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Geometry and Arithmetic of a Simple Cellular Automaton*

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Abstract. This paper presents the results of a study of the geometric and arithmetic properties of the graph of a simple cellular automaton, considered as a mapping of the unit interval to itself. The graph provides an example of a strictly self-similar figure and exhibits some numeric properties relating to Fermat numbers. In addition an interesting density result is proved: the predecessor set of any number in $[0, 1]$ is dense in the interval, and the set of kth-order predecessors of any number is uniformly distributed over a partition of the interval into 2^k uniform segments.

1. Introduction

A number of studies of cellular automata have focused on one-dimensional nearest neighbor automata defined over Z_2 [1, 2, 3, 4]. In the set of all such automata those that are additive are of interest because of their mathematical tractability. Additive automata rules may be represented as either real or complex polynomials, as sums of powers of the shift operator, and as circulant matrices.

Table 1 lists the forms of the seven non-trivial additive nearest neighbor rules over Z_2.

Algebraic properties of rules 90 and 150 have been exhaustively studied [5] and the entropy-reducing properties of all rules in Table 1 are known [2, 5, 6]. It is also known that the space-time outputs of these additive rules exhibit self-similar properties and, under an appropriate mapping, define fractals [7, 8, 9].

In this paper attention is focused on the asymmetric rule 102, represented as in Table 1 by the operator $D = I + \sigma$. In particular, this rule is considered

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Rule	$p(t)$	$p^*(\omega)$	Operator	Shifts	Circulant
204	1	1	I	$I=\sigma^0$	circ(10...0)
170	t	ω	σ	σ	circ(010...0)
240	t^{-1}	ω^{-1}	σ^{-1}	σ^{-1}	circ(0...01)
102	$1+t$	$1+\omega$	D	$I+\sigma$	circ(110...0)
60	$1+t^{-1}$	$1+\omega^{-1}$	D^-	$I+\sigma^{-1}$	circ(10...01)
90	$t+t^{-1}$	$\omega+\omega^{-1}$	δ	$\sigma+\sigma^{-1}$	circ(010...01)
150	$t+1+t^{-1}$	$\omega+1+\omega^{-1}$	Δ	$\sigma+I+\sigma^{-1}$	circ(110...01)

Table 1: Additive nearest neighbor rules over Z_2.

as it operates on the space E^+ of all right half-infinite sequences with entries in Z_2. Each element μ of this space defines a number in the interval $[0,1]$ by

$$\mu = \sum_{i=1}^{\infty} \mu_i 2^{-i} \tag{1.1}$$

In this way the map $D : E^+ \to E^+$ defines a map $D : [0,1] \to [0,1]$. The properties of this map are studied in this paper.

2. Algebraic properties of $\mathbf{D} : \mathbf{E}^+ \to \mathbf{E}^+$

The fundamental algebraic properties of the operator D can be quickly summarized. Let E_n denote the subset of E^+ consisting of all sequences that are periodic with period a divisor of n. Elsewhere [2] it has been shown that D satisfies

$$[D^k(\mu)]_i = \sum_{j=1}^{k+1} \binom{k+1}{j} \mu_{i+j-1} \tag{2.1}$$

where $\binom{k+1}{j}$ is the jth entry in the $(k+1)$st row of the mod(2) Pascal triangle.

Lemma 1. *Let $n = 2^m d$ with d odd. Under iteration of D every element of E_n maps to a cycle or fixed point in at most 2^m iterations.*

Proof. If $n = 2^m$ and $\mu \in E_n$, then by (2.1) and the properties of the mod(2) Pascal triangle,

$$[D^k(\mu)]_i = \mu_i + \mu_{i+2^m} = 0 \bmod (2)$$

Thus, for $d = 1$, all elements of E_n map to $\mathbf{0}$ in at most 2^m iterations.

Now suppose that $d > 1$ and let k be the smallest integer such that $2^k = 2^m \bmod (n)$. Then

$$[D^{2^k}(\mu)]_i = \mu_i + \mu_{i+2^k} = \mu_{i+2^m} = [D^{2^m}(\mu)]_i$$

which yields

$$D^{2^k-2^m}(D^{2^m}(\mu)) = D^{2^m}(\mu)$$

indicating that $D^{2^m}(\mu)$ is on a cycle. ■

We will say that $\mu \in E^+$ *terminates* at n if $\exists n > 0$ such that $\mu_n = 1$ and $\mu_i = 0 \, \forall \, i > n$.

