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Exponential Transient Classes of Symmetric Neural Networks for Synchronous and Sequential Updating

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Abstract. We exhibit a class of symmetric neural networks which synchronous iteration possesses an exponential transient length. In fact if $\{1,\ldots,n\}$ is the set of nodes we prove the transient length satisfies $\tau \geq 2^{n/3}$. For sequential updating we get the bound $\bar{\tau} \geq 2^{n/6}$. This behavior shows that the dynamics of these class of networks is complex while the steady states are simple: only fixed points or orbits of period 2.

1. Introduction

Neural networks with symmetric connections (a_{ij}) have been developed and applied in several areas including associative memories and pattern recognition [1,2]. Extremely simple behavior appears for the steady-state when the dynamics of this class of networks is studied: the limit orbits are cycles of length 1 or 2 [3], and if the matrix of connections is positive-definite only fixed points are obtained [4].

But this does not mean the dynamics of the network is in itself simple because the transient can be very complicated. More precisely we shall prove there exists a class of symmetric networks with exponential transient, i.e., possessing initial conditions which transient length is $\geq 2^{\alpha n}$ being n the number of cells of the network and α some positive constant.

This result is shown for synchronous and sequential update. We prove for synchronous iteration there exist classes which transient length verifies $\tau \geq 2^{n/3}$ and for sequential iteration we obtain transient length of order $\bar{\tau} \geq 2^{n/6}$.

A first approach to the exponential behavior was studied in [5] in the context of synchronous update. The construction used in this reference was

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based in self-dual networks which, in opposition to the present work, makes the dynamics of the construction difficult to follow and interpret.

Recently an analogous class of network evolution was exhibited in [6] but the update was not synchronous neither sequential. In fact, in this last work the choice of the node to update needed to evaluate a minima value of a certain potential of the network which implied a global knowledge about the state of the network and is not compatible with the distributed computing paradigm. In any case we refer to the discussion made in reference [6] about relation among complexity and exponential transient classes of symmetric neural networks.

2. Exponential transient classes for synchronous iteration

Let $I = \{1, \ldots, n\}$ be the set of nodes of the networks, $A = (a_{ij} : i, j \in I)$ be the matrix of connections and $b = (b_i : i \in I)$ be the set of thresholds. We note by (A, b, I) this neural network, it is called symmetric if A is symmetric, i.e., $a_{ij} = a_{ji}$ for any $i, j \in I$.

The synchronous update of the network (A, b, I) is given by the following evolution equation:

$$x_i(t+1) = \mathbf{1}\left(\sum_{j \in I} a_{ij} x_j(t) - b_i\right), i \in I, t \geq 0 \tag{2.1}$$

where $\mathbf{1}(u) = 1$ if $u \geq 0$ (0 otherwise).

A stable configuration is a finite periodic sequence $(x(t_0), \ldots, x(t_0 + l) = x(t_0))$.

If $l \geq 1$ is the minimal number for which $x(t_0 + l) = x(t_0)$ it is called the period of the sequence. When A is symmetric any stable configuration has period $l \leq 2$ [2].

If $x(0) = (x_i(0) : i \in I)$ is an initial condition its synchronous transient length τ is the time it takes evolving under equation (2.1) and departing from $x(0)$ to enter for the first time to a stable configuration.

Exponential transients for symmetric neural networks are built by a recursive procedure. Before to formalize it we illustrate the construction for $n = 2$ and $n = 5$ as follows. Let us take the couple (A, b) given by $a_{11} = -1$, $a_{12} = a_{21} = 1$, $a_{22} = 1$; $b_1 = -1/2$, $b_2 = 1$. It is direct that the synchronous trajectory associated to (A, b) is $(0,0) \to (1,0) \to (0,1) \to (1,1)$, where the vector $(1,1)$ is a fixed point. It is important to point out that the trajectory travel through all the vertex of the 2-hypercube. Now, by adding three cells to previous network one may travel on the vertex of the 3-hypercube. It suffices to take the couple $(\tilde{A}, \tilde{b})$:

$$\tilde{A} = \left[\begin{array}{cc|ccc} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 1 & -2 & 1 \\ & & 1 & -3 & 2 \\ \hline 1 & 1 & 0 & 6 & 0 \\ -2 & -3 & 6 & 0 & 4 \\ 1 & 2 & 0 & 4 & 0 \end{array}\right] \qquad \tilde{b} = \begin{pmatrix} b \\ 2 \\ 1 \\ 4 \end{pmatrix}$$

