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Cellular Automata with Regular Behavior

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1. Introduction

The study of cellular automata (CA) was motivated recently by their application to systems whose complex behavior arises from the interaction among simple identical components. Actually, a CA consists of a linear biinfinite array of cells, each one connected with the r cells that precede it on the left-hand side and the r cells that follow it on the right-hand side *neighborhood*. Each cell is in one of finitely many states. The new state of a cell is computed according to a local rule that is a function of the states of the cells in the neighborhood, besides the old state of the cell.

All cells are assumed to change state simultaneously.

In [5] CA are classified with respect to their behavior. The great part of CA falls in the third class, that is, the one whose evolution leads to a chaotic pattern. Recently, however, Wilson [3] and Culik [1] exhibited some CA belonging to this class and having a very regular behavior, fractal-like on particular initial configurations.

In this paper, we study a class we will call pseudototalistic cellular automata (PTCA).

In PTCA, the local rule defines the new state of a cell as a linear combination of the state of its neighborhood. They have, in one case, the additive CA studied by Culik in [1], where it is shown that they have a highly regular behavior on an arbitrary finite configuration as initial seed. Here we will prove that PTCA whose number of states is a prime number show regular behavior when the initial configuration is a single non-idle state.

In section 4 we study the class of CA with radius r and code $0^n 1^{2m} 0^n$ where $n + m = r + 1$, $n \geq 1$, $m \geq \epsilon_r \geq 1$, where ϵ_r is some constant depending on r. Culik conjectured in [1] that they behave in a regular way for finite configuration as initial seed. In [1] it is proven that CA with code 001100 and 011110 behave regularly on seeds of the form $(00 + 11)^*$. Here we prove that CA with code 0110 behave regularly on seeds belonging to the set $I = \sum^* - \Gamma$ where $\sum = \{0,1\}$ and $\Gamma = \sum^* 000 \sum^* \cup \sum^* 111 \sum^*$.

2. Preliminaries

Formally a CA is a triple (S, f, r) where S is a finite set of states, r is the neighborhood radius, and $f : S^{2r+1} \to S$ is the local function called CA-rule. At time zero, an arbitrary configuration is assumed: initial seed; then, all cells change state simultaneously according to the local function.

A configuration is a function $\alpha : Z \to S$; the set of configurations is denoted by S^Z and $\alpha(i)$ represents the state of the ith cell for every $\alpha \in S^Z$; $G_f : S^Z \to S^Z$ is the *global function* such that if α and β are two configurations, then

$$G_f(\alpha) = \beta \text{ iff } f[\alpha(i-r), \ldots, \alpha(i), \ldots, \alpha(i+r)] = \beta(i) \text{ for every } i \in Z;$$

$G_j^k(\alpha)$ represents the configuration obtained after k steps from α.

A CA $A = (S, f, r)$ is called a totalistic cellular automata (TCA) if S is a subset of Z and there exists a new function f' such that $f(s_{-r}, \ldots, s_0, \ldots, s_r) = f'(s_{-r} + \ldots + s_0 + \ldots + s_r)$.

If $S = \{0, 1\}$, then the transition function can be expressed as $b_{2r+1}, b_{2r}, \ldots, b_0$, i.e.,

$$f(s_{-r}, \ldots, s_0, \ldots, s_r) = b_i \text{ iff } s_{-r} + \ldots + s_0 + \ldots + s_r = i$$

An interesting case of TCA are those defined by XOR-rule; with this rule a cell at time $t+1$ will be active iff the number of cells active in its neighborhood at time t is odd.

The class of PTCA is obtained by considering all the CA $A = (S, f, r)$ such that $S = \{0, 1, \ldots, n-1\}$ and for every $x_{-r}, \ldots, x_0, \ldots, x_r \in S$, $f(s_{-r}, \ldots, s_0, \ldots, s_r) = (s_{-r}x_{-r} + \ldots + s_0x_0 + \ldots + s_rx_r) \bmod n$ where $s_i \in S$ for $-r \le i \le r$.

From now on let $s_i(t)$ be the state of the ith cell at time t.

With this assumption the XOR-rule can be expressed as $s_i(t) = (s_{i-r}(t) + \ldots + s_i(t) + \ldots + s_{i+r}) \bmod 2$. Note that this is a particular case of a rule of PTCA.

