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Robust Quasihomogeneous Configurations in Coupled Map Lattices*

L. Battiston
Instituto Studio Dinamica Grandi Masse–CNR,
S. Polo 1364, 30125 Venice, Italy

L. Bunimovich
Institute of Oceanology of the Academy of Sciences of the USSR,
Krasikova 23, 117218 Moscow, USSR

R. Lima
Centre de Physique Théorique, Unité Propre de Recherche 7061,
CNRS-Luminy, Case 907, 13288 Marseille Cedex 9, France

Abstract. We study the evolution of a coupled map lattice in two dimensions with strong coupling. We show the tendency of such system to assume (quasi) homogeneous space configurations evolving in time. We describe the mechanism of this phenomenon.

Coupled Map Lattices (CMLs), which were introduced rather recently [1, 2], now serve as one of the most useful and powerful instruments for understanding the dynamics of spatially extended systems. The main activity in this field is directed toward the study of the behavior of Coupled Map Lattices for small or moderate values of the spatial interactions [2]. This is natural because one of the main reasons for introducing this class of models is to use the information about the motion of the finite-dimensional dynamical systems that are used as local maps (i.e., those that act at each site of the lattice) in order to understand some features of the dynamics of CMLs. A new approach to this problem was developed in [3], where for some CML with diffusive coupling a thermodynamic formalism was constructed that allows representation of the corresponding infinite-dimensional dynamical system as some lattice model of statistical mechanics. The presence of space variables in CMLs implies that this model of statistical mechanics is at least two-dimensional, in contrast to point (non-extended) dynamical systems where the thermodynamic formalism always leads to one-dimensional models of

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statistical mechanics. It was proved in [3] that the corresponding model of statistical mechanics in the region of high temperature (corresponding to weak spatial interactions in CMLs) has no phase transitions. The conjecture formulated in [3] is that, for strong spatial interactions, this model has some phase transition and that the emerging new phases could be interpreted as coherent structures in the corresponding CML. This conjecture was confirmed in [4], where the first investigation was performed of this phenomenon in the region of local chaotic dynamics, with strong spatial interactions for one-dimensional CMLs generated by the logistic map with a diffusive coupling.

Here we study the region of the strong spatial interactions for a two-dimensional CML of the same type. The dynamics of the system under investigation is given by

$$x_{i,j}^{(n+1)} = (1-\varepsilon) f(x_{i,j}^{(n)}) + \frac{\varepsilon}{4} \sum_{k_1,k_2} \left(f(x_{k_1,j}^{(n)}) + f(x_{i,k_2}^{(n)}) \right) \tag{1}$$

where $k_1 = i \pm 1$, $k_2 = j \pm 1$, and $f(x)$ is the logistic map, namely, $f(x) = ax(1-x)$, $0 \leq a \leq 4$. It is known [5] that this map exhibits stable periodic as well as chaotic behavior, depending on the value of a. We are interested in the case where the system is chaotic, that is, $a_c < a \leq 4$, where some periodic windows have to be excluded. here $a_c = 3.5699456\ldots$ is the accumulation point for the period-doubling cascade [5].

For such values of a and small ε the system exhibits space-time chaos [2] in both one- and two-dimensional lattices. It was discovered in [4] that, for large values of ε, there is an interval $(\varepsilon_{\mathrm{cr}\,1}(a), \varepsilon_{\mathrm{cr}\,2}(a))$ of values of the parameter ε where the one-dimensional CML has a stationary solution that is a standing wave in space with period 2. This solution is stable [4, 6] in some more narrow range of parameters ε, and it can be considered the simplest coherent structure of the corresponding CML (spatially homogeneous solutions are unstable for these values of the parameter [4, 6]).

In our computer experiments we have considered a square lattice with 100×100 sites. The parameter value $a = 3.6$ corresponds to the existence of a single invariant measure for the one-dimensional transformation that is absolutely continuous with respect to the Lebesgue measure on $[0,1]$. This measure is concentrated in two disjoint subintervals

$$I_1, I_2 \subset [0,1] \text{ such that } f(I_1) = I_2,\ f(I_2) = I_1.$$

We found that in the range $(\varepsilon_{\mathrm{cr}\,1}, \varepsilon_{\mathrm{cr}\,2})$ the system (1) again has a stationary state that has the form of a chess board (figure 1). This means that, for all $n = 0, 1, 2, \ldots$, $x_{i,j}^{(n)} = a_1 = a_1(\varepsilon)$ if $(i+j)$ is even and $x_{i,j}^{(n)} = a_2 = a_2(\varepsilon)$ if $(i+j)$ is odd (or vice versa), where $a_1(\varepsilon) \in I_1$ and $a_2(\varepsilon) \in I_2$. These values satisfy the equations

$$\begin{aligned} (1-\varepsilon) f(a_1) + \varepsilon\, f(a_2) &= a_1 \\ (1-\varepsilon) f(a_2) + \varepsilon\, f(a_1) &= a_2 \end{aligned} \tag{2}$$