Lemma 2. *Let $\mu \in E^+$ terminate at n. Then D iterated on μ yields a cycle of period 2^k where k is the smallest integer such that $n \leq 2^k$.*

Proof. Application of (2.1) yields $[D^{2^k}(\mu)]_i = \mu_i + \mu_{i+2^k}$, which taken with the condition that μ terminates at $n \leq 2^k$ tells us that $[D^{2^k}(\mu)]_i = \mu_i \, \forall \, i$. Suppose that $s > 0$ such that $D^{2^k - s}(\mu) = \mu$. Making use of (2.1) to write this in component form and canceling terms that sum to zero yields the set of equations

$$\begin{aligned}
0 &= \binom{2^k - s + 1}{2}\mu_2 + \cdots + \binom{2^k - s + 1}{r}\mu_r \\
0 &= \binom{2^k - s + 1}{2}\mu_3 + \cdots + \binom{2^k - s + 1}{r - 1}\mu_r + \binom{2^k - s + 1}{r}\mu_{r+1} \\
0 &= \binom{2^k - s + 1}{2}\mu_{n-1} + \binom{2^k - s + 1}{3}\mu_n \\
0 &= \binom{2^k - s + 1}{2}\mu_n
\end{aligned} \tag{2.2}$$

where

$$r = \begin{cases} n & d^k - s + 1 \geq n \\ 2^k - s + 1 & 2^k - s + 1 < n \end{cases}$$

Since $\mu = 1$ the last equation of (2.2) requires that $\binom{2^k - s + 1}{2} = 0$. Substitution of this condition into the penultimate equation of (2.2) then requires that $\binom{2^k - s + 1}{3} = 0$. Continuation of this process indicates that the full set of equations in (2.2) will be satisfied if and only if $2^k - s + 1 \geq n$ and

$$\binom{2^k - s + 1}{j} = 0 \qquad 2 \leq j \leq n \tag{2.3}$$

The $2^k - s + 1$ row of Pascal's triangle has $2^k - s + 1$ entries, and k has been choosen as the smallest integer such that $n \leq 2^k$. Hence $2^{k-1} < n$ and satisfaction of (2.3) requires that, after the leading one, more than the first half of this row must consist of zeros. This occurs if and only if $s = 0$ and the proof is done. ■

The additivity of D in combination with Lemmas 1 and 2 now yields:

Theorem 1. *Let $\mu \in E^+$ have the form $\mu = \mu' + \beta$ where for some k, n*

$$\begin{aligned}
\mu'_k &= 1 & i > k \\
\mu'_i &= 0 & i > k \\
\beta_i &= \begin{cases} 0 & i \leq k \\ \beta_{i+n} & i > k \end{cases}
\end{aligned}$$

Let $\pi_n(\beta)$ be the period of the cycle to which β goes under iteration of D, and let p be the smallest integer such that $k \leq 2^p$. Then D iterated on μ yields a cycle with period at most $2^p\pi_n(\beta)$.

Sequences of the form indicated in Theorem 1 will be called *eventually periodic* with period n. Note that eventually periodic sequences with period 1 either are terminating, or end in an infinite string of ones. Sequences that are not terminating or eventually periodic will be called non-periodic. Notationally, eventually periodic sequences will be indicated in the standard way by underlining their periodic part.

In what follows it will also be necessary to invert the map $D : E^+ \to E^+$.

Theorem 2. *[10] Let $\beta \in E^+$. The general solution to the equation $D(\mu) = \beta$ is given by*

$$\mu = \mu_1 B(\alpha_1) + B(\sigma^{-1}(\beta)) \tag{2.4}$$

where α_1 is the sequence $1\underline{0}$, σ^{-1} is defined by

$$\sigma^{-1}(\mu) = \begin{cases} 0 & i = 1 \\ \beta_{i-1} & \textit{otherwise} \end{cases}$$

and the operator B is defined by

$$[B(\beta)]_i = \sum_{r=1}^{i} \beta_r$$

The operator B in the preceding theorem is known to have the following period-doubling property [10]: if $\beta \in E^+$ is periodic with period n, then there is a k such that $B^k(\beta)$ has period $2n$.