3. PTCA

In this section we will prove that the behavior of PTCA $A = (S, f, r)$ with $S = \{0, 1, \ldots, p-1\}$ where p is a prime number, pPTCA, is very regular at least when the initial seed has only one nonquiescent state.

First we show how to compute the speed-up rule for PTCA, by giving a closed form formula for $s_i(t+k)$ for every k.

Theorem 1. *Let $A = (S, f, 1)$ be a PTCA. Then there exist $t_{-k}, \ldots, t_k$ such that*

$$s_i(t+k) = \left(\sum_{j=-k}^{k} t_j s_{i+j}(t) \right) \bmod n \tag{3.1}$$

Moreover,

$$
\begin{aligned}
t_j &= \sum_{i=0}^{\lfloor\frac{k-|j|}{2}\rfloor} \binom{k}{i}\binom{k-i}{i+j} x_1^{i+\frac{|j|-j}{2}} x_2^{k-|j|-2i} x_3^{i+\frac{|j|+j}{2}} \\
& \sum_{i=0}^{\lfloor\frac{k-|j|}{2}\rfloor} C(k,i,j)X(k,i,j)
\end{aligned}
$$

Proof. The proof is by induction on k.It is omitted because it is very long and tedious. ∎

We start by studying pPTCA with initial seed $^{\omega}0s0^{\omega}$ when $x_1 = x_2 = x_3 = 1$ and $s = 1$. It is easy to generalize to case $s > 1$.

Lemma 2. *Given a pPTCA* $A = (S, f, 1)$ *with initial seed* $^{\omega}010^{\omega}$ *and* $f(x,y,z) = (x+y+z)$ *mod p, then*

$$s_i(kp) = 0 \text{ for } k \geq 1 \text{ and } (j-1)p < i < jp \quad \forall j : -k+1 \leq j \leq k$$

Proof. In figure 1 the behavior claimed by lemma 2 is shown.

From theorem 1 we have

$$s_i(k) = \left\{ \sum_{h=0}^{\lfloor\frac{k-|i|}{2}\rfloor} \binom{k}{h}\binom{k-h}{h+|i|} \right\} \bmod p \qquad -k \leq i \leq k$$

We provide the proof only for $i \geq 0$, since the proof for $i < 0$ is symmetrical.

Let $i > 0$:

$$
\begin{aligned}
s_i(kp) &= \left\{ \sum_{h=0}^{\lfloor\frac{kp-|i|}{2}\rfloor} \binom{kp}{n}\binom{h}{i} \right\} \bmod p \\
&= \left\{ \sum_{h=0}^{\lfloor\frac{kp-|i|}{2}\rfloor} \frac{(kp)!}{h!(kp-2h-i)!(h+i)!(h+i)!} \right\} \bmod p \, (*)
\end{aligned}
$$

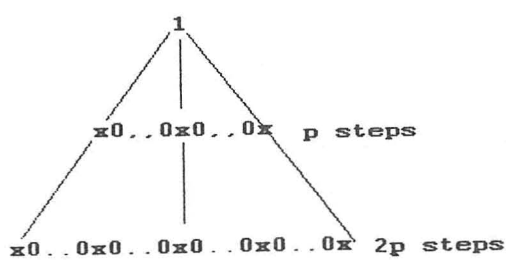

Figure 1.

```
        1
       111
      12321
     1312131
    140141041
   10000100001
  1110011100111
 123211232112321
13121441214412131
1401424424244241041
100001000010000100001
```

Figure 2.

First we note that the factor n can be found k times in $(kp)!$.

Since the factor p can appear at most $k-1$ times in $h!(k-2h-i)!(h+i)!$, in each term of $(*)$ the factor p occurs and cannot disappear because p is a prime number.

Then:

$$s_i(kp) = \left\{ \sum_{h=0}^{\lfloor \frac{kp-|j|}{2} \rfloor} t_h p \right\} \bmod p = 0 \qquad t_h \in N$$

Suppose now $i = j$ with $-k \le j \le k$. When $j \ge 0$,

$$\begin{aligned} s_i(kp) &= \left\{ \sum_{h=0}^{\lfloor \frac{(k-1)p}{2} \rfloor} \binom{kp}{h} \binom{kp-h}{h+jp} \right\} \bmod p \\ &= \left\{ \sum_{h=0}^{\lfloor \frac{(k-1)p}{2} \rfloor} \frac{(kp)!}{h![(k-j)p-2h]!(h+jp)!} \right\} \bmod p \end{aligned}$$

The number of occurrences of the factor p in $h![(k-j)-2h]!(h+jp)!$ is $k-1-I(2x) \le k-1$ if $x > 0$, k otherwise, where $I(c) = a$ if $p \le c \le (a+1)p$ and $x = h - I(h) \times p$.

