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Phase-Space Study of Bistable Cellular Automata

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Abstract. We find the basins of attraction and local stability properties of the limit states for three two-valleyed elementary cellular automata rules for all lattice sizes. The basins of attraction and the unstable directions of the limit states are determined by the bits near the boundaries.

1. Introduction

Cellular automata (CAs) have recently become a popular subject of study for mathematicians and physical, natural, and computer scientists [1–8]. CAs are mathematical models in which space and time are discontinuous, and the state variables can only take on values from a finite set. The rules by which the state variables change are local; that is, they depend on a small number of neighbors. The transition rules can be a deterministic or a probabilistic function of the neighborhood. We only consider the first of these cases in the present paper.

While much of the literature has concentrated on the spatiotemporal structures generated by CAs, some papers have dealt with the more abstract phase-space structure of the models. In particular, the number and size of attractors and their basins of attraction are often quantities of interest. The phase-space literature is reviewed in reference [9]; other recent papers appear in reference [8].

This paper is motivated by a recent numerical study of elementary CAs with fixed boundaries [9]; these are nearest-neighbor rules with two states per site. Among others, three rules were found to have exactly two basins each, of constant relative size. This is an unusually small number of basins, as a typical random mapping of 2^L integers has L limit cycles. Although these rules do not appear to have an energy invariant [10], one can make an analogy to physical systems with two energy wells—such as low-temperature ferromagnets—or to dynamical systems with two attractors—such as a forced pendulum—and study the stability of the "equilibrium" or limit states.

In this paper we study the size and structure of these basins, as well as the stability of the limit states. For the latter, we use a criterion developed by Kauffman [11] to determine the stability of randomly connected automata to mutations. In a similar vein, Hogg and Huberman [12] have studied the stability of more abstract discrete dynamical systems, in particular those with multiple fixed points and a tree-like transient structure. It is rather fortunate that two basins of attraction is the strict minimum necessary to perform a stability analysis in the Kauffman sense.

Section 2 describes the three rules. In section 3 we derive rigorous results for the size and shape (in phase space) of the basins of attraction for these rules (for *all* lattice sizes) as well as for the local stability of their limit states. In particular, if phase space is represented as an L-dimensional cube one can easily visualize basins of attraction and the attractors themselves; we show examples of this. We make some comments about the formal language theory of the limiting states of these rules in section 4, and discuss the results in section 5.

2. Bistable rules

The three rules we have studied consist of a one-dimensional lattice of L sites, each of which can take the values 0 or 1, denoted by $s_1^t, s_2^t, \ldots, s_L^t$ at time t. The nodes are updated simultaneously: the new states $s_1^{t+1}, \ldots, s_L^{t+1}$ are deterministic functions of the value of each node and its nearest neighbors to the left and right. We denote this function by $f(N)$. By extension, N can be a region of 3 or more sites, for which $f(N)$ is unique. We take s_0 and s_{L+1} to be fixed boundary conditions, as described below. We will use an asterisk $(*)$ to denote a wild card (unknown or arbitrary) bit.

Rule 24. *The local evolution of this rule is given by* $f(100) = f(011) = *1*$, $*0*$ *otherwise. Or, in modulo 2 arithmetic,* $s_i^{t+1} = s_{i+1}^t(1 + s_i^t)(1 + s_{i-1}^t) + s_i^t s_{i-1}^t(1 + s_{i+1}^t)$. *The nomenclature of the rules follows Wolfram [2, 7].*

Rule 27. *The local evolution of this rule is given by* $f(*00) = f(0*1) = *1*$, $*0*$ *otherwise. Or in modulo 2 arithmetic,* $s_i^{t+1} = 1 + s_i^t + s_{i+1}^t(s_{i-1}^t + s_i^t)$.

Rule 40. *The local evolution of this rule is given by* $f(011) = f(101) = *1*$, $*0*$ *otherwise. Or, in modulo 2 arithmetic,* $s_i^{t+1} = s_{i+1}^t(s_{i-1}^t + s_i^t)$.