Theorem 3. *Let $\mu \in E^+$ be non-periodic. Then $D(\mu)$ is also non-periodic.*

Proof. Suppose μ is non-periodic and $D(\mu) = \beta$ is eventually periodic with period n. By Theorem 2 and the period-doubling property of B, however, the predecessors of β, one of which is μ, must be eventually periodic with period n or $2n$. This contradicts the assumption that μ is non-periodic. ∎

Theorem 4. *Let $\mu \in E^+$ be on a cycle of D. Then μ is eventually periodic.*

Proof. If μ is on a cycle then $\exists\, k$ such that $D^k(\mu) = \mu$. From (2.1) and the equality $\binom{k+1}{1} = \binom{k+1}{k+1} = 1$ we obtain the set of equations

$$\mu_{i+k} = \binom{k+1}{1}\mu_{i+1} + \cdots + \binom{k+1}{k}\mu_{i+k-1} \qquad i \geq 1 \tag{2.5}$$

Thus for all $r > k$, μ_r is determined in terms of $\mu_2, \ldots, \mu_k$ by a simple recursion relation. ∎

From the above we see that D maps terminating sequences to terminating sequences, eventually periodic sequences to eventually periodic sequences, and non-periodic sequences to non-periodic sequences. Further, all terminating and eventually periodic sequences are either on cycles or iterate to cycles. In terms of the map $D : [0,1] \to [0,1]$ this means that rational numbers map to rationals, irrationals map to irrationals, and no irrationals map to cycles.

By analogy to Taylor's theorem the operator D can be thought of as a derivative with respect to sequence index [10]. Operationally D is defined by the command "shift and add without carrying."

The map $D : E^+ \to E^+$ is a function since $D(\mu)$ is uniquely defined for all μ. This is not true for $D : [0,1] \to [0,1]$. The reason is found by considering the sequences μ and μ' defined by

$$\begin{aligned} \mu_i &= \begin{cases} \text{arbitrary} & i < n \\ 1 & i = n \\ 0 & i > n \end{cases} \\ \mu'_i &= \begin{cases} \mu_i & i < n \\ 0 & i = n \\ 1 & i > n \end{cases} \end{aligned} \tag{2.6}$$

Both of these sequences define the same point of $[0,1]$ but

$$[D(\mu')]_i = \begin{cases} [D(\mu)]_i & i \neq n-1 \\ 1 + [D(\mu)]_i & i = n-1 \end{cases}$$

Every sequence of the form (2.6) corresponds to a point of $[0,1]$ having denominator a power of two. Thus $D : [0,1] \to [0,1]$ is double-valued at all points with rational expression $K/2^n$.

3. Geometric properties of $\mathbf{D} : [\mathbf{0},\mathbf{1}] \to [\mathbf{0},\mathbf{1}]$

The graph of $D : [0,1] \to [0,1]$ is shown in Figure 1. In appearence this graph is composed of multiple triangular structures of different sizes, is reflection symmetric about $x = 1/2$, and is self-similar. These properties will be demonstrated in what follows.

Theorem 5. *The graph of $D : [0,1] \to [0,1]$ is reflection symmetric about $x = 1/2$.*

Proof. Let $x \in [0,1/2]$ and take μ and μ' such that, under the map of (1.1), $\mu \to (1/2) - x$ and $\mu' \to (1/2) + x$. Then $\mu + \mu' = \mathbf{1}$ so that $D(\mu + \mu') = \mathbf{0}$. Since D is additive this means that $D(\mu) = D(\mu')$, hence $D((1/2) - x) = D((1/2) + x)$. ∎

As a result of Theorem 5 it is only necessary to consider values of x in the interval $[0,1/2]$, and this restriction will be observed hereafter.

Coordinates can be defined on the graph shown in Figure 1 that have a natural correspondence to the structure of this graph. On the interval $[0,1/2]$