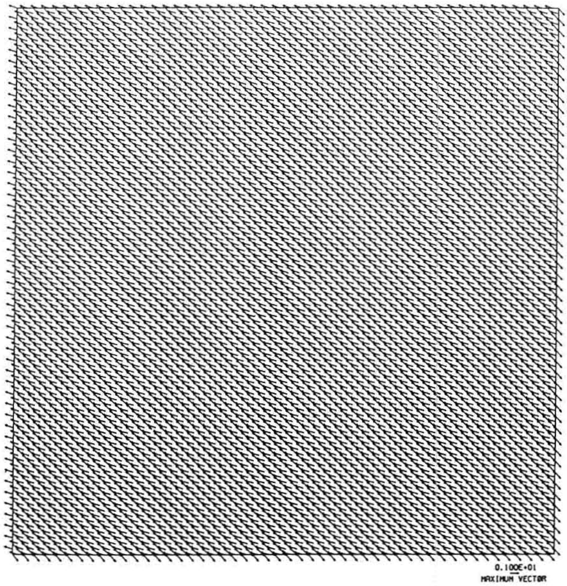


Figure 1: Chess-board configuration. At each point the value of the variable is represented by the direction of the corresponding vector. Directions in the upper half-plane correspond to I_1 and directions in the lower half-plane to I_2.

The domain on the plane of parameters (a, ε), where the corresponding chess-board configurations exist, is in fact the same as for the above-mentioned standing waves in the one-dimensional case [4, 6]. The analytical expression for the wave has already been given. In the case $a = 3.6$ we have $\varepsilon_{\mathrm{cr}\,1} \approx 0.86$, $\varepsilon_{\mathrm{cr}\,2} \approx 0.95$. These two chess-board-like coherent structures are stable in some more narrow interval $(\varepsilon'_{\mathrm{cr}\,1}(a), \varepsilon'_{\mathrm{cr}\,2}(a))$. Clearly, because of some symmetry in (1), this time-independent chess-board solution corresponds to the cycle of period 2 with the same spatial structure, but alternating in time (from black to white and vice versa). The reason for this is that we can read (2) as an equation for the same system, but where ε is replaced by $(1 - \varepsilon)$.

Coming back to the time-independent, chess-board-like solution of (1), this is preserved as soon as the initial conditions are inside an interval $\Gamma_1 \supset I_1$ ($\Gamma_2 \supset I_2$) that does not contain a_2 (a_1). In other words, a perturbed value in the "black" ("white") state of the chess board cannot yield the state of the chess board with opposite color. This condition defines the basin of attraction of the chess-board state of the lattice.

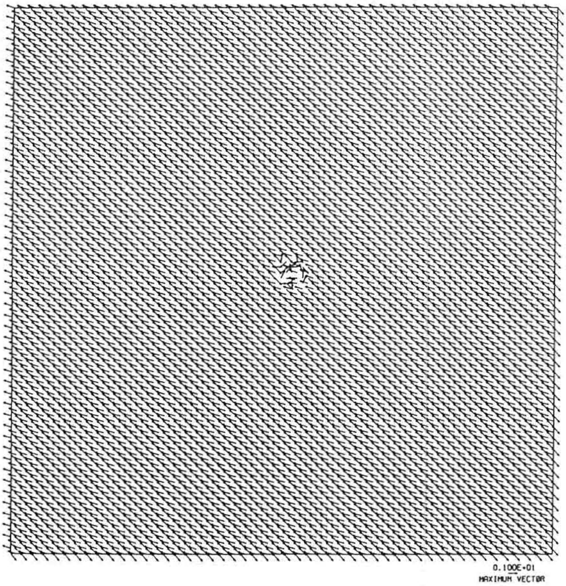
The generic initial state (i.e., the one where, at each site, the value is randomly taken in $[0, 1]$) does not evolve to this coherent structure. Therefore we performed other types of computer experiments to study the regime of spatial coexistence of the chess-board structure with the state corresponding to randomly chosen initial conditions in $[0, 1]$. In those experiments the sites of a square of side L were initiated randomly in the interval $[0, 1]$, whereas the remaining sites of the lattice were in the chess-board configuration (see figure 2). We found that there is a critical value L_c such that, if $L < L_c$, the chess-board state survives; otherwise it dies.