3. Phase space study

3.1 Rule 24

We consider the fixed boundary conditions $s_0 = s_{L+1} = 1$.

Theorem 1. *There are two period-three limit cycles,*

$$\text{(A)}\quad ..(010)^n0 \rightarrow ..(001)^n0 \rightarrow ..(100)^n0$$

and

$$\text{(B)} \quad ..(010)^n01 \rightarrow ..(001)^n01 \rightarrow ..(100)^n01$$

where the dots stand for an incomplete repeating sequence and n is the integer part of $(L-1)/3$ and $(L-2)/3$, respectively. The two cycles (A) and (B) attract all initial configurations.

Lemma 1. *The sequence 11 cannot occur after $t = 0$.*

Proof. $f(*100*) = *010*$, $f(*011*) = *010*$. These would be the only neighborhoods capable of producing two consecutive 1s. For $t > 0$, rule 24 is dynamically equivalent to rule 16 ($f(100) = *1*$, and $*0*$ otherwise). In this rule, isolated 1s shift to the right with unit speed. ■

Lemma 2. *For $t > 1$ the sequence* 101 *cannot exist.*

Proof. $f^{-1}(*101*) = 10011, 01100$, which cannot exist at $t > 0$ by Lemma 1. Therefore, for $t > 1$, 1s must be isolated by at least two 0s, except next to the fixed boundaries. ■

Proof of Theorem 1.

1. From Lemmas 1 and 2, it is clear that any initial configuration will be washed away after L time steps and replaced by a sequence $(001)^n$ shifting to the right with speed one.

2. If the initial configuration has $s_{L-1} = 0$ and $s_L = 1$, these two sites remain unchanged for all times, and $s_L = 1$ acts as a "sink" for the "1s" produced at the left boundary:

$$f(*011) = *011$$

$$f(1011) = 0011$$

   where the last bit is the right boundary. Therefore, we have cycle (B) in this case.

3. Otherwise, s_L eventually becomes (and remains) zero:

$$f(*01) = f(*11) = *01$$

   and s_{L+1} acts as a "sink" for the right-shifting isolated 1s. Here again, the last bit is s_{L+1}.

Therefore, initial configurations of the form $S01$ (one quarter of the total) evolve to limit cycle (B), and all others to limit cycle (A). ■

Figure 1 shows a projection of the space of initial conditions into the $s_{L-1} - s_L$ cube, with the allowed flows.

We now study the *stability* of the cycles (A) and (B) to the flipping of a random bit s_n, $1 \leq n \leq L$. We wish to know if a "final" state in the cycle (A) or (B) returns to the same cycle, or whether it is pushed to the other attractor.

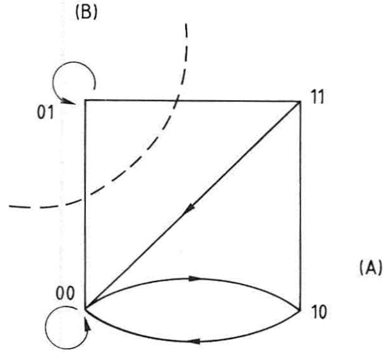

Figure 1: Evolution and basins of attraction for rule 24 in $s_{L-1}s_L$ space (horizontal and vertical directions, respectively). (A), (B) as in Theorem 1.

Theorem 2. *States in limit cycle (B) have two unstable directions each; only two states in cycle (A) have one unstable direction, and one state does not have any. These results are independent of lattice size.*

Proof: As shown in the previous proof, a perturbation in the sites $s_1, s_2, \ldots,$ s_{L-2} eventually washes away, so that the limit states are stable in those "directions." It can be shown by direct application of the rule that there is no "phase change" in these cases; that is, a perturbed state catches up to the state in the cycle to which the unperturbed state would evolve.

We now consider perturbations to the sites s_{L-1} and s_L in turn.