The new cells $\{3,4,5\}$ are used as control units to repeat twice the dynamics of the 2-hypercube. The dynamics of $(\tilde{A},\tilde{b})$ for the initial condition $x(0) = (0,0,0,0,0)$ is the following:

t	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$	$x_5(t)$	
0	0	0	0	0	0	
1	1	0	0	0	0	
2	0	1	0	0	0	
3	1	1	0	0	0	
4	1	1	1	0	0	
5	1	1	1	1	0	
6	0	0	1	1	1	
7	1	0	1	1	1	
8	0	1	1	1	1	
9	1	1	1	1	1	fixed point

The cells $\{1,2,3\}$ travel through the 3-hypercube. For that the fourth cell is on if and only if the first travel of the 2-hypercube is finished. Also, it switches off the cells $\{1,2\}$. By doing so, these cells repeat the initial trajectory of the 2-hypercube.

Extending previous construction to networks of any size allows us to get the following result:

Theorem 1. *For any n there exists a symmetric neural network (A,b) such that its synchronous transient length verifies $\tau(A,b) \geq 2^{n/3}$.*

Proof. Recall it suffices to show for any n of the form $n = 3m+2$ we can construct a symmetric network verifying $\tau(A,b) \geq 3(2^{m+1}-1)$. In fact if $n = 3m+4$ (or $n = 3m+3$) we can bound its transient by the transient of the case $3m+2$. As $m = n - 4/3$ we deduce $\tau(A,b) \geq 3(2^{n-4/3+1}-1)$ which is $\geq 2^{n/3}$, for $n \geq 10$. Now for $n < 10$ it follows directly from the network we shall exhibit that $\tau(A,b) \geq 2^{n/3}$. Hence assume $n = 3m+2$ with $m \geq 0$.

The symmetric neural network we shall construct on the set of nodes $I^{(m)} = \{1,\ldots,3m+2\}$ will contain a trajectory at least as large as the following one:

$$\begin{aligned}(0,0,\ldots,0) \;\in\; & \{0,1\}^{m+2} \to (0,1,\ldots,0) \to (0,0,1,\ldots,0) \to \ldots \\ & \to (1,1,1,\ldots,1),\end{aligned}$$

which contains 2^{m+2} different points. The other $2m$ sites being used to control the network, i.e., their connections will make possible that such an evolution can be realized.

The construction of the network will be made recursively then at each step we add three nodes, one of them allows to increase the length of the above orbit, the other two being used for control.

First take $k = 0$, $I^{(0)} = \{1,2\}$, we shall construct $A^{(0)}$ (a 2×2 symmetric matrix) and $b^{(0)}$ (a 2-vector). After we suppose we have constructed a symmetric matrix $A^{(k)}$ and a vector $b^{(k)}$ for $I^{(k)} = \{1,\ldots,3k+2\}$ and we give an algorithm to construct a symmetric matrix $A^{(k+1)}$ and a vector $b^{(k+1)}$ on $I^{(k+1)} = \{1,\ldots,3(k+1)+2\}$. The sequence $(A^{(k)}, b^{(k)})$ defined by the algorithm will verify:

i. $A^{(k+1)}$ restricted to $I^{(k)} \times I^{(k)}$ is equal to $A^{(k)}$.

ii. $b^{(k+1)}$ restricted to $I^{(k)}$ coincide with $b^{(k)}$.

iii. The initial condition $x(0) = (0,\ldots,0) \in \{0,1\}^{3k+2}$ possesses a transient length $\tau_k = 3(2^{k+1}-1)$ when we make evolve it in a synchronous way with matrix $A^{(k)}$ and vector $b^{(k)}$.