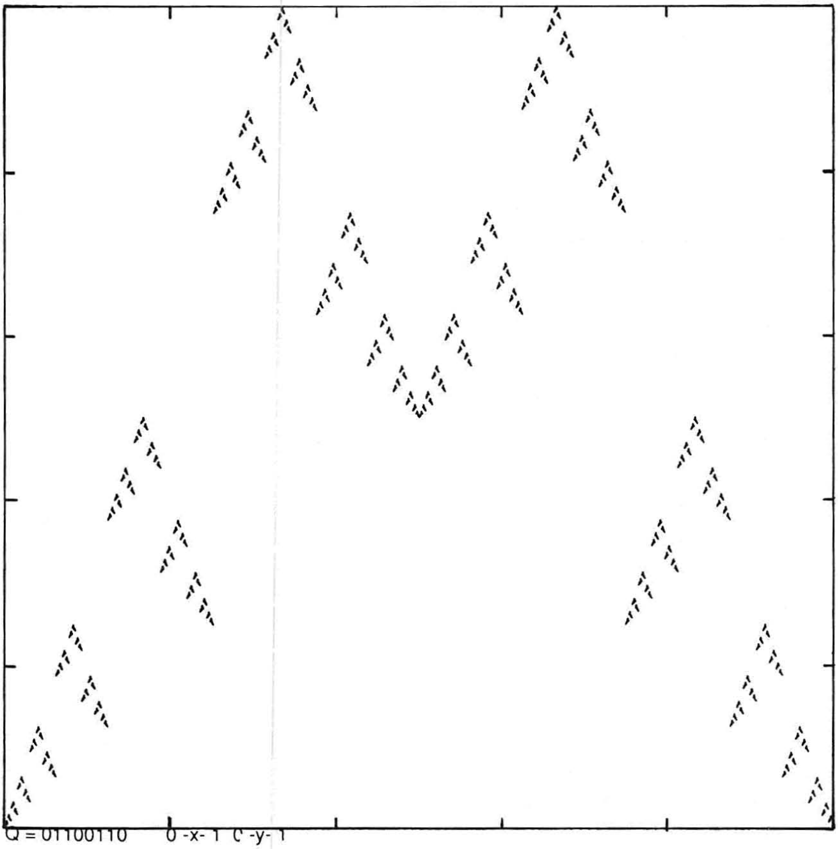

Figure 1: Graph of $D : [0,1] \to [0,1]$.

this structure is based on the sequence defined in (3.1), which approaches $(0,0)$ as a limit.

$$\{S(a_1)\} = \{(2^{-a_1}, 2^{-a_1}) \mid 1 \le a_1 < \infty\} \tag{3.1}$$

Every point of this sequence is, in turn, the limit point of a second sequence $\{S(a_1, a_2)\}$, and so on *ad infinitum.*

To make this intuitive image precise, we define a subset of $[0, 1/2] \times [0,1]$ consisting of those points on the graph of D having x-coordinates with denominator a power of two. (Recall that D is double-valued on precisely these points.) Points of this subset are labeled $S(a_1, \ldots, a_n)$ for some finite n. The x and y coordinates of $S(a_1, \ldots, a_n)$ will be denoted as $x(a_1, \ldots, a_n)$ and $y(a_1, \ldots, a_n)$. The binary sequence $x^*(a_1, \ldots, a_n)$ corresponding to a point $x(a_1, \ldots, a_n)$ is defined by the following algorithm:

1. $x^*(\overbrace{1,\ldots,1}^{n})$ is the sequence having the first n terms alternating 0 and 1 starting with 0. If n is odd the remaining terms of the sequence are one and if n is even the remaining terms are zero. Thus $x^*(1,1) = 01\underline{0}$ and $x^*(1,1,1) = 010\underline{1}$.

2. For $1 \le i \le n$, $x^*(a_1,\ldots,a_n)$ is generated from $x^*(\overbrace{1,\ldots,1}^{n})$ as follows:

   a) If i is odd replace the zero in the ith position of $x^*(\overbrace{1,\ldots,1}^{n})$ by a block of zeros of length a_i; b) If i is even replace the one in the ith position of $x^*(\overbrace{1,\ldots,1}^{n})$ by a block of ones of length a_i.

For example, $x^*(2,3,3) = 00111000\underline{1}$ and $x^*(2,3,1,2) = 00111011\underline{0}$.

Lemma 3. *Every number in $[0,1/2]$ of the form $K/2^s$ has an expression $x^*(a_1,\ldots,a_n)$ for finite n.*

Lemma 4. *Every number in $[0,1/2]$ that is not of the form $K/2^s$ has a unique expression*

$$x^* = \lim_{n\to\infty} x^*(a_1,\ldots,a_n)$$

As an example of Lemma 4, $1/3$ is the limit as n becomes infinite of the sequence $x^*(\overbrace{1,\ldots,1}^{n})$. Since every point of $[0,1/2]$ can be approximated arbitrarily closely by a point $k/2^s$, only these latter need be considered in computing properties of the graph shown in Figure 1. Points of this graph are characterized by two structural lemmas:

