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Confirmation in Experimental Mathematics: A Case Study

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Abstract. On the basis of experiment, two groups arrived independently at similar conjectures on the nature of optimal strategies employed in large random games. The first conjecture was made by Kuhn and Quandt in 1963; the second was made by the present authors in 1987. In traditional mathematics such duplication is superfluous. In experimental mathematics it may provide mutual confirmation.

1. Introduction

The thesis of this note is that independent experimental confirmation may play a role in mathematics that is close to its role in traditional experimental science. However there is this difference: Experimental confirmations may be made almost obsolete at the moment a proof is found.

The case begins with a 1963 paper by Kuhn and Quandt [1]. (See also [2] for related work.) Their paper deals with the properties of large random matrices, which may be regarded as payoff matrices in a game between two opponents. Each matrix has a *value*, in the sense of game theory. Their paper contains a theorem on the asymptotic convergence of the value to a constant. It also presents the results of a numerical experiment that gives evidence on the average number of strategies given non-zero weight by each player, provided that the players are playing optimally.

For square matrices the result is the following. Each player has an optimal mixed strategy, given by a probability vector. The expected fraction of non-zero components of this vector in the random game experiment approaches one-half for large games. (The conjecture of Kuhn and Quandt is actually more general as it deals with rectangular games.)

Our subsequent work [3] was done without knowledge of these results; in fact its inspiration came from some ecological results of Cohen and Newman [4]. We used the methods of Cohen and Newman to prove a theorem

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giving a bound on large deviations of the value of the random game. This bound did not appear to be optimal, so we undertook numerical investigations on the standard deviation of the value of the random game. In the course of this numerical work we rediscovered the apparent fact about the asymptotics of the average number of strategies given non-zero weight by each player. Furthermore we found numerical evidence for asymptotics of higher order.

Our other research on random matrices has dealt mainly with spectral properties, and we had not been dealing with their game-theoretic aspects. Thus we went to some effort to perform a computer search to find relevant literature. We also checked other sources, such as the bibliography in the book [5]. However we did not learn of the earlier work of Kuhn and Quandt until we were informed of it in a letter from Kuhn [6].

On learning of this, we realized that the conjecture on the average number of strategies had received independent experimental confirmation. The chances of programming or statistical error thus seemed much reduced. This is a situation that is familiar from traditional experimental science; it is likely to become increasingly frequent in the mathematics of complex systems.

2. The Kuhn-Quandt experiment

The theoretical result of Kuhn and Quandt applies to an n-by-n random matrix with entries independently distributed by a uniform distribution that is symmetric about zero. For such a matrix the expected value of the game is zero, by symmetry. Kuhn and Quandt proved that the standard deviation of the value goes to zero as n approaches infinity. They obtained this by using a bound obtained in the situation where one player uses a uniform strategy.

Data generated in the course of their numerical experiments led to a conjecture. We quote from their paper [1]: "A count was kept of the number of slack variables appearing in the solution, that is, the number of constraints which are satisfied as strict inequalities at the maximum. Denoting the number of slack variables in the solution as a fraction of the total number m of nonzero variables in the solution of a problem with m constraints and n variables by F_{mn}, we state as a conjecture ... *If* $\lim_{m,n\to\infty} m/(m+n) = L$, *then* $\lim_{m,n\to\infty} E(F_{mn}) = L$." In the context of game theory, with square game matrices, this conjecture predicts that the mean fraction of non-zero weights a player will use in an optimal mixed strategy converges to one-half. (In general it converges to the proportion of the total number of strategies available to both players that are available to the other player.)

Kuhn and Quandt regard the statistical evidence for the conjecture as "very good." However the conjecture is "somewhat counterintuitive." In the course of their discussion they suggested: "If the conjecture is true, it appears likely therefore that convergence will be from below." We shall return to this remark below.

We quote the following historical comments from Kuhn's letter.

> I have stated this conjecture [that a rational player in a square random game tends to employ a strategy that has half its components zero] many times in scientific meetings as a result in stochastic programming that is easy to state, has substantial computational evidence in its favor, and seems to be very difficult to prove.
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> As a sidelight on the history of this problem, one of the reasons that we undertook this experiment was the availability of 40 hours a week free time on one of the first CDC 1600 computers. The catch was that the only input was via paper tape and the only language was machine language. When our programs did not run, we stepped them through 3×3 examples on the console and thereby discovered two *wiring* errors in the machine.

In a subsequent letter [7] Kuhn adds the following.

> A major point of our paper was that an experimental study had led to a conjecture in pure mathematics that was counterintuitive and yet simple to state. This point is strengthened when the conjecture was rediscovered 28 years later by another experimental study but, as yet, although the conjecture has been widely publicized, no one has come forward to prove it.

3. The Faris-Maier experiment

In the subsequent work of the present authors [3] there is a theoretical result that sharpens the Kuhn-Quandt result on convergence of the value of the game to zero by giving a bound on the probability of a fluctuation. This result is for the general case of independent, uniformly distributed entries of the game matrix. The bound is sharp for the situation where one player uses a uniform strategy. It should be remarked that such a large deviation bound gives more information than is contained in a moment bound. In our work there is also a rigorous bound on the probability of strategy vectors that have all components non-zero.

Our numerical experiment was also begun in part to test out a new computer, namely a mini-supercomputer manufactured by Scientific Computing Systems. The question of original interest was the standard deviation of the value of the random game. For simplicity all game matrices were n-by-n square matrices. Experiments were done in which the entries were uniform, as in the Kuhn-Quandt experiment, and in which the entries were Gaussian. In both cases the standard deviation was found numerically to approach zero at a rate proportional to $1/n$. This gives a quantitative aspect to the previous results on approach to zero.

During this experiment it was noticed that the proportion of strategies actually employed by each player tended to be about $1/2$. The numerical evidence gathered supports an asymptotic behavior $1/2 - 1/(4n)$. This experimentally discovered higher-order term lends support to the Kuhn-Quandt hypothesis that convergence will be from below. Presumably further numerical work will reveal the next few higher-order terms in an asymptotic series in $1/n$ for this quantity. A rigorous proof of the existence of such a series is lacking, however.

4. Conclusions

In retrospect the numerical experiments of Kuhn and Quandt and of the present authors make clear that mathematics has a component that is very similar to experimental science. In the absence of a rigorous proof of a conjecture, one can only "confirm" a conjecture by subjecting it to repeated experimental test. These confirmations may give rise to new conjectures, such as our conjecture on the existence of an asymptotic series in $1/n$.

Experimental confirmation may become obsolete at the moment a proof is found. This is not, however, an absolute matter. A proof is mechanically checkable in principle; however most mathematicians take at least a few intuitive shortcuts. One is justified in feeling more confident about a proof if its conclusion agrees with simulations.

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