Then when we put $A = A^{(m)}$, $b = b^{(m)}$ the initial configuration $x(0) = (0,\ldots,0) \in \{0,1\}^m$ will possess a transient length $\tau_m = 3(2^{m+1}-1) = 3(2^{n-2/3+1}-1)$. Then it fulfills the properties we have asserted. So take $I^{(0)} = \{1,2\}$. We construct $A^{(0)} = (a_{ij} : i,j \in I^{(0)})$, $b^{(0)} = (b_1, b_2)$ in order than $x(0) = (0,0)$ has the following dynamics:

$$\begin{array}{llll} t & x_1(t) & x_2(t) & \\ 0 & 0 & 0 & \text{transient behavior} \\ 1 & 1 & 0 & \text{transient behavior} \\ 2 & 0 & 1 & \text{transient behavior} \\ \tau_0 = 3(2^1-1) = 3 & 1 & 1 & \text{fixed point} \\ 5 & 1 & 1 & \text{fixed point} \end{array} \tag{2.2}$$

It is easy to see that when we improve conditions:

$$a_{11} < b_1 < 0 < b_2 < \inf(a_{12}, a_{22}), b_1 < a_{11} + a_{12}, a_{21} = a_{12} \tag{2.3}$$

we get the above dynamics of $x(0)$. Then the transient length of initial condition $x(0)$ is $\tau_0 = 3$.

Now call $C^{(0)}$ the matrix which contains the transient evolution of $x(0) = (0,0)$ and the first time it attained the fixed point:

$$C^{(0)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.4}$$

Now suppose we have constructed $A^{(k)} = (a_{ij} : i,j \in I^{(k)})$, $b^{(k)} = (b_i : i \in I^{(k)})$ in such a way that $x(0) = (0,\ldots,0) \in \{0,1\}^{3k+2}$ evolves as follows:

$$\begin{array}{llll} t & x_1(t) & \cdots & x_{3k+3}(t) \\ 0 & & & \\ \vdots & & C^{(k)} & \\ \tau_k = 3(2^{k+1}-1) & & & \\ \tau_k + 1 & 1 & \cdots & 1 \end{array} \tag{2.5}$$

with transient length $\tau_k = 3(2^{k+1} - 1)$.

Now we will add three elements: $3k+3$, $3k+4$, $3k+5$. We will explicit the restrictions of the symmetric matrix of connections $A^{(k+1)}$ and the threshold vector $b^{(k+1)}$ in order that the initial configuration $x(0) = (0, \dots, 0) \in \{0,1\}^{3(k+1)+2}$ evolves in the following way:

$$\begin{array}{cccccccc}
t & x_1(t) & \cdots & x_{3k+2}(t) & x_{3k+3}(t) & x_{3k+4}(t) & x_{3k+5}(t) \\
0 & & & & 0 & 0 & 0 \\
\vdots & & C^{(k)} & & \cdots & \cdots & \cdots \\
\tau_k = 3(2^{k+1}-1) & & & & 0 & 0 & 0 \\
\tau_k + 1 & 1 & \cdots & 1 & 1 & 0 & 0 \\
\tau_k + 2 & 1 & \cdots & 1 & 1 & 1 & 1 \\
 & & & & 1 & 1 & 1 \\
\vdots & & C^{(k)} & & \vdots & \vdots & \vdots \\
\tau_{k+1} = 3(2^{k+2}-1) & & & & 1 & 1 & 1 \\
\tau_{k+1} + 1 & 1 & \cdots & 1 & 1 & 1 & 1
\end{array} \tag{2.6}$$

In the above process we deduce $\tau_k = 3(2^{k+1} - 1)$ because it is the solution to equation $\tau_{k+1} = 2\tau_k + 2 + 1$ with initial condition $\tau_0 = 3$. Also from the recurrence construction it follows that the row $\tau_{k+1} = 3(2^{k+2} - 1)$ is equal to the row $\tau_{k+1} + 1$, and also that the first $t = \tau_{k+1}$ for which we enter to a steady state is for $\tau_{k+1} = 3(2^{k+2} - 1)$ which is the transient length of $x(0)$. We take $C^{(k+1)}$ equal to the matrix formed from row $t = 1$ till the row $t = \tau_{k+1}$ and we continue the process:

$$C^{(k+1)} = \left[\begin{array}{cccccc}
 & & & 0 & 0 & 0 \\
 & C^{(k)} & & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 \\
1 & \cdots & 1 & 1 & 0 & 0 \\
1 & \cdots & 1 & 1 & 1 & 1 \\
 & C^{(k)} & & \vdots & \vdots & \vdots \\
 & & & 1 & 1 & 1
\end{array}\right]
\begin{array}{c}
1 \\ \vdots \\ \tau_k \\ \tau_k + 1 \\ \tau_k + 2 \\ \vdots \\ \tau_{k+1}
\end{array} \tag{2.7}$$

Hence we must explicit the constraints of $A^{(k+1)}$, $b^{(k+1)}$ in order that the above evolution of $x(0)$ could be possible and we must also show this system of constraints admits a solution.