Then not all terms are 0. ∎

In figure 2 the evolution of pPTCA $A = (S, f, 1)$ when $S = \{0, 1, \ldots, 4\}$ is shown.

Lemma 3. *Given a pPTCA* $A = (S, f, 1)$ *with initial seed* $^{\omega}010^{\omega}$, *and* $f(x, y, z) = (x + y + z) \bmod p$ *then*

$$s_{jp}(kp) = s_j(k) \bmod p \tag{3.2}$$

Proof. It can be noted that (3.2) is equivalent to the following:

$$s_{jp}(kp) = (s_{(j-1)p}[(k-1)p] + s_{jp}[(k-1)p] + s_{(j+1)p}[(k-1)p]) \bmod p$$

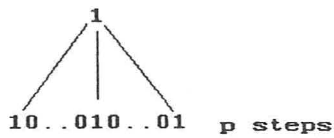

Figure 3.

When $k = 1$ we only need to prove that $G_j^p\,({}^{\omega}010^{\omega}) = (10^{p-1})^3$ (see figure 3). We prove the assertion only for $0 \le i \le p$ since the proof is symmetrical for $i < 0$.

$$s_p(p) = \left\{ \sum_{h=0}^{\lfloor \frac{p-|p|}{2} \rfloor} \binom{p}{h} \binom{p-h}{h+p} \right\} \bmod p = 1$$

$$s_0(p) = \left\{ \sum_{h=0}^{\lfloor \frac{p}{2} \rfloor} \binom{h}{p} \binom{p-h}{h} \right\} \bmod p$$

Since p is a prime number $\lfloor p/2 \rfloor = (p-1)/2$,

$$\begin{aligned} s_0(p) &= \left\{ \sum_{h=0}^{\frac{p-1}{2}} \binom{p}{h} \binom{p-h}{h} \right\} \bmod p \\ &= \left\{ 1 + \sum_{h=1}^{\frac{p-1}{2}} \frac{p!}{h!(p-2h)!h!} \right\} \bmod p \end{aligned}$$

But

$$\frac{p!}{h!(p-2h)!h!} \bmod p = 0 \quad \forall h : 1 \le h \le (p-1)/2$$

since h, $(p-2h) < p$. Then $s_0(p) = 1$.

Moreover, $s_i(p) = 0$ for $i \neq -p, 0, p$ as proved before. Now we prove that (3.2) is true $\forall k > 1$. From theorem 1,

$$s_i(t+q) = \left\{ \sum_{l=-q}^{q} \left[\sum_{h=0}^{\lfloor \frac{q-|l|}{2} \rfloor} \binom{q}{h} \binom{q-h}{h+|l|} \right] s_{i+l}(t) \right\} \bmod p$$

From the last expression ($t = (k-1)p$ and $q = p$) we obtain

$$s_i(kp) = \left\{ \sum_{l=-p}^{p} \left[\sum_{h=0}^{\lfloor \frac{p-|l|}{2} \rfloor} \binom{p}{h} \binom{p-h}{h+|l|} \right] s_{i+l}[(k-1)p] \right\} \bmod p$$

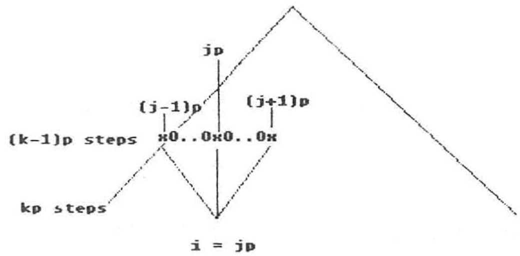

Figure 4.

As can be seen in figure 4, we obtain

$$\begin{aligned} s_{jp}(kp) &= \Bigg\{ \binom{p}{0}\binom{p}{p} s_{(j-1)p}[(k-1)p] \\ &\quad + \sum_{h=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{h}\binom{p-h}{h} s_{jp}[(k-1)p] \\ &\quad + \binom{p}{0}\binom{p}{p} s_{(j+1)p}[(k-1)p] \Bigg\} \bmod p \\ &= \{ s_{(j-1)p}[(k-1)p] + s_{jp}[(k-1)p] + s_{(j+1)p}[(k-1)p] \} \bmod p \; \blacksquare \end{aligned}$$

The results obtained for $x_1 = x_2 = x_3 = 1$ can be generalized for all possible values of x_1, x_2, x_3.

