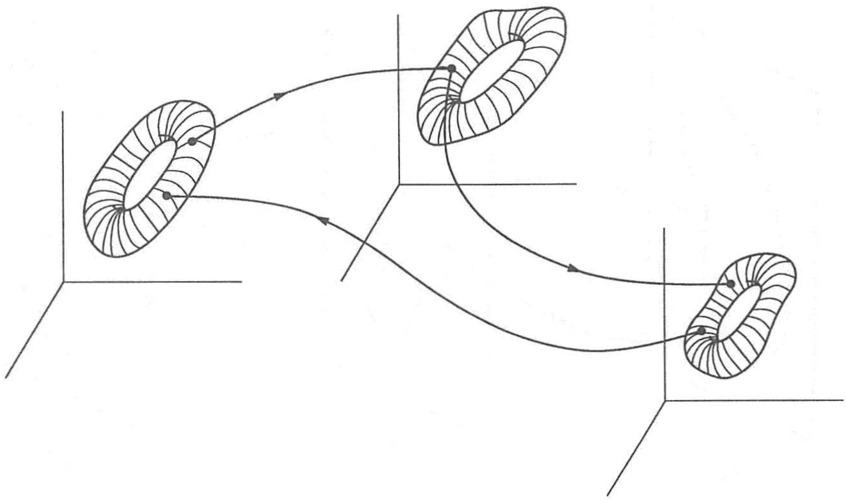

Figure 6: Spontaneous changes of multi-periodic attractors.

by the same system (4.4,4.5). As in the previous case, the spontaneous relocations of the static attractors for equation (4.4) will cause the corresponding relocations of multiperiodic chaotic attractors in the configuration subspaces $u_1 = \text{const}$.

Thus, we have introduced self-developing, dynamical systems which are able to spontaneously change their structure, i.e., locations and parameters of their attracting sets. Despite the fact that these systems are fully deterministic, their behavior as well as their structure is totally unpredictable. Although one can argue that maybe the sequence of chaotic attractors spontaneously created by the self-developing systems possesses some hidden order and can be considered as a more complex type of attraction, so far we have no reasons to support such an assumption. It should be recalled that these new effects which are essentially different from the chaotic behavior are due to failure of the Lipschitz condition (1.4) which is not violated in classical dynamics, and therefore, in chaotic systems.

What is the usefulness of the self-developing systems if they are totally unpredictable? Let us recall that we introduced such systems as an alternative to the systems with "rigid" behavior in order to develop a mathematical framework for modeling the biological systems. So far these systems have not yet been equipped by an internal logic. That is why they do not "know" how to use the freedom they have. In the next item, we will incorporate an internal logic in the form of an objective which should be reached by the system.

5. Systems with objective

Let us return to the one-neuron-one-synapsis dynamical system written in the form (3.23), (3.24) and introduce a global objective by requiring that the system settle at a point attractor $u = \tilde{u}$. It is implied that the system will find itself the corresponding synapsis $\tilde{T}$ in the course of its spontaneous activity. First, we will modify equation (3.24) as follows:

$$\dot{v}_1 = [v_2(u - \tilde{u}) + v_1(1 - v_1^2 - v_2^2)] \tag{5.1}$$

$$\dot{v}_2 = [-v_1(u - \tilde{u}) + v_2(1 - v_1^2 - v_2^2)] \tag{5.2}$$

Now, instead of (3.25), the stable limit cycle is

$$v_1 = \cos\, \omega(t), \quad v_2 = \sin\, \omega(t), \tag{5.3}$$

and

$$\dot{\omega} = -(u - \tilde{u}) \tag{5.4}$$

Obviously, the spontaneous activity of equation (3.23) ends when $v_2 =$ const > 0, because all the points of the curve $Tu = 1$ become terminal attractors. However, $v_2 =$ const only if $\dot{\omega} = 0$, i.e., when $u = \tilde{u}$. Consequently, the system eventually will approach the desirable structure with the prescribed point attractor (figure 7).

It is important to emphasize that neither the value of $\tilde{T}$ nor the strategy for defining this value was prescribed in advance.