Lemma 5.

$$\begin{aligned} x(a_1,\ldots,a_n,1) &= x(a_1,\ldots,a_n+1) \\ y(a_1,\ldots,a_n,1) &= y(a_1,\ldots,a_n+1) + 2^{-a(1,n)} \end{aligned} \tag{3.2}$$

where

$$a(k,n) = \sum_{s=k}^{n} a_s \tag{3.3}$$

Proof. Suppose that n is even. Then the sequence $x^*(a_1,\ldots,a_n+1)$ has the form

$$(M_1,\ldots,M_{a_{n-1}},\overbrace{1,\ldots,1}^{a_n+1},\underline{0})$$

where each M_{a_i} is a block of a_i ones (i even) or zeros (i odd). On the other hand, the sequence $x^*(a_1,\ldots,a_n,1)$ will have the form

$$(M_1,\ldots,M_{a_{n-1}},\overbrace{1,\ldots,1}^{a_n},0,\underline{1})$$

and the numerical values of these two sequences are the same. A similar argument for odd n finishes proof of the first of the claimed equalities. The second is demonstrated by observing the way in which the sequences $x^*(a_1, \ldots, a_n, 1)$ and $x^*(a_1, \ldots, a_n + 1)$ map under D. Writing these out and applying D we find that, regardless of whether n is even or odd,

$$D(x^*(a_1, \ldots, a_n, 1)) = D(x^*(a_1, \ldots, a_n + 1)) + (0, \ldots, 0, 1, \underline{0})$$

where the one in the second term on the right is located in the $a(1, n)$th position. Mapping this back to $[0, 1/2]$ then yields the second equality. ∎

Lemma 6.

$$\begin{aligned} x(a_1, \ldots, a_{n+1} + 1) &= (1/2)[x(a_1, \ldots, a_n + 1) + x(a_1, \ldots, a_n)] \\ y(a_1, \ldots, a_{n+1} + 1) &= (1/2)[y(a_1, \ldots, a_n + 1) + y(a_1, \ldots, a_n)] \end{aligned}$$

Proof. Assume n is odd. Then

$$\begin{aligned} x^*(a_1, \ldots, a_n) &= (M_1, \ldots, M_{a_n}, \overbrace{1, \ldots, 1}^{a_{n+1}}, \underline{1}) \\ x^*(a_1, \ldots, a_{n+1}) &= (M_1, \ldots, M_a, M_{a_{n+1}}, \underline{0}) \\ x^*(a_1, \ldots, a_{n+1} + 1) &= (M_1, \ldots, M_a, M_{a_{n+1}}, 1, \underline{0}) \end{aligned} \tag{3.4}$$

where M_{a_n} is a block of a_n zeros and $M_{a_{n+1}}$ is a block of a_{n+1} ones. By (3.4) $x(a_1, \ldots, a_n) = x(a_1, \ldots, a_{n+1}+1)+2^{-a(1,n+1)-1} = x(a_1, \ldots, a_{n+1})+2^{-a(1,n+1)}$ and this proves the first equation. The second follows since $y^*(a_1, \ldots, a_n) = D(x^*(a_1, \ldots, a_n))$ and the map $D : E^+ \to E^+$ is linear. Similar arguments hold for n even and the proof is done. ∎

Theorem 6. *The (x, y) coordinates of $S(a_1, \ldots, a_n)$ are given by*

$$\begin{aligned} x(a_1, \ldots, a_n) &= 1 - 2^{-a(1,n)} \left\{(-1)^n + \sum_{k=1}^{n}(-1)^{k-1}2^{a(k,n)}\right\} \\ y(a_1, \ldots, a_n) &= 2^{-a(1,n)} \left\{1 + \sum_{k=2}^{n} 2^{a(k,n)}\right\} \end{aligned} \tag{3.5}$$

Proof. For $n = 1$ the claim is valid by inspection. Assume that (3.5) are valid for $S(a_1, \ldots, a_n)$. By (3.3) and Lemma 5

$$x(a_1, \ldots, a_n, 1) = 1 - 2^{-a(1,n)-1} \left\{(-1)^n + \sum_{k=1}^{n}(-1)^{k-1}2^{1+a(k,n)}\right\} \tag{3.6}$$

while iteration of the first equation of Lemma 6 yields

$$x(a_1, \ldots, a_{n+1}) = 2^{1-a(1,n+1)}x(a_1, \ldots, a_n, 1) + x(a_1, \ldots, a_n) \sum_{s=1}^{a_{n+1}-1} 2^{-s} \tag{3.7}$$

Noting that $\sum_{s=1}^{r} 2^{-s} = 2^r(2^r - 1)^{-1}$ we substitute (3.6) and the first equation of (3.5) into (3.7) to obtain the claimed result for $x(a_1, \ldots, a_{n+1})$, after a suitable rearrangement of terms.