Lemma 4. *Given a pPTCA $A = (S, f, 1)$ with initial seed ${}^\omega 010^\omega$ and $f(x, y, z) = (x_1 x + x_2 y + x_3 z)$ mod p, then*

(a) $s_i(kp) = 0 \quad \forall k \geq 1, \ -k+1 \leq j \leq k, \ (j-1)p < i < jp,$

(b) $s_{jp}(kp) = s_j(p) \quad \forall k \geq 1, \ -k+1 \leq j \leq k.$

Proof. (a) is easily proved analogously to lemma 2. (b) is equivalent to

$$s_{jp} = (x_1 s_{(j-1)p}[(k-1)p] + x_2 s_{jp}[(k-1)p] + x_3 s_{(j+1)p}[(k-1)p]) \bmod p$$

From theorem 1 (for $t = k-1$ and $q = p$) we obtain

$$\begin{aligned} s_{kp} &= \Bigg\{ \sum_{l=-p}^{p} \left[\sum_{h=0}^{\lfloor \frac{p-|l|}{2} \rfloor} \binom{p}{h}\binom{p-h}{h+|l|} x_1^{h+\frac{|l|-l}{2}} x_2^{p-|l|-2h} x_3^{h+\frac{|l|+l}{2}} \right] \\ &\quad s_{i+l}[(k-1)p] \Bigg\} \bmod p \end{aligned}$$

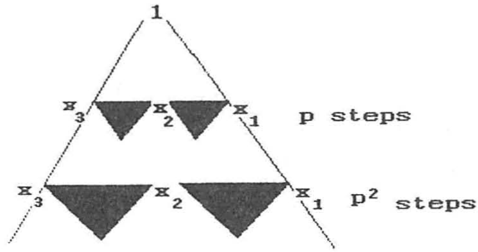

Figure 5.

For $i = jp$ we obtain

$$\begin{aligned} s_{jp}(kp) &= \Bigg\{ \binom{p}{0}\binom{p}{p} x_1^p s_{(j-1)p}[(k-1)p] \\ &+ \left[\sum_{h=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{h}\binom{p-h}{h} x_1^h x_2^{p-2h} x_3^h \right] s_{jp}[(k-1)p] \\ &+ \binom{p}{0}\binom{p}{p} x_3^p s(j+1)[(k-1)p] \Bigg\} \bmod p \end{aligned}$$

but, as before, we know that for $0 < h < p$,

$$\binom{p}{h}\binom{p-h}{h} \bmod p = 0$$

and that $x^p \bmod p = x$, so that

$$s_{jp} = (x_1 s_{(j-1)p}[(k-1)p] + x_2 s_{jp}[(k-1)p] + x_3 s_{(j+1)p}[(k-1)p]) \bmod p$$

■

Theorem 5. *Given a pPTCA $A = (S, f, 1)$ with initial seed $^\omega 010^\omega$ and $f(x, y, z) = (x_1 x + x_2 y + x_3 z)$ mod p, it holds that*

$$G_f^{p^k}(^\omega 010^\omega) = {}^\omega 0 x_3 0^{p^{k-1}} x_2 0^{p^{k-1}} x_1 0^\omega \quad \forall k \geq 0$$

Proof. The proof is easily obtained from the last lemma. ■

An analogous result can be obtained when the initial seed is $^\omega Os0^\omega$ for every possible value of s.

In figure 5, the evolution of a pPTCA is shown. Triangles represent zeros. In the case of a PTCA $A = (S, f, 1)$ such that $f(x, y, z) = (x_1 x + x_2 y + x_3 z)$ mod n, where n is not a prime number, we can find examples having very different behavior.