> Cycle (A): possible initial conditions are of the form $S00$ and $S10$. When the last site is perturbed, these become $S01$ and $S11$, respectively, which evolve to (B) and (A), respectively. When the second-to-last bit is perturbed, the initial conditions become $S10$ and $S00$, which evolve to cycle (A). Therefore, the two limit states of the form $S00$ have one unstable direction each, and the $S10$ state is stable under perturbations.
>
> Cycle (B): the initial conditions, of the form $S01$, can be perturbed to $S11$ or $S00$; either way they evolve to attractor (A). ∎

We see then that only a perturbation of the last two bits can cause a jump between attractors, and as $L \to \infty$ the perturbation of a random bit in a limit state will have no long-term effect in most cases.

It can be seen in figure 2 that the unstable directions just found correspond to nearby limit states in phase space (e.g., A1 and B2 or A3 and B1).

In figure 3 we show the main features of this rule as they are actually seen during the evolution of the automaton. The first 120 time steps show the dynamical approach to one of the limit cycles, in this case cycle (A).The

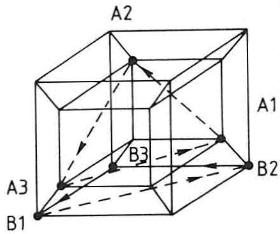

Figure 2: Limit cycle structure for rule 24. Directions: front-back, s_{L-3}; left-right, s_{L-2}; up-down, s_{L-1}; in-out, s_L. Points are labeled according to which limit cycle they belong to. Notice the proximity of points in different limit cycles (A1 and B2, A3 and B1).

s_1^0	s_2^0	s_3^0	s_L^0	t even	t odd
0	0	—	0	0^L	1^L
0	0	—	1	$0^{L-1}1$	$1^{L-1}0$
0	1	0	0	$1^{L-1}0$	$0^{L-1}1$
0	1	0	1	1^L	0^L
0	1	1	0	0^L	1^L
0	1	1	1	$0^{L-1}1$	$1^{L-1}0$
1	—	—	0	$1^{L-1}0$	$0^{L-1}1$
1	—	—	1	1^L	0^L

Table 1: Basins of attraction for rule 27.

next 89 steps show the response to a small perturbation in a central bit (45 in this case). The right side of the picture shows the difference between the unperturbed and perturbed evolutions; the fact that the disturbance disappears shows that the system is stable to this perturbation. The final 40 time steps show the response to a perturbation to the Lth bit. It is clear both from the evolution of the system and from the difference between the perturbed and the unperturbed cases that a permanent change has set in; in fact, the system is now in limit cycle (B).

3.2 Rule 27

We consider fixed boundary conditions $s_0 = 1$ and $s_{L+1} = 0$.

Theorem 3. *Under the above boundary conditions, rule 27 has two possible period-2 attractors: (A) $0^L \leftrightarrow 1^L$ and (B) $0^{L-1}1 \leftrightarrow 1^{L-1}0$, determined by four bits in the initial configuration as shown in table 1.*