It is natural to compare this result with those that were obtained for the model (1) in the region of small and moderate space interactions [7]. In the latter case the motion of the CML is chaotic. It was found [7] that there are three different cases for the behavior of the system (1) (in the square with side L, and with periodic boundary conditions). These cases are the following: (i) $1 \leq L \leq 8$; (ii) $9 \leq L \leq 14$; and (iii) $L \geq 15$. In the first and third regions the largest Lyapunov exponent of the system is strictly positive and has some values well separated from zero. On the boundaries of the intermediate region the Lyapunov exponent suddenly falls to zero. Our results show that, to survive, the length of the chaotic state has to be at least of the order of the low boundary of the intermediate region. The reason is that, to destroy the chess-board (regular) structure, one needs a sufficiently large set of sites moving chaotically in time. In other words, in the region of the parameter for which the chess-board configuration has a basin of attraction corresponding to its stability domain, the spatial size of the chaotic perturbation needed to get out of this basin needs to be larger than the first threshold found in [7].

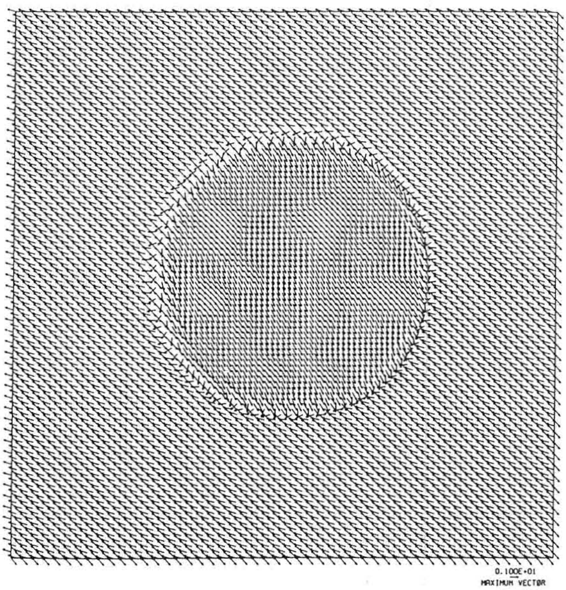
The last issue that we want to address refers to the region of the strongest spatial interaction (ε close to 1) where the chess-board coherent structure loses its stability. We considered the range of parameters $0.94 \leq \varepsilon \leq 1$. The corresponding results are represented in figure 3. Let us be reminded that we are in the situation where, for the map f, there are two intervals that attract almost all points of $[0, 1]$ and transform one into the other. Therefore it is natural also to expect to have in the CML (1) some of these features, namely period two in time and the presence of subsets in the phase space that correspond to the jumps from one interval to the other.

To test these ideas we used in our experiments initial conditions of the following three types: (i) random uniform distribution in the whole segment $[0, 1]$ for the initial value at each site; (ii) at sites of the upper (lower) half lattice the initial values were taken with uniform distribution in two intersecting segments $\Gamma_1 \supset I_1$ ($\Gamma_2 \supset I_2$); (iii) at sites of the upper (lower) half of the lattice, the initial values were taken with uniform distribution in I_1 (I_2).

The main property that we discover was that in all three cases there appear large domains where at each time step the state of the lattice is (quasi) homogeneous (figure 3). In case (i) this domain fills the whole lattice (figure 3(a)); in the opposite case (iii) as expected there are two regions in the lattice—the upper and lower halves—and, at each step, each region appears in a (quasi) homogeneous configuration but with different phases (figure 3(b)). Finally, in the intermediate case (ii) the lattice is divided into domains of (quasi) homogeneity with more complicated shapes but with rather

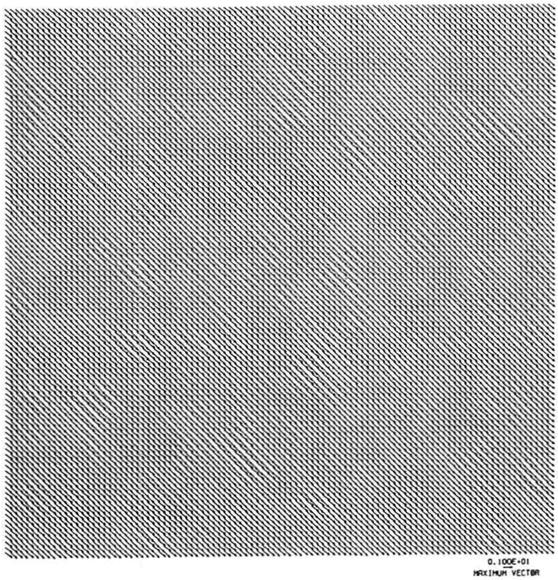


(a)