This approach can be generalized to the case of n neurons and n^2 synaptic interconnections (see equations (4.1) and (4.2)) if ω in equation (4.2) is considered as a prescribed function of $u_1, u_2, \ldots u_n$, i.e.,

$$\omega_i = \omega_i(u_1, u_2 \ldots u_n), \quad i = 1, 2 \ldots n$$

Clearly, the system will stop at such a point attractor whose coordinates $\tilde{u}_i$ satisfy the following equations

$$\omega_i(u_1, u_2 \ldots u_n) = 0, \quad i = 1, 2, \ldots n \tag{5.5}$$

Hence, depending on selections of ω_i, the system can approach a single point attractor, a countable set of possible point attractors, and continuous hypersurfaces of possible point attractors. If equations (5.5) do not have a solution, then the system will never stop.

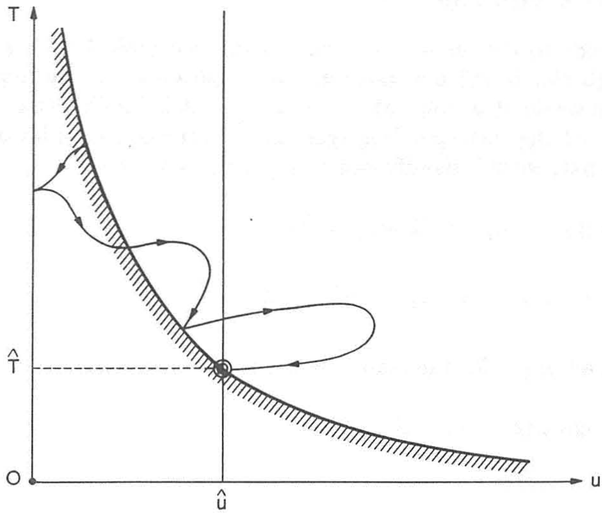

Figure 7: System with objective.

If there exists such a function E that

$$\frac{\partial E}{\partial u_i} = \omega_i, \tag{5.6}$$

then the point attractor approached by the system corresponds to a minimum of this function and therefore, the dynamical system (4.1,4.2,5.5) performs optimization of this function.

It should be emphasized that the incorporation of the objective into a self-developing system does not impose any limitations upon the strategy for reaching this objective: the strategy is developed by the system itself. Because of this, however, one does not have any control over the time of convergence of the system to the desirable state. That is why in the next section we will introduce self-developing systems with a microstructure which allows a flexible guidance of their behavior.

6. Guided self-developing systems

Let us return to the one-neuron-one-synapsis dynamical system (see equations (3.23,3.24)). We will slightly modify equations (3.23) by introducing an infinitesimal bias as follows:

$$\dot{u} = -(u - Tu^2 + \varepsilon_0)^{1/3} v_2, \quad \dot{T} = -\dot{u}^2 \tag{6.1}$$

in which

$$|\varepsilon_0| \to 0, \text{ but } |\varepsilon_0 \lambda_2| \to \infty \tag{6.2}$$

where λ_2 is given by equation (3.8) and v_2 is defined by equations (3.6).

This bias can be ignored when the system is stable, but it becomes significant during the periods of instability. Indeed, in the last case the solution to equation (3.12) with the bias ε_0:

$$\frac{dT}{du} = -(u - Tu^2 + \varepsilon)^{1/3} v^2 \tag{6.3}$$

in the neighborhood of a terminal repeller has the following structure (compare to equation (3.19)):

$$T = \frac{\varepsilon_0}{\lambda_2} e^{\lambda_2 u}, \ \lambda_2 \to \infty \tag{6.4}$$

Now the escape from the terminal repeller is controlled by the bias ε_0, and the changes in the structure of the system become predictable.

A compromise between these two extremes can be reached if one sets up

$$\varepsilon = \varepsilon_0 \sin \gamma t \tag{6.5}$$

Then, the unpredictable structural changes will appear only when $\sin \gamma t$ and $\sin \omega t$ vanish simultaneously (which depends on the ratio γ/ω). In other words, here one can control the degree of unpredictability.

More complex situations can occur in a two-neuron dynamical system:

$$\dot{u}_1 = -[u_1 - T_1 V(u_1) + \varepsilon_0 \text{ sign } f_1(u_2)]^{1/3} v_1, \quad \dot{T}_1 = -\dot{u}_1^2 \tag{6.6}$$

$$\dot{u}_2 = -[u_2 - T_2 V(u_2) + \varepsilon_0 \text{ sign } f_2(u_1)]^{1/3} v_2, \quad \dot{T}_2 = -\dot{u}_2^2 \tag{6.7}$$

in which v_1 and v_2 are defined by equations (4.2) at $i = 1, 2$, while f_1 and f_2 are prescribed functions.