Example 1.

$$
\begin{aligned}
A &= (S, f, r) \qquad S = \{0,1,2,3\} \quad r = 1 \\
f(x,y,z) &= (2x + y + 2z) \bmod 4 \\
G_f({}^{\omega}010^{\omega}) &= {}^{\omega}2120^{\omega} \\
G_f^2({}^{\omega}010^{\omega}) &= {}^{\omega}010^{\omega} \\
&\cdots\cdots\cdots \\
G_f^{2i}({}^{\omega}010^{\omega}) &= {}^{\omega}010^{\omega} \\
G_f^{2i+1}({}^{\omega}010^{\omega}) &= {}^{\omega}2120^{\omega}
\end{aligned}
$$

```
  0 1 0
0 2 1 2 0
  0 1 0
0 2 1 2 0
```

Example 2. *Suppose $n = x^2$ and let $A = (S, f, r)$*

$$
\begin{aligned}
S &= \{0,1,2,\ldots,n-1\} \\
r &= 1 \\
f(a,b,c) &= (xa + b + xc) \bmod n \\
G_f^k({}^{\omega}010^{\omega}) &= \begin{cases} {}^{\omega}010^{\omega} & \text{if } k = 0 \bmod \sqrt{n} \\ {}^{\omega}0px1px0^{\omega} & \text{if } k = p \bmod \sqrt{n} \end{cases} \\
G_f({}^{\omega}0px1px0^{\omega}) &= {}^{\omega}0px^2(k+1)x1(k+1)xkx^20^{\omega} = \\
{}^{\omega}0(k+1)x1(k+1)x0^{\omega}
\end{aligned}
$$

since $kx^2 \equiv 0 \bmod n$.

Let $n = 9$ and $f(a,b,c) = (3a + b + 3c) \bmod 9$. The evolution of this PTCA with initial seed 1 is shown in the following:

```
  0 1 0
0 3 1 3 0
0 6 1 6 0
  0 1 0
```

Example 3. *Let us consider a PTCA defined as follows:*

$$
\begin{aligned}
A &= (S, f, r) \\
S &= \{0,1,\ldots,n-1\} \\
n &= k^2y \\
r &= 1 \\
f(a,b,c) &= (n/ka + b + n/kc) \bmod n
\end{aligned}
$$

The evolution of this CA with initial seed ${}^{\omega}010^{\omega}$ *is shown in the following:*

$$\begin{array}{ccccc}
 & 0 & 1 & 0 & \\
0 & n/k & 1 & n/k & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & (k-1)n/k & 1 & (k-1)n/k & 0 \\
 & 0 & 1 & 0 &
\end{array}$$

The following PTCA has a particular behavior.

Example 4. $A = (S, f, r)$, $n = 4$, $S = \{0, 1, 2, 3\}$, $f(a, b, c) = (3a + 2b + 2c) \bmod 4$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 2 & 2 & 3 & & & & \\
 & & 0 & 0 & 0 & 0 & 1 & & \\
 & 0 & 0 & 0 & 0 & 2 & 2 & 3 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Example 5. $A = (S, f, r)$, $r = 1$, $n = 4$, $S = \{0, 1, 2, 3\}$, $f(a, b, c) = (2a + 2b + 2c) \bmod 4$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 2 & 2 & 2 & & \\
 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

In conclusion, pPTCA can have regular behavior starting with an initial seed composed by a single state.

It is quite possible that it is true for random initial seed also, but we have no proof of this.

4. CA with code 0110

It is easy to see that $s_i(t + k)$ depends on the state of cells at time t that are distant at most k cells from the ith and then the following property can be easily proved (see figure 6).

Property 1.

$$\begin{array}{lll}
\textit{If} & G_f^k(w) = \gamma & |\gamma| = |w| + 2k \\
\textit{and} & G_f^k(w') = \gamma' & |\gamma'| = |w'| + 2k
\end{array}$$

then

$$G_f^k(w0^x w') = \gamma 0^{x-2k} \gamma'$$

In particular for CA with code 0110 the following lemma can be proved.

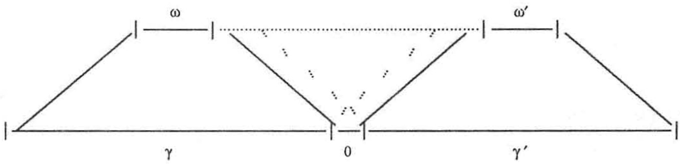

Figure 6.