Lemma 3. $s_L^t = t + s_L^0$ *(mod 2).*

Proof. $f(*10) = *0*$ and $f(*00) = *1*$. ∎

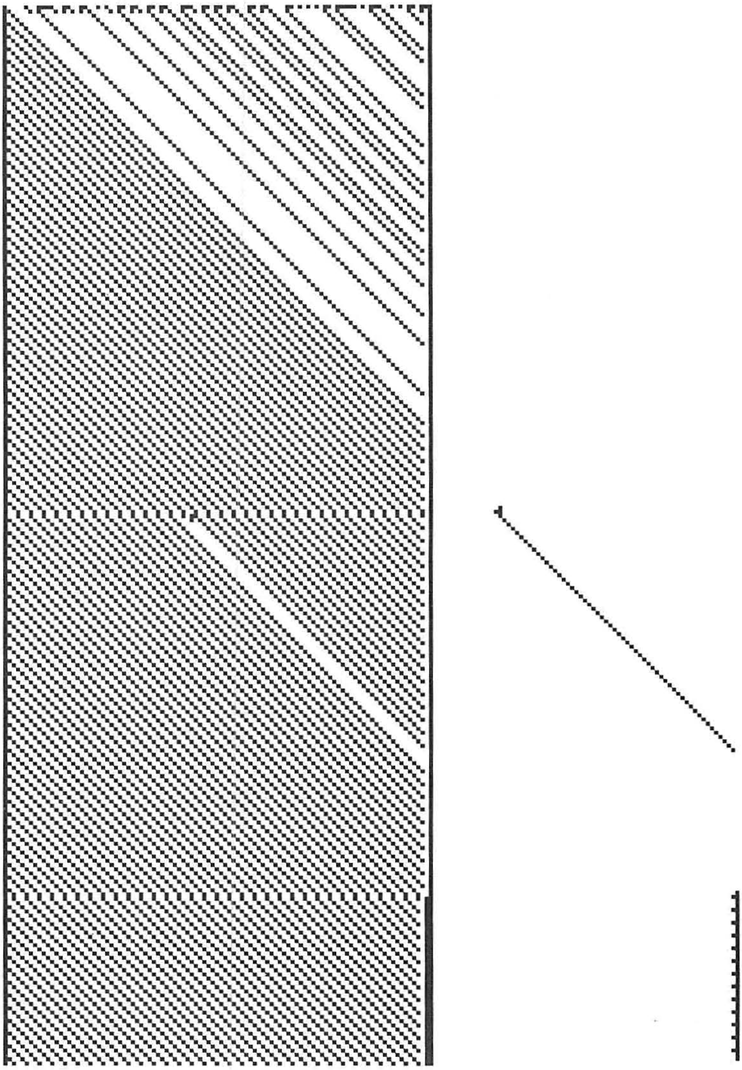

Figure 3: Time evolution of rule 24 for a lattice of size $L = 100$. Also shown are the left and right boundary bits. Left: time steps 1–120, approach to limit cycle (A) from a random initial condition; time steps 121–209, when a central bit (45) is altered, the system returns to limit cycle (A); time steps 210–239, when bit 100 is altered, the system jumps to limit cycle (B). Right: the "damage" or distance between the altered and the unaltered limit cycle (A): time steps 121–209, the damage shifts to the right and disappears; time steps 210–239, damage in the right side of the system remains, as the system moves to limit cycle (B).

Lemma 4. *For $t > 3$, either $s_1 = s_2 = s_3 = t$ or $s_1 = s_2 = s_3 = t + 1$ (mod 2).*

Proof. $f^4(11{*}{*}{*}{*}) = 1111{*}{*}$; $f^4(100{*}{*}{*}) = 1000{*}{*}$; $f^4(1010{*}{*}) = 1111{*}{*}$; $f^4(1011 * *) = 1000 * *$. Here the leftmost 1 corresponds to s_0. To complete the proof we note that $f(1111 * *) = 1000 * *$ and $f(1000 * *) = 1111 * *$. ∎

Lemma 5. *Under these "alternating" boundary conditions $s^t_M = t$ and $s^{t+1}_M = t + 1$; after at most 3 time steps $s^t_{M+1} = t$ and $s^{t+1}_{M+1} = t + 1$.*

Proof. $f(01*) = 110$; $f^2(10*) = 11*$. ∎

The iterated function is to be understood with an alternating first bit. The two cases above are those in which an alternating "front" meets on its right a bit of a different value.

Proof of Theorem 3. Applying Lemma 5 inductively, we conclude that eventually (after at most $2L$ steps) the sites $s_1, s_2, \ldots, s_{L-1}$ will oscillate in phase, $0^{L-1} \leftrightarrow 1^{L-1}$. The phase itself is determined by Lemma 4. As site s_L evolves according to Lemma 3, this concludes the Proof of Theorem 3. ∎

It can be seen from table 1 that, for lattice sizes $L \geq 5$, there are two basins of attraction of equal size.