Equations (6.6) and (6.7) possess a very interesting property: they are coupled only at the moments of escape from terminal repeller. Indeed, only at that moment the vanishing terms with ε_0-factor cannot be ignored: when the system (6.6) approaches the terminal repeller, the choice of the escape scenario depends upon the sign of its last term, i.e., upon the state of the system (6.7), and vice versa. Such an "impulsive" coupling represents a typical cause-and-effect relationship between two dynamical systems: each of these systems is independent up to a certain "turning point" when it has to choose from several available scenarios. In contradistinction from the situation described in section 2, this choice is fully determined by the state of the other system. Therefore, the dynamical systems (6.6) and (6.7) can be considered as a possible model for a "nonrigid" behavior which is typical for biological systems.

Let us assume now that

$$f_1 = 0 \tag{6.8}$$

Then, the system (6.6) becomes totally independent and unpredictable, while the system (6.7) is still dependent on it, i.e., one arrives at a master-slave relationship. This situation can be generalized to the following chain of the master-slave subordination:

$$\dot{u}_1 = -[u_1 - T_1 V(u_1)]^{1/3} v_1, \quad \dot{T}_1 = -\dot{u}_1^2 \tag{6.9}$$

$$\dot{u}_2 = -[u_2 - T_2 V(u_2) + \varepsilon_0 f_2(u_1)]^{1/3} v_2, \quad \dot{T}_2 = -\dot{u}_2^2 \tag{6.10}$$

$$\dot{u}_3 = -[u_3 - T_3 V(u_3) + \varepsilon_0 f_3(u_2)]^{1/3} v_3, \quad \dot{T}_3 = -\dot{u}_3^2 \tag{6.11}$$

$$\dot{u}_n = [u_n - T_n V(u_n) + \varepsilon_0 f_n(u_{n-1})]^{1/3} v_n, \quad \dot{T}_n = -\dot{u}_n^2 \tag{6.12}$$

in which $v_i (i = 1, 2 \ldots n)$ are defined by equations (4.2).

The elements of this chain are not necessarily the one-neuron-one-synapsis dynamical systems. They can be presented in a more general form:

$$\dot{u}_{i_1} = -\left[u_{i_1} + \sum_{j_1=1}^{n_1} T_{i_1 j_1} V(u_{j_1})\right]^{1/3} v_{i_1}, \quad \dot{T}_{i_1 j_1} = -\dot{u}_{i_1} \dot{u}_{j_1} \tag{6.13}$$

$$\dot{u}_{i_2} = -\left[u_{i_2} + \sum_{j_2=1}^{n_2} T_{i_2 j_2} V(u_{j_2}) + \varepsilon_0 f_{i_2}(u_1, \ldots u_{n_1})\right]^{1/3} v_{i_2}, \; \dot{T}_{i_2 j_2} \tag{6.14}$$

$$= -\dot{u}_{i_2} \dot{u}_{j_2} \text{ etc.} \tag{6.15}$$

It should be noticed again that the guided self-developing systems introduced above fall between the classical (rigid) dynamical systems and totally unpredictable dynamical systems discussed in sections 2 and 3. It seems reasonable to assume that such systems may provide a proper mathematical framework for modeling the biological systems.

7. Guided systems with objective

In this section, we will simply combine the results of the two previous sections and discuss the guided systems with objective. Starting with the one-neuron-one-synapsis dynamical system, let us write it in the form:

$$\dot{u} = -[u - Tu^2 + \varepsilon_0 \text{ sign } (u - \tilde{u})]^{1/3} v_2, \quad T = -\dot{u}^2 \tag{7.1}$$

$$\dot{v}_1 = [v_2(u - \tilde{u}) + v_1(1 - v_1^2 - v_2^2)] \tag{7.2}$$

$$\dot{v}_2 = [-v_1(u - \tilde{u}) + v_2(1 - v_1^2 - v_2^2)] \tag{7.3}$$

As follows from equations (7.2) and (7.3), the dynamical system has a global objective: to settle at the point attractor $u = \tilde{u}$ (compare with equations (5.1,5.2)). Besides that, it is guided by the bias in equations (7.1). Because of this guidance, the system in its critical points selects such a branch among the available solutions which decreases the distance $|u - \tilde{u}|$ between the current state u and the desirable point attractor $\tilde{u}$. Indeed, as follows from equations (7.1), if $u < \tilde{u}$, then ε_0 sign $(u - \tilde{u})$ is negative, and therefore, at the critical point $\dot{u} = \varepsilon_0 > 0$, i.e., the selected branch corresponds to the decrease of the difference $\tilde{u} - u$.