Lemma 6. *Given a CA with code 0110, we have*

$$G_f^{2^{i-1}}(1^{2^i}) = 1^{2^i}$$

Proof. From now on we will write $\alpha \xrightarrow{k} \beta$ whenever $G_f^k(\alpha) = \beta$.
For $i = 1$ the proposition is true because

$$1^2 \rightarrow 1^4$$

Consider now 1^{2^i}. It is easy to prove that $1^{2^i} \xrightarrow{1} 1^2 0^{2^i-2} 1^2$. For every $k \geq k+1$, from property 1 (since $2^i - 2^k \geq 22^{k-1}$) and by inductive hypothesis, it follows that

$$1^{2^k} 0^{2^i-2^k} 1^{2^k} \xrightarrow{2^{k-1}} 1^{2^{k+1}} 0^{2^i-2^{k+1}} 1^{2^{k+1}}$$

And then for $k = i - 1$,

$$1^{2^{i-1}} 0^{2^i-2^{i-1}} 1^{2^{i-1}} \xrightarrow{2^{i-2}} 1^{2^i} 0^{2^i-2^i} 1^{2^i} = 1^{2^{i+1}}$$

Finally,

$$\begin{aligned} 1^{2^i} &\rightarrow 1^2 0^{2^i-2} 1^2 \\ &\rightarrow 1^4 0^{2^i-4} 1^4 \\ &\xrightarrow{2} 1^8 0^{2^i-8} 1^8 \\ &\rightarrow \ldots \rightarrow 1^{2^{i-1}} 0^{2^{i-1}} 1^{2^{i-1}} \\ &\xrightarrow{2^{i-1}} 1^{2^{i+1}} \end{aligned}$$

and then the assertion holds because

$$1 + \sum_{j=0}^{i-2} 2^j = 2^{i-1} \blacksquare$$

Lemma 7. *Given a CA with code 0110, we have*

$$1^{2^i-1} \xrightarrow{2^{i-1}} 1^{2^{i+1}-1} \quad \forall \quad i > 1$$

Proof. The proof is analogous to the one of lemma 6.

Actually, from lemma 6 it follows that for $k \leq i-2$,

$$1^{2^k}0^{2^i-2^k-1}1^{2^k} \overset{2^{k-1}}{\longrightarrow} 1^{2^{k+1}}0^{2^i-2^{k+1}-1}1^{2^{k+1}}$$

We know that $1^{2^{i-1}} \overset{2^{i-2}}{\longrightarrow} 1^{2^i}$, but since $2^{i-1}-1 < 22^{i-2}$ property 1 cannot be used to obtain the next configuration reached from $1^{2^{i-1}}0^{2^i-2^{i-1}-1}1^{2^{i-1}}$.

Note that by lemma 6,

$$1^{2^{i-1}}0^{2^{i-1}}1^{2^{i-1}} \overset{2^{i-2}-1}{\longrightarrow} w00w \rightarrow 1^{2^{i+1}}$$
$$1^{2^{i-1}}0^{2^{i-1}-1}1^{2^{i-1}} \overset{2^{i-2}-1}{\longrightarrow} w0w \rightarrow 1^{2^{i+1}-1}$$

Finally, $1^{2^i-1} \overset{2^{i-1}}{\longrightarrow} 1^{2^{i+1}-1}$ and then the assertion holds. ∎

Let us see the evolution of the CA with code 0110 between the two configurations 1^{2^i} and $1^{2^{i+1}}$ (analogous considerations can be made for 1^{2^i-1} and $1^{2^{i+1}-1}$).

We know that $1^2 \overset{2^{i-1}-1}{\longrightarrow} 1^{2^i}$ and that $1^{2^i} \overset{1}{\rightarrow} 1^2 0^{2^i-2}1^2$. The two 1's in $1^2 0^{2^i-2}1^2$ are distant more than 2^{i-1} cells and then by property 1 the configuration obtained from $1^2 0^{2^i-2}1^2$ is the concatenation of the two obtained from 1^2 after $2^{2^{i-1}-1}$ steps and itself. Since this observation can be recursively repeated, lemmas 6 and 7 show that CA with code 0110 and initial seeds 1^{2^i} and 1^{2^i-1} have a fractal evolution. In fact, looking only at a part of the figure, we know the structure of the entire figure and so we obtain a so-called self-similar figure. In this case, the figure can be obtained from the following well-known fractal construction:

Step 1 take an equilateral triangle;

Step 2 picture it in a new equilateral triangle whose vertices are on the middle of each side of the initial triangle;

Step 3 repeat Step 2 (see figure 7).

Obviously, configurations like the following generate fractal figures also: 100101, 101001, ... as they give in one step 1^8. In the following, we will prove that CA with code 0110 behave regularly on initial seed 1^n, for every n.