Theorem 4. *If one of the limit states of (A) has site s_L altered, the system jumps to attractor (B), and vice-versa.*

Proof. This follows from Lemma 3. ∎

From the previous section (Lemmas 3 through 5) we also deduce that the bit s_1 is unstable in both (A) and (B) states, and the bit s_2 is unstable in states 0^L and $0^{L-1}1$. All other bits in limit states are stable to changes. As in rule 24, changes in these bits do not produce phase changes. We then see that the unstable directions in this case always correspond to states that are contiguous in phase space.

3.3 Rule 40

We consider the fixed boundaries $s_0 = s_{L+1} = 1$. In this rule, 1s are shifted to the left or destroyed, but never created.

Theorem 5. *There are two fixed points 0^L and $0^{L-1}1$, to be called (A) and (B). Initial configurations of the form $S00$ or $S111$ evolve to (A).*

Lemma 6. *The 111 sequence can only occur at $t = 0$.*

Proof. $f(1011*) = *110*$. ∎

This would be the only neighborhood capable of producing three consecutive 1s. We only need to consider single and double 1s shifting to the left.

Lemma 7. *A double 0 spreads to the left with speed one, and to the right with speed one-half until it reaches the boundary.*

Proof. $f(**00*) = *000*$; $f^2(**00*) = 00000$. ∎

Lemma 8. *A traveling "11" creates a 0 pair when it hits $s_0 = 1$.*

Proof. $f(111*) = 100*$. ∎

From Lemma 9, a pair of 1s will eventually cause a spreading pattern of 0s. Only the case of alternating single 0s and 1s remains.

Lemma 9. *An initial condition $(01)^n$ produces a double 1 on the right boundary.*

Proof. $f(1011) = (0111)$, where the last bit is s_{L+1}. Therefore, *all* initial conditions eventually produce an expanding pattern of 0s. ∎

Proof of Theorem 5. The fixed points can be verified by direct application of the rule; we see from Lemmas 7 through 9 that these two fixed points attract all initial configurations:

$$f(*001) = 0001 \qquad f(*1111) = **001$$

All other initial configurations evolve to (B). The basins of attraction occupy 3/8 and 5/8 of phase space, respectively. ∎

From the previous subsection, we see that only changes in the last three bits may result in fixed point changes.

Theorem 6. *Fixed point (A) has two unstable directions, while (B) has one unstable direction.*

Proof. Changes in the last three bits of 0^L lead to ..001, ..010, or ..100. By Theorem 5, these evolve to (B), (B), and (A), respectively.

Changes in the last three bits of $0^{L-1}1$ lead to ..000, ..011, or ..101, which evolve to (A), (B), and (B), respectively. In this proof the leading dots correspond to multiple 0s. ∎

The unstable direction in (B) and one of those in (A) are related to the proximity of the two fixed points in phase space.

4. Formal language theory

The limit sets of configurations for each rule can be generated from the graphs of figure 4. Each limiting configuration on a system of size L (and reading from right to left) corresponds to a path that starts from the specified node and traverses L arcs. Conversely, every path starting from the specified node, however long, corresponds to a limiting configuration for some L. The regular languages thus described can be compared with those obtained at finite times for the same rules and without boundaries [13, 14]. For instance, with boundaries the state transition graph corresponding to rule 40 has the limit shown in Figure 4(c), while in the unbounded case [14] the size of the graph appears to grow with time.

(*a*)

(*b*)

(*c*)

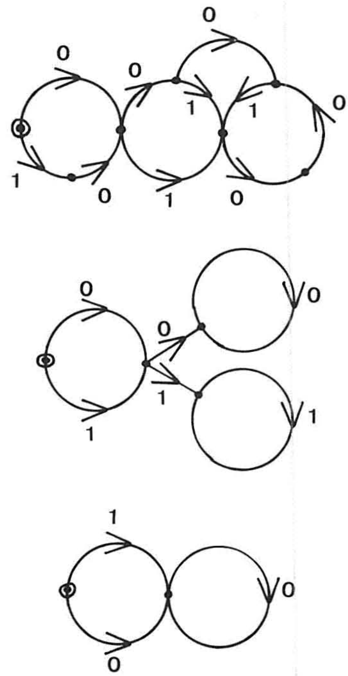

Figure 4: The state transition graphs for the deterministic finite automata that generate the regular languages obtained in the large-time limit for the three rules. From the top, they correspond to rules 24, 27, and 40.