First the coefficients a_{ij} of $A^{(k+1)}$ for $i, j \in I^{(k)}$ are the same as those for $A^{(k)}$, also the coefficients b_i of $b^{(k+1)}$ for $i \in I^{(k)}$ is the same as those of $b^{(k)}$.

In order that $x_i(\tau_k + 1) = 1$ for any $i \in I^{(k)}$, we must have:

$$\gamma_i^{(k)} = \sum_{j \in I^{(k)}} a_{ij} > b_i \text{ for any } i \in I^{(k)} \tag{2.8}$$

Recall (2.8) is verified for $k = 0$ (see (2.3)), we suppose it is verified by recurrence hypothesis for k.

On the other hand the condition $x_{3k+3}(t) = 0$ for $t = 0, \ldots, \tau_k$ is implied by the stronger condition:

$$b_{3k+3} > \kappa_{3k+3} \text{ where } \kappa_{3k+3} = \sup_{L' \subset I^{(k)}, L' \neq I^{(k)}} \sum_{j \in L'} a_{3k+3,j} \tag{2.9}$$

Equality $x_{3k+3}(\tau_k + 1) = 1$ is implied by the condition:

$$\sum_{j \in I^{(k)}} a_{3k+3,j} > b_{3k+3} \tag{2.10}$$

Now $x_i(\tau_k + 2) = 1$ for any $i \in I^{(k)}$ follows from the inequality:

$$\sum_{j \in I^{(k)}} a_{ij} + a_{i,3k+3} > b_i \quad \forall i \in I^{(k)} \tag{2.11}$$

In order to verify conditions (2.9), (2.10), (2.11) we take: $a_{i,3k+3} = a_{3k+3,i} > 0$, so $\kappa_{3k+3} < \sum_{j \in I^{(k)}} a_{3k+3,j}$, so we choose b_{3k+3} verifying (2.9), (2.10) which do always exist. Inequality (2.11) follows from positiveness of $a_{i,3n+3}$ and (2.8).

To get $x_{3k+3}(\tau_k + t + 1) = 1$ for $t = 1, 2, \ldots, (\tau_{k+1} - \tau_k)$, we improve

$$a_{3k+3,3k+3} > \sup_{L' \subset I^{(k)}} (-\{\sum_{j \in L'} a_{3k+3,j} + a_{3k+3,3k+4} + a_{3k+3,3k+5}\} + b_{3k+3}) \tag{2.12}$$

for any choose we make of $a_{3k+3,3k+4}, a_{3k+4,3k+5}$ will not depend on $a_{3k+3,3k+3}$.

To get the above dynamics of node $3k+4$, i.e., $x_{3k+4}(t) = 0$ for $t = 0, \ldots, \tau_k + 1$; $x_{3k+4}(\tau_k + 2) = 1$ and $x_{3k+4}(\tau_k + t + 2) = 1$ for $t = 1, 2, \ldots, (\tau_{k+1} - \tau_k + 1)$ we require for the following inequalities to be verified:

$$b_{3k+4} > \kappa_{3k+4} \text{ where } \kappa_{3k+4} = \sup_{L' \subset I^{(k)}} \sum_{j \in L'} a_{3k+4,j} \tag{2.13}$$

$$a_{3k+4,3k+3} = a_{3k+3,3k+4} > b_{3k+4} - \sum_{j \in I^{(k)}} a_{3k+4,j} \tag{2.14}$$

$$a_{3k+4,3k+4} > \sup_{L' \subset I^{(k)}} (-\{\sum_{j \in L'} a_{3k+4,j} + a_{3k+4,3k+3} + a_{3k+4,3k+5}\} + b_{3k+4}) \tag{2.15}$$

The conditions on coefficients $a_{3k+4,i} = a_{i,3k+4}$ come from the equality $x_i(\tau_k + 3) = 0$ for any $i \in I^{(k)}$. Then we also need the condition: $k \sum_{j \in I^{(k)}} a_{ij} + a_{i,3k+3} + a_{i,3k+4} < b_i$. Hence we impose:

$$a_{3k+4,i} = a_{i,3k+4} < -(\gamma_i^{(k)} + a_{i,3k+3}) \text{ for } i \in I^{(k)} \tag{2.16}$$

where the $\gamma_i^{(k)}$ was defined in (2.8). So $a_{3k+4,i}$ is strictly negative, $a_{3k+4,3k+3}$ is strictly positive. There always exist solution for the above requirements (2.8)–(2.11).