Lemma 8. *Given a CA with code 0110, we have*

$$1^{2i-2} \overset{2^{i+k+1}}{\longrightarrow} 1^{2^i-2}0^{(2^{k+2}-2)2^i+2}1^{2^i-2} \quad \forall k \geq 0$$

Proof.

$$1^{2^i-2} \overset{1}{\rightarrow} 1^2 0^{2^i-4}1^2$$

since $1^2 \overset{2^{i-1}-1}{\longrightarrow} 1^{2^i}$ and from property 1 it follows that

$$1^2 0^{2^i-4}1^2 \overset{2^{i-1}-2}{\longrightarrow} ww$$

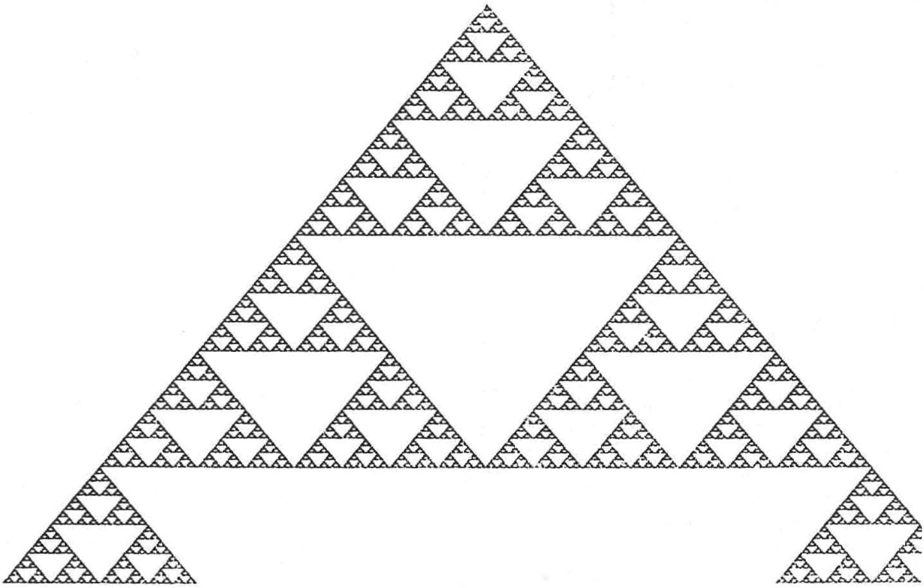

Figure 7.

where w is defined as follows:

$$w \xrightarrow{1} 2^i$$

hence

$$\begin{array}{rcl} w & = & (1^2 0^2)^k 1^2 \quad k = (2^i - 4)/4 \\ ww & \xrightarrow{1} & 1^{2^i-2} 0^2 1^{2^i-2} \end{array}$$

Finally

$$\begin{array}{rcl} 1^{2^i-2} & \xrightarrow{2^{i-1}} & 1^{2^i-2} 0^2 1^{2^i-2} \\ 1^{2^i-2} 0^2 1^{2^i-2} & \rightarrow & 1^2 0^{2^i-4} 1^4 0^{2^i-4} \\ & \xrightarrow{2^{i-1}-2} & w 1^{2^i} w \\ & \rightarrow & 1^{2^i-2} 0^{2^i+2} 1^{2^i-2} \end{array}$$

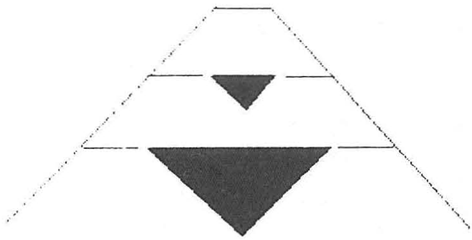

Figure 8.

that is,

$$\begin{aligned}
1^{2^i-2} &\xrightarrow{2^i-1} 1^{2^i-2}0^2 1^{2^i-2}\\
&\xrightarrow{2^{i-1}} 1^{2^i-2}0^{2^i+2}1^{2^i-2}\\
&\xrightarrow{2^{i-1}-2} ww0^4ww\\
&\rightarrow 1^{2^i-2}0^2 1^{2^i-2}0^2 1^{2^i-2}0^2 1^{2^i-2}\\
&\rightarrow 1^2 0^{2^i-4}1^4 0^{2^i-4}1^4 0^{2^i-4}1^4 0^{2^i-4}1^2\\
&\xrightarrow{2^{i-1}-2} w1^{32^i}w\\
&\rightarrow 1^{2^i-2}0^{32^i+2}1^{2^i-2}
\end{aligned}$$