5. Summary and discussion

In this paper we have determined the basin size, and the structure and stability properties of the limit states, for three cellular automata rules each with two basins of attraction. The attractors are fixed points or limit cycles of length 2 or 3. Because of the small number of attractors—and the independence of this number on lattice length—we have been able to determine these properties rigorously. The results are summarized in Theorems 1 through 6 in section 3.

A first observation is that only a few bits near the boundaries determine to which attractor an initial condition is going to evolve. In rules that become purely shift rules after a few time steps, such as 24 and 40, only the first few or the last few bits matter. In more complicated rules like 27, bits near both boundaries are important. These three rules, then, can be seen to act as "pattern recognizers" for certain properties of the initial condition.

A second observation is that the limit states are extremely stable under small perturbations. Only perturbations to the bits near the boundaries will cause the system to jump from one limit cycle to another, and most other perturbations will not even cause a change in the "phase" of the limit cycle.

There are two final remarks about the consequences of discrete phase space. One is that the structure of the basins of attraction is very different from what is observed in the continuum. For example, the hypercubic projection of figure 2 is quite different in structure from the fractal basin boundaries that have been found in a forced pendulum (figure 1 in reference [15]). The final remark is that this discrete structure seems to play a role in the existence of the few unstable "directions" in the limit states. (See for example figure 2 in this paper).

Acknowledgments

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References

[1] J. von Neumann, *The Theory of Self-Reproducing Automata*, edited by A. Burks (University of Illinois Press, Urbana, 1966).

[2] S. Wolfram, "Statistical Mechanics of Cellular Automata," *Reviews of Modern Physics*, **55** (1983) 601.

[3] J. D. Farmer, T. Toffoli, and S. Wolfram, editors, *Cellular Automata* (North-Holland, Amsterdam, 1984).

[4] S. Wolfram, editor, *Theory and Applications of Cellular Automata* (World Scientific, Singapore, 1986).

[5] T. Toffoli and N. H. Margolus, *Cellular Automata Machines* (MIT Press, Cambridge, 1987).

[6] R. Livi, *et al.*, editors, *Chaos and Complexity* (World Scientific, Singapore, 1988).

[7] P. Manneville, *et al.*, editors, *Cellular Automata and Modeling of Complex Physical Systems* (Springer, Berlin, 1989).

[8] H. Gutowitz, editor, *Cellular Automata: Theory and Applications* (North-Holland, Amsterdam, 1990).

[9] P. M. Binder, "Topological Classification of Cellular Automata," *Journal of Physics A: Mathematical and General*, **24** (1991) L31.

[10] Y. Pomeau, "Invariant in Cellular Automata," *Journal of Physics A: Mathematical and General*, **17** (1984) L415.

[11] S. Kauffman, "Metabolic Stability and Epigenesis in Randomly Connected Nets," *Journal of Theoretical Biology*, **22** (1969) 437.

[12] T. Hogg and B. A. Huberman, "Attractors on Finite Sets: The Dissipative Dynamics of Computing Structures," *Physical Review A*, **32** (1985) 2338.

[13] S. Wolfram, "Computation Theory of Cellular Automata," *Communications in Mathematical Physics*, **96** (1984) 15.

[14] L. P. Hurd, "Regular Language Complexities," in reference [4], pages 523–526.

[15] C. Grebogi, E. Ott, and J. A. Yorke, "Basin Boundary Metamorphoses: Changes in Accessible Boundary Orbits," *Physica*, **24D** (1987) 243.