Similarly, iterative reduction of the second equation of Lemma 6 gives

$$\begin{aligned} y(a_1, \ldots, a_{n+1}) &= 2^{1-a_{n+1}} \Big\{ y(a_1, \ldots, a_n, 1) \\ &\quad + (2^{a_{n+1}-1} - 1) y(a_1, \ldots, a_n) \Big\} \end{aligned} \tag{3.8}$$

and substitution from Lemma 5 and the induction hypothesis again yields the desired result. ■

Inspection of Figure 1 suggests that the sequence of points

$$s(i, n) = \{S(a_1, \ldots, a_n) \mid a_r \text{ fixed } r \neq i, 1 \leq a_i < \infty\}$$

lies along a straight line. Computation of the slope of this line shows this to be true.

Theorem 7. *All points of the sequence $s(i, n)$ lie on a straight line with slope*

$$m(i, n) = -\left[1 + \sum_{k=i+1}^{n} 2^{a(k,n)}\right] \Big/ \left[\sum_{k=i+1}^{n} (-1)^{k-1} 2^{a(k,n)} + (-1)^n\right]^{-1} \tag{3.9}$$

Proof. Pick any two values of a_i and apply Theorem 6 to compute the slope of the straight line joining the corresponding points of $s(i, n)$. The result is given by (3.9). Since (3.9) is independent of a_i, all points of $s(i, n)$ are connected by straight lines of the same slope and the proof is done. ■

Remarks:

1. $m(n, n) = (-1)^{n-1}$
   $m(n-1, n) = (-1)^{n-1}(2^{a_n} + 1)/(2^{a_n} - 1) = \pm 3$ if $a_n = 1$

2. The slope of a sequence $s(i, n)$ depends only on the a_r for $r > i$.

Therefore any two sequences $s(i, n)$ and $s(j, k)$ such that $n - i = k - j$, and with all a_r equal for the final $n - i$ terms, will lie along distinct straight lines having the same slope.

Strict self-similarity has been defined by Hutchinson [11] as implying that all components of a figure map identically onto all other components by a similarity transformation (isometry and scaling).

The graph indicated in Figure 1 is composed of infinite sequences grouped into triangular patterns. Any given triangle will have vertices defined by $S(a_1, \ldots, a_n)$, $S(a_1, \ldots, a_{n+1})$, and $S(a_1, \ldots, a_{n+1}, 1)$ for some $a_1, \ldots, a_{n+1}$. By Theorem 7 the slopes of the three sides of these triangles will be

$$m(S(a_1,\ldots,a_n),S(a_1,\ldots,a_{n+1}))=\pm 1$$

$$m(S(a_1,\ldots,a_n),S(a_1,\ldots,a_{n+1},1))=\pm 3$$

$$m(S(a_1,\ldots,a_{n+1}),S(a_1,\ldots,a_{n+1},1))=-(\pm 3)$$

hence all such triangles are similar. Further, the nature of the infinite sequences composing these triangles, as determined in Lemmas 5 and 6, insures that they are pointwise identical up to isometries and scalings. Thus we have:

Theorem 8. *The graph of $D:[0,1]\to[0,1]$ is strictly self-similar.*

4. Arithmetic of $\mathbf{D}:[\mathbf{0},\mathbf{1}]\to[\mathbf{0},\mathbf{1}]$

The fixed points of D are $\mathbf{0}=0$ and $1\underline{0}=1/2$. The point $0\underline{1}=1/2$ goes to $1\underline{0}$ in one iteration. In this section some of the numerical properties of D iterated on $[0,1]$ are presented.