In conclusion,

$$\begin{aligned}
1^{2^i-2} &\xrightarrow{2^i-2} 1^{2^i-2}0^2 1^{2^i-2}\\
&\xrightarrow{2^i} 1^{2^i-2}0^{32^i+2}1^{2^i-2}\\
&\xrightarrow{2^{i+1}} 1^{2^i-2}0^{72^i+2}1^{2^i-2}\\
&\vdots \quad \vdots\\
&\xrightarrow{2^{i+k}} 1^{2^i-2}0^{(2^{k+2}-2)2^i+2}1^{2^i-2}
\end{aligned}$$

Figure 8 shows the evolution of such a CA.

Lemma 9. *Given a CA with code 0110 it holds that*

$$1^{2^i-3} \xrightarrow{2^{i+k+1}} 1^{2^i-2}0^{(2^{k+2}-2)2^i+1}1^{2^i-2} \text{ for } k \geq 0$$

Proof.

$$1^{2^i-3} \rightarrow 1^2 0^{2^i-5}1^2 \xrightarrow{2^{i-1}-3} y0y$$

where y is defined as follows:

$$\begin{aligned}
y &\xrightarrow{2} 1^{2^i} & |y| &= 2^i-4\\
y &= (1^4 0^4)^k 1^4 & k &= (2^i-8)/8
\end{aligned}$$

$$\begin{aligned} y0y &= (1^4 0^4)^k 1^4 0 1^4 (0^4 1^4)^k \\ &\rightarrow (1^2 0^2)^{2k+1} 1^3 (0^2 1^2)^{2k+1} \\ &\rightarrow 1^{8k+6} 0 1^{8k+6} = 1^{2^i-2} 0 1^{2i-2} \end{aligned}$$

Then

$$\begin{aligned} 1^{2i-3} &\rightarrow 1^{2^i-2} 0 1^{2i-2} \\ 1^{2^i-2} 0 1^{2^i-2} &\rightarrow 1^2 0^{2^i-4} 1^3 0^{2^i-4} 1^2 \\ &\xrightarrow{2^i-2} wzwz = 1^{2^i-1} \text{ as } 1^3 \xrightarrow{1^{2^{i-1}-2}} 1^{2^i-1} \\ w &\rightarrow 1^{2^i} \quad |w| = 2^i - 2 \\ w &= (1^2 0^2)^k 1^2 \quad k = (2^i - 4)/4 \\ wzw &= (1^2 0^2)^k 1^{2^i+3} (0^2 1^2)^k \\ &\rightarrow 1^{4k+2} 0^{2^i+1} 1^{4k+2} = 1^{2^i-2} 0^{2^i+1} 1^{2^i-2} \end{aligned}$$

In conclusion,

$$1^{2^i-3} \xrightarrow{2^{i-1}} 1^{2^i-2} 0 1^{2^i-2} \xrightarrow{2^{i-1}} 1^{2^i-2} 0^{2^i+1} 1^{2^i-2}$$

The lemma can be proved continuing in this way. ∎

Lemma 10. *Every CA with code 0110 behaves regularly with initial seed 1^n for every n.*

Proof. The lemma is proved using lemmas 8 and 9 and observing that each configuration 1^{2^i-k} for even k and $2 \leq k \leq 2^{i-1}$ evolves like 1^{2^i-2} and those with odd k like 2^{i-3}. ∎

Configurations like $(10)^n$, and the ones in which at most two contiguous Os or 1's can be found, behave regularly as they evolve in one step to 1^n, so that we can state the following theorem.

Theorem 11. *Every CA with code 0110 behaves regularly with initial seed belonging to $1^+ \cup (\sum^* - \Gamma)$ where $\sum = \{0, 1\}$ and $\Gamma = \sum^* 000 \sum^* 111 \sum^*$.*

Example 6. *We list all configurations of length 4 and 5 that evolve regularly.*

$n = 4$

1	1	1	1
1	0	1	1
1	0	0	1
1	1	0	1

$n = 5$

1	1	1	1	1
1	1	0	0	1
1	1	0	1	1
1	0	1	0	1
1	0	10	1	1

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