In an analogous way to get $x_{3k+5}(t) = 0$ for $t = 0, \ldots, \tau_k + 2$, $x_{3k+5}(\tau_k + 3) = 1$, $x_{3k+5}(\tau_k + t + 3) = 1$ for $t = 1, 2, \ldots, (\tau_{k+1} - \tau_k - 2)$, we impose:

$$b_{3k+5} > \kappa_{3k+5} \text{ where } \kappa_{3k+5} = \sum_{L' \subset I^{(k)} \cup \{3k+3\}} \sum_{j \in L'} a_{3k+5,j} \tag{2.17}$$

$$a_{3k+5,3k+4} = a_{3k+4,3k+5} > b_{3k+5} - \sum_{j \in I^{(k)} \cup \{3k+3\}} a_{3k+5,j} \tag{2.18}$$

$$a_{3k+5,3k+5} > \sup_{L' \subset I^{(k)}} \Big(-\{\sum_{j \in L'} a_{3k+5,j} + a_{3k+5,3k+3} + a_{3k+5,3k+4}\} + b_{3k+5}\Big) \tag{2.19}$$

The last evolution equations, which will only involve conditions on $a_{3k+5,j} = a_{j,3k_5}$ for $j \in I^{(k)}$, are the following:

$$x_i(\tau_k + 3 + t) = x_i(t) \text{ for } i \in I^{(k)}, t = 0, \ldots, \tau_k \tag{2.20}$$

In order to satisfy (2.20) let us make the following choice of $a_{3k+5,j}$:

$$a_{3k+5,j} = a_{j,3k+5} = -(a_{3k+4,j} + a_{3k+3,j}) \text{ for } j \in I^{(k)} \tag{2.21}$$

Remark that (2.21) implies the equalities:

$$\sum_{j \in L' \cup \{3k+3, 3k+4, 3k+5\}} a_{ij} = \sum_{j \in L'} a_{ij} \qquad \begin{array}{l} \forall i \in I^{(k)} \\ \forall L' \subset I^{(k)} \end{array} \tag{2.22}$$

We claim property (2.20) follows from expressions (2.16), (2.11). This is shown by recurrence on $t \geq 0$. In fact for $t = 0$ the equality (2.20) is implied by condition (2.16) and we apply recurrence on $t \geq 1$ by using condition (2.22).

As there always exists a solution for (2.17), (2.18), (2.19), (2.21) we have proved that the evolution of initial condition $x(0) = (0, \ldots, 0) \in \{1, 1\}^{3(k+1)+2}$ is the one asserted in (2.6). Finally we must remark that condition (2.22) together with (2.12), (2.15), (2.19) imply: $\gamma_i^{(k+1)} = \sum_{j \in I^{(k+1)}} a_{ij} > b_i$ for any $i \in I^{(k+1)}$, then (2.8) can be assumed by recurrence hypothesis. So the result. ∎

Remark. The fact to add three cells in the inductive process is imposed to maintain the symmetry of matrix A. Otherwise it is easy to obtain very large transients.

3. Exponential transient classes for sequential evolution

Now we shall prove that the sequential iteration of symmetric neural networks also contain classes with exponential transient. The proof of this fact will use a construction made by Tchuente [7] which allows to simulate any synchronous neural network update by a sequential one.