This result can be easily generalized to a dynamical system with n neurons:

$$\dot{u}_i = -\left[u_i + \sum_{j=1}^{n} T_{ij} V(u_j) + \varepsilon_0 \text{ sign } (u - \tilde{u})\right]^{1/3} v_i, \; \dot{T}_{ij} = -\dot{u}_i \dot{u}_j \tag{7.4}$$

$$\begin{aligned} \dot{v}_i &= -w_i(u_i - \tilde{u}_i) + v_i(1 - v_1^2 - w_1^2), \\ \dot{w}_i &= v_i(u_i - \tilde{u}_i) + w_i(1 - v_1^2 - w_1^2) \end{aligned} \tag{7.5}$$

The system (7.4,5) has an obective: to settle at the point attractor $\tilde{u}_i$. At the critical points where the solution is branching, the system is "pushed" by the bias terms toward its attractor.

One should notice that, strictly speaking, both the system (7.1,7.3) and (7.4,5) are characterized by fully deterministic behavior and objective, although they are extremely sensitive to infinitesimal excitations. In the next example, we will introduce a guided system with an implicit objective which is not fully deterministic.

Suppose that a two-neuron dynamical system has the following form:

$$\dot{u}_1 = -[u_1 + T_1 V(u_1) + \varepsilon_0 \text{ sign } (u_1 - u_2)]^{1/3} v_1, \quad \dot{T}_1 = -\dot{u}_1^2 \tag{7.6}$$

$$\dot{u}_2 = -[u_2 + T_2 V(u_2) + \varepsilon_0 \text{ sign } (u_2 - u_1)]^{1/3} v_2, \quad \dot{T}_2 = -\dot{u}_2^2 \tag{7.7}$$

$$\begin{aligned} \dot{v}_i &= -w_i(u_1 - u_2) + v_i(1 - v_i^2 - w_i^2), \\ \dot{w}_i &= v_i(u_1 - u_2) + w_i(1 - v_i^2 - w_i^2), i = 1, 2, \end{aligned} \tag{7.8}$$

The objective of the system is to settle at the point attractor whose position is not fully determined; it can be located at any point of the straight line:

$$u_1 = u_2 \tag{7.9}$$

of the configuration space. At the critical points the system will be "pushed" by the bias terms toward this line. The exact location of the attractor can not be predicted (figure 8).

All the previous examples can be generalized by the following model:

$$\begin{aligned} \dot{u}_i &= -\left[u_i + \sum_{j=1}^{n} T_{ij} V(u_j) + \varepsilon_0 \text{ sign } (u_i - f_i)\right]^{1/3} v_i, \\ \dot{T}_{ij} &= -\dot{u}_i \dot{u}_j \end{aligned} \tag{7.10}$$

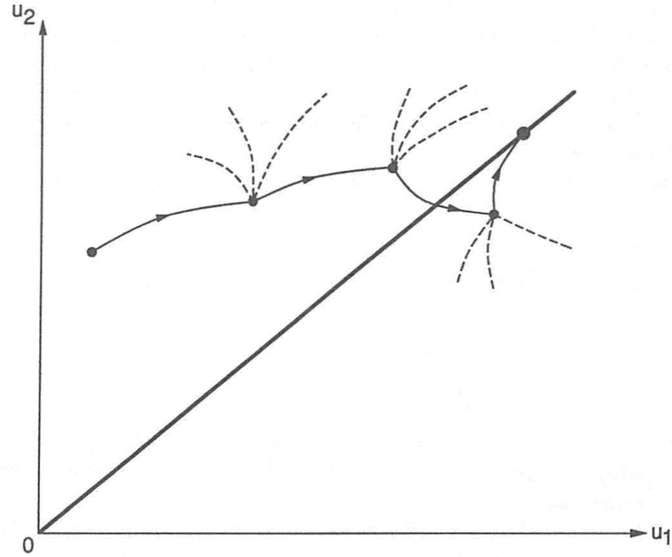

Figure 8: System with objective that is not fully determined.

$$\dot{v}_i = -w_i\omega_i + v_i(1 - v_i^2 - w_i^2), \quad \dot{w}_i = v_i\omega + w_i(1 - v_i^2 - w_i^2) \tag{7.11}$$

in which ω_i are prescribed functions

$$\omega_i = \omega_i(u_1, u_2 \ldots u_n) \tag{7.12}$$

and

$$f_i(u_1, \ldots u_{i-1}, u_{i+1}, \ldots u_n) = u_i \tag{7.13}$$

is the explicit expression for u_i from the equation

$$\omega_i(u_1, \ldots u_n) = 0 \tag{7.14}$$

As in the case of unguided systems, discussed in section 4, this system will stop at the point whose coordinates $\tilde{u}_i$ satisfy equations (7.14). However, in addition, at each critical point the system will select those branches of the solution which are directed toward the desirable attractor $\tilde{u}_i$.