Let $\mu\in E^+$ be periodic with period 2^k. Then μ defines the point

$$x(\mu)=(2^{2^k}-1)^{-1}\sum_{s=0}^{2^k-1}\mu_{2^k-s}2^s \tag{4.1}$$

If μ_i is one when $i=2^ks$ and is zero otherwise, then $x(\mu)=(2^{2^k}-1)^{-1}$ and iterates to 0 in $2k$ steps. For $m<2k$

$$D^m(x(\mu))=(2^{2^k}-1)^{-1}\sum_{s=0}^{m}[D^m(\mu)]_{2^k-s}2^s$$

which on substitution from (2.1) becomes

$$D^m(x(\mu))=(2^{2^k}-1)^{-1}\sum_{s=0}^{m}\binom{m+1}{s+1}2^s \tag{4.2}$$

The sum in (4.2) can be evaulated using a result of Hewgill [12].

Lemma 7. *[12]*

$$\sum_{s=0}^{m}\binom{m+1}{s+1}2^s=F_{m+1}^{\binom{m+1}{1}}F_m^{\binom{m+1}{2}}\cdots F_0^{\binom{m+1}{m+1}} \tag{4.3}$$

where $F_r=2^{2^r}+1$ is the rth Fermat number.

The denominator of (4.2) is given by the expansion

$$2^{2^k}-1=F_{k-1}F_{k-2}\cdots F_0 \tag{4.4}$$

Every point of $[0,1]$ has exactly two immediate predecessors, which can be computed by use of (2.4). Figure 2 shows a portion of the binary tree of predecessors of 0.

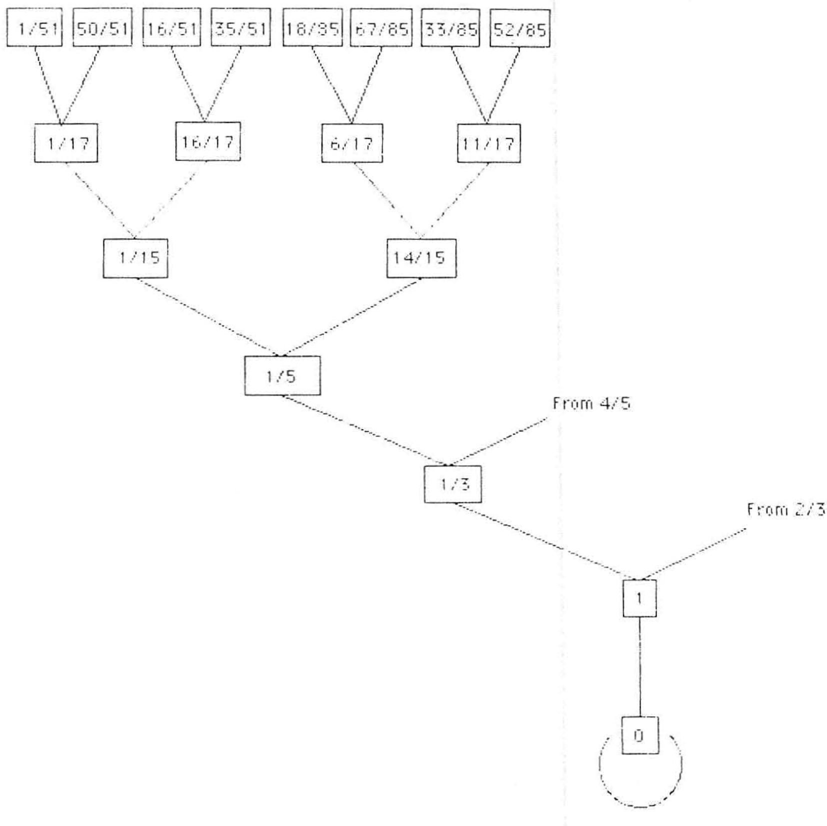

Figure 2: Some predecessors of 0.

More generally, if $\mu \in E^+$ has period $n = 2^k m$ then μ is either on a cycle or iterates to a cycle. For such a μ

$$x(\mu) = (2^{2^k m} - 1)^{-1} \sum_{s=0}^{2^k m-1} \mu_{2^k m-s} 2^s \tag{4.5}$$

and the denominator of (4.5) is given by

$$2^{2^k m} - 1 = (2m-1) \prod_{r=0}^{k-1} F_r(m) \tag{4.6}$$

where $F_r(m) = (F_r - 1)^m + 1$ are generalized Fermat numbers.