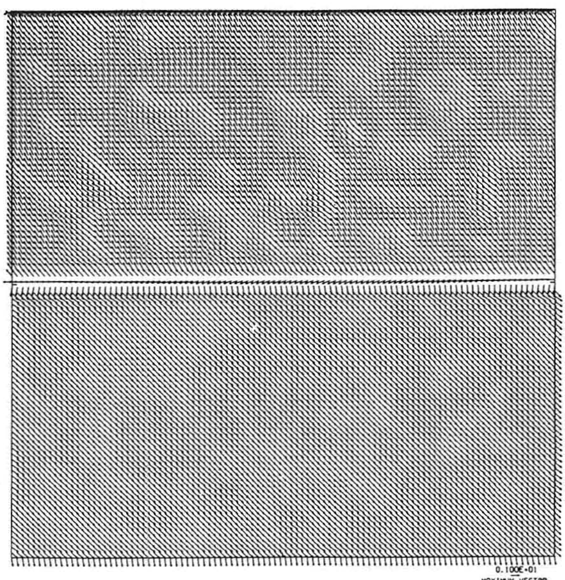


(b)

Figure 2: For $\varepsilon = 0.940$. (a) Transient regime of a chess-board configuration after a random perturbation in a square of size 5×5. After this transient regime the system returns to the chess-board configu-



(a)



(b)

Figure 3: (a) Quasihomogeneous configuration in the entire lattice. (b) Coexistence of two regions of the lattice with a quasihomogeneous configuration in two different phases.

smooth boundaries. These results show that the CML (1) has, for strong spatial interactions, the evident tendency to end up in space-homogeneous states. This is rather natural because of the diffusive coupling that makes the state of the system more homogeneous. On the other hand, since f represents a (local) force in (1)—which tends to make the motion at each site chaotic—the final issue should be the result of this balance. The analysis of such a mechanism is discussed in the remainder of this paper. Let us give the condition of the success of the first of these tendencies. Then we can show that these conditions are satisfied in our case.

Let us begin, for simplicity, with a one-dimensional lattice and let us take, instead of f, the so-called tent map, which is uniformly expanding:

$$f(x) = 2x \quad \text{if} \quad 0 \leq x \leq \frac{1}{2} \tag{3}$$

and

$$f(x) = 2 - 2x \quad \text{if} \quad \frac{1}{2} < x \leq 1.$$

Let $\varepsilon = 1$. Then we get

$$x_i^{(n+1)} = \frac{1}{2} f(x_{i-1}^{(n)}) + \frac{1}{2} f(x_{i+1}^{(n)}). \tag{4}$$

Therefore

$$x_{i+2}^{(n+1)} - x_i^{(n+1)} = \frac{1}{2} f(x_{i+3}^{(n)}) - \frac{1}{2} f(x_{i-1}^{(n)}). \tag{5}$$

It is easy to check that if

$$0 < x_{i-1}^{(n)} \quad \text{and} \quad x_{i+3}^{(n)} \leq \frac{1}{2}$$

or

$$\frac{1}{2} < x_{i-1}^{(n)} \quad \text{and} \quad x_{i+3}^{(n)} \leq 1 \tag{6}$$

then in the relation

$$|x_{i+1}^{(n+1)} - x_i^{(n+1)}| \leq |x_{i+3}^{(n)} - x_{i-1}^{(n)}| \tag{7}$$

the equality holds. In all the other cases the right side of (7) is strictly larger than the left side. Hence the values at different sites of the odd sublattice tend to be equal, as do those of the even sublattice. Let us mention that for the logistic map this process fails in the vicinity of points where the derivative is in modulus less than 1, but is even more intense for the remaining points.

Now we shall show how the whole lattice tends toward the equilibrium state. Let

$$\cdots = x_i^{(n)} = x_{i+2}^{(n)} = \cdots = a_1^{(n)}$$

and

$$\cdots = x_{i+1}^{(n)} = x_{i+3}^{(n)} = \cdots = a_2^{(n)}.$$

Then the corresponding map

$$\begin{pmatrix} a_1^{(n)} \\ a_2^{(n)} \end{pmatrix} \to \begin{pmatrix} a_1^{(n+1)} \\ a_2^{(n+1)} \end{pmatrix}$$

preserves the Lebesgue measure in the unit square. Therefore, according to the Poincaré recurrence theorem, at some moment $n+m$ we may have $a_1^{(n+m)} \approx a_2^{(n+m)}$. But then, if ε is not strictly equal to 1 because of diffusive coupling, the system evolves to a (quasi) homogeneous equilibrium state. The (maximal) derivatives of f^2 are equal to 2.8 and 2.4 at the ends of our interval. Therefore this mechanism can work perfectly since inside this interval the modulus of the derivative tends toward zero.

It is also easy to see that for two-dimensional lattices it works even better (it was mentioned before [8] that in two or higher dimensions the CML has stronger homogeneous properties than in one dimension).

Notice, finally, the presence of time period 4 in all the cases shown here. This corresponds [4] to the bifurcation that occurs in the map (2) after the chess-board state (or standing wave in the one-dimensional CML) loses its stability.

Acknowledgments

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