Recall that if $J = \{1, \ldots, p\}$ is the set of nodes, $C = (c_{ij} : i, j \in J)$ the connection matrix, $d = (d_i : i \in J)$ the threshold vector, then the sequential evolution of the neural network (C, d, J) iterated with the usual order $\leq$ in J is

$$y_i(t+1) = \mathbf{1}\left(\sum_{j<i} c_{ij} y_j(t+1) + \sum_{j \geq i} c_{ij} y_j(t) - d_i\right) \quad \text{for } i \in J, t \geq 0 \quad (3.1)$$

Now suppose the neural network (A, b, I) defined on the set of nodes $I = \{1, \dots, n\}$ is iterated synchronously. On the set of nodes $\bar{I} = \{1, \dots, 2n\}$ construct the following neural network $(\bar{A}, \bar{b}, \bar{I})$:

$$\bar{a}_{ij} = \begin{cases} 0 & \text{if } |i-j| \leq n \\ a_{i,j-n} & \text{if } 1 \leq i \leq n, n+1 \leq j \leq 2n \\ a_{i-n,j} & \text{if } n+1 \leq i \leq 2n, 1 \leq j \leq n \end{cases} \qquad (3.2)$$

Consider the usual order $\leq$ in $\bar{I}$ and iterate $(\bar{A}, \bar{b}, \bar{I})$ sequentially.

Now take $x(0) \in \{0,1\}^n$ an initial condition and let $(x(t) : t \geq 0)$ be its synchronous evolution for (A, b, I). Now pick the initial condition $\bar{x}(0) = (x(0),\ x(0)) \in \{0,1\}^{2n}$ and note $(\bar{x}(t) : t \geq 0)$ its orbit for the sequential evolution of $(\bar{A}, \bar{b}, \bar{I})$. Then it is easy to see that $\bar{x}(t) = (x(t), x(t))$. so the transient T of $x(t)$ is equal to the transient $\bar{T}$ of $\bar{x}(t)$. Hence by the construction made in section 2 we get the following result:

Theorem 2. *On the set of nodes $\bar{I} = \{1, \dots, p\}$ there exists a symmetric neural network $(\bar{A}, \bar{b}, \bar{I})$ and an initial condition $\bar{x}(0)$ which sequential iteration has a transient length $\bar{\tau}$ satisfying $\bar{\tau} \geq 2^{p/6}$.*

Proof. Suppose p is even, take $n = p/2$. Construct A, b of section 2, and define $\bar{A}, \bar{b}$ on $\{1, \dots, p = 2n\}$ as we made in the above paragraph. Then the transient length of the initial condition $\bar{x}(0) = (x(0), x(0)) = (0, \dots, 0, 0, \dots, 0) \in \{0,1\}^{2n}$ is $\bar{\tau} \geq 2^{n/3} = 2^{p/6}$. On the other hand recall that the definition made in (3.2) is such that A symmetric implies $\bar{A}$ symmetric. Then we conclude the theorem. For p odd it is easy to show that the result is also verified. ∎

4. Conclusion

Instead of the simplicity of the steady state behavior of symmetric neural networks (fixed points and/or two cycles) we proved that the transient phase may be complex, i.e., large time to reach the stationary regime.

This fact is important to understand the performances of neural networks in applications such as associative memories, optimization strategies, etc. In fact in such cases the convergence is fast because the matrix entries are not very different in size or the matrix verify same hypothesis that implies short transient times (i.e., positive–definite, regular connections, etc.).

On the other hand, numerical experiences with symmetric matrices show that the exponential behavior appear rarely and from a statistical point of view convergence is very fast; usually $0(n)$.

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References

[1] J.H. Hopfield, "Neural networks and physical systems with emergent collective computational abilities," *Proc. Natl. Acad. Sci. USA*, **79** (1982) 2554–2558.

[2] E. Bienstock, F. Fogelman, and G. Weisbuch (eds.), *Disordered Systems and Biological Organization*, vol. 20, Nato ASI Series F: Computer and Systems Sciences (Springer-Verlag, 1986).

[3] E. Goles and J. Olivos, "Comportement périodique des fonctions à seuil binaires et applications," *Disc. Appl. Math.*, **3** (1981) 95–105.

[4] E. Goles, "Fixed point behavior of threshold functions on a finite set," *SIAM J. on Alg. and Disc. Meth.*, **3(4)** (1982) 529–531.

[5] E. Goles and J. Olivos, "The convergence of symmetric threshold automata," *Inf. and Control*, **51** (1981) 98–104.

[6] A. Haken and M. Luby, "Steepest descent can take exponential time for symmetric connection networks," *Complex Systems*, **2** (1988) 191–196.

[7] M. Tchuente, "Sequential simulation of parallel iterations and applications," *Theor. Comp. Sci.*, **48** (1986) 135–144.