If

$$\mu_i = \begin{cases} 1 & i = 2^k ms \quad 1 \leq s < \infty \\ 0 & \text{otherwise} \end{cases}$$

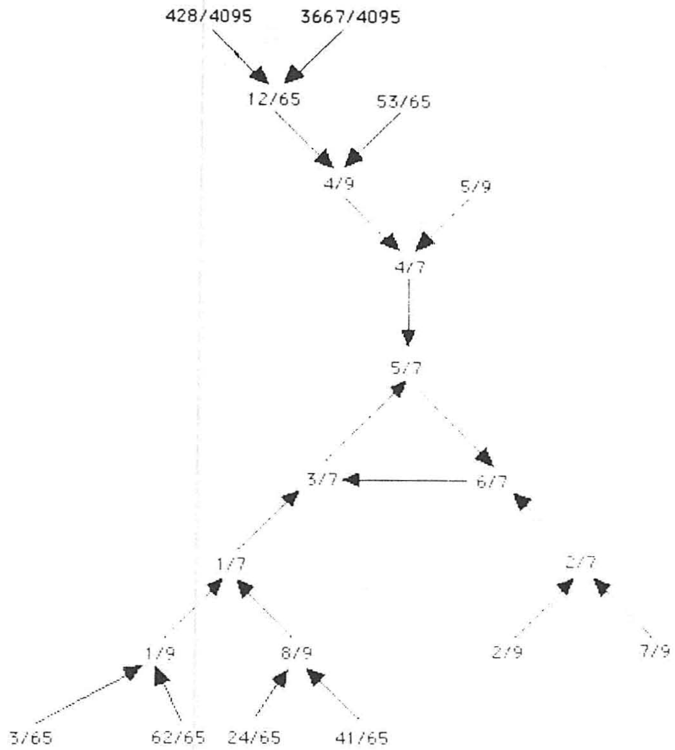

Figure 3: Some predecessors of a 3-cycle.

then $x(\mu)$ iterates to a cycle in exactly 2^k steps, and for $r \leq 2^k$

$$D^r(x(\mu)) = (2^{2^k m} - 1)^{-1} \sum_{s=0}^{r} \binom{r+1}{s+1} 2^s \tag{4.7}$$

which again can be evaulated in terms of (4.4) and (4.6).

Figure 3 shows a portion of the predecessor trees for the 3-cycle $\cdots \to 3/7 \to 5/7 \to 6/7 \to \cdots$

5. Density of predecessor set

As might be anticipated from Figures 2 and 3, the basins of attraction for cycles of D have a highly complicated structure. Iteration of (2.4) yields the general solution of $D^k(\mu) = \beta$ as [10]

$$\mu = B^k \sigma^{-k}(\beta) + \sum_{s=1}^{k} \mu_s B^s(\alpha^{(s)}) \tag{5.1}$$

in which $[\alpha^{(s)}]_r$ is one if $s = r$ and zero otherwise. Note that the first k terms $\mu_1, \ldots, \mu_k$ of these kth order predecessors are free parameters. In particular, if μ' is any element of E^+ we can choose $\mu_i = \mu'_i$ for $1 \le i \le k$. This proves

Theorem 9. *Let $\beta, \mu' \in E^+$ be arbitrary. Then β has a kth order predecessor μ such that $|x(\mu) - x(\mu')| \le 2^{-k}$.*

Corollary. The set $P(x(\beta))$ of all predecessors of β is dense in $[0, 1]$.

In fact a stronger result is possible. Consider a partition of $[0, 1]$ into 2^k segments of equal length so that the endpoint of each segment is $r/2^k$ for $0 \le r \le 2^k - 1$. Write the kth order predecessors of β as

$$\mu = (\mu_1 \cdots \mu_k \underline{0}) + (0 \cdots 0 \mu_{k+1} \cdots) \equiv \mu^* + \mu'$$

where μ^* represents the initial conditions. Then $x(\mu) = x(\mu^*) + x(\mu')$ and $x(\mu^*)$ takes the values $r/2^k$. Denote these as $x(\mu^*; r)$ and the corresponding values of $x(\mu)$ by $x(\mu; r)$. It is easy to see that $x(\mu^*; r) \le x(\mu; r) \le x(\mu^*; r + 1)$. Thus each segment of the partition contains exactly one kth order predecessor of $x(\beta)$. This proves

Theorem 10. *Let $x(\beta) \in [0, 1]$. The set $P_k(x(\beta))$ of kth order predecessors of β is uniformly distributed with respect to the uniform partition of $[0, 1]$ into 2^k segments.*

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