If, in particular, the functions (7.12) have a potential E (see equations (5.6)), then the attractor $\tilde{u}_i$ will correspond to the minimum of this potential.

8. Discussion and conclusions

This paper has introduced a substantially new type of dynamical system which spontaneously changes its structure, i.e., locations of its attracting sets. The approach was motivated by an attempt to remove one of the most fundamental limitations of artificial computational systems—their rigid behavior compared with even simplest biological systems. This approach exploits a novel paradigm in nonlinear dynamics based upon the concept of terminal attractors and repellers. Incorporation of these new types of attractors and repellers into dynamical systems required a revision of some fundamental concepts in theory of differential equations associated with the failure of Lipschitz condition, such as uniqueness of solutions, infinite time of approaching of attractors, bounded Lyapunov exponents, and so on. In the course of this revision, it was demonstrated that non-Lipschitzian dynamics based upon the failure of Lipschitz condition exhibits a new qualitative effect: a multi-choice response to periodic external excitations. It appeared that dynamical systems which possess such a property can serve as an underlying idealized framework for neural nets with "creativity." Based upon this property, a substantially new class of self-developing dynamical systems was introduced and discussed. These systems are represented in the form of coupled activation and learning dynamical equations whose ability to be spontaneously activated are based upon two pathological characteristics. First, such systems have zero Jacobian. As a result, they have an infinite number of equilibrium points which occupy curves, surfaces, or hypersurfaces. Second, at all these equilibrium points, the Lipschitz condition fails, so the equilibrium points become terminal attractors or repellers depending on the sign of the periodic excitation. Both of these pathological characteristics result in self-developing properties of dynamical systems.

Four types of self-developing dynamical systems were introduced and discussed. The first type is represented by totally unpredictable systems which are characterized by unpredictable behavior, unpredictable location of their attracting sets, and unpredictable terminal state. It should be emphasized that, in contradistinction to chaotic systems (which are structurally stable, and therefore, whose averaged properties are predictable), these systems have an unpredictable structure. One should also recall that in the chaotic systems, the unpredictability of a particular trajectory is caused by a supersensitivity to the initial conditions, while the uniqueness of the solution for fixed initial conditions is guaranteed by the Lipschitz condition. In contrast, the unpredictability of self-developing dynamical systems is caused by the failure of the uniqueness of the solution at some of the attracting sets. It is still unclear whether the sequence of attracting sets created by a self-developing system has a hidden order and can be considered as a more complex attracting object.

From the practical viewpoint, self-developing systems of this type can be regarded as a mathematical framework for modeling "nonrigid" dynamical behavior.

The second type of self-developing systems is characterized by a global objective which makes the terminal state of the system fully predictable, although the strategy for approaching this objective is not prescribed: the system must "create" its own strategy. Hence, these systems are self-programmed. However, the price paid for that is an unpredictable time required for approaching the desired terminal state.

The third type of self-developing systems (the guided systems) has a microstructure: it contains infinitesimal bias terms which control the system behavior at the critical points where the system must make a choice between several different available scenarios of motion. In contrast to the previous case, the behavior of such systems is fully deterministic, although their final state is not prescribed in advance. However, one has to realize that the determinism of the guided systems is as shaky as those in chaotic systems, because they are supersensitive to infinitesimal changes of the bias terms. Obviously, the type of instability in guided self-developing systems is different from the chaotic ones: it is characterized by an instantaneous jump from one branch of the solution to another at the critical points, while in chaotic motions the shift from one trajectory to another develops gradually.

The last type of self-developing dynamical system has both global objective and a microstructure. Its behavior is deterministic, but nonrigid: several subsystems can be uncoupled for most of the time, and they effect each other only during a vanishingly short interval. That is why these systems can model cause-and-effect relationships.

Thus, it has been demonstrated that self-developing dynamical systems which spontaneously change their own structure can be utilized for modeling more complex relationships than those modeled by classical dynamics. From the viewpoint of neural networks, these systems suggest the way of minimizing pre-programming by entrusting this procedure to the dynamical system itself.

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