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# Period-Doublings to Chaos in a Simple Neural Network: An Analytical Proof

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**Abstract.** The dynamics of discrete-time neural networks with the sigmoid function as neuron activation function can be extraordinarily complex, as some authors have displayed in numerical simulations. Here we consider a simple neural network of *only* two neurons, one excitatory and the other inhibitory, with no external inputs and no time delay as a parameterized family of two-dimensional maps, and give an analytical proof for the existence of period-doublings to chaos and strange attractors in the network.

## 1. Introduction

Chaotic dynamical behavior in the brain has recently been observed and discussed [12, 18, 36]. Whether chaotic behavior of neural networks has any application in biological modeling (e.g., learning and information processing) is a very controversial issue [2, 4, 9, 18, 19, 20, 36, 40]. An important question is the biological implications of chaos in neural networks. Nevertheless, a lot of effort has gone into modeling and analyzing the chaotic behavior of biological systems, especially for the Hodgkin-Huxley axon model [12].

In this paper, we are interested in analytically exploring the possible existence and essential causes of chaos in neural networks with sigmoidal functions as neuron activation functions. These kinds of networks, as continuous approximations of McCulloch-Pitts Boolean networks, have received much attention in recent years and are widely studied in neural computation (for example, see [16, 23, 31, 35]). When used as associative memories, the discrete-time networks with symmetric connection weights have been shown [28] to have periodic behavior of periods at most 2 (i.e., fixed points, or fixed points and periodic orbits of period 2). But in general the dynamics of both continuous-time (defined by differential equations) and discrete-time (defined by difference equations) neural networks can be extraordinarily complex.

In the literature, many researchers [1, 8, 14, 24, 27, 33, 34] have performed numerical simulations on both continuous-time and discrete-time neural networks, most of which are driven by external inputs and/or have time delays, and observed various bifurcations and chaos. Although computer simulation and physical experiments are probably the most widely used techniques for understanding complex dynamical behavior in nonlinear systems, they are seldom adequate for a full understanding unless used in conjunction with analytical techniques. Sompolinsky, Crisanti, and Somers [37] demonstrated that continuous-time networks with random asymmetric connection will be chaotic asymptotically as the number of neurons $n \to \infty$, provided that the origin is not a stable fixed point. But it is not clear how "spontaneous" chaos occurs in autonomous neural networks of finite neurons with no time delay and no external inputs. Renals and Rohwer claimed in [33], according to their limited computer simulations, "as would be expected, very small networks ($n = 2, 3$) do not display complex dynamical behavior in any region of parameter space." We shall show this claim is not true.

In this paper, we consider a discrete-time neural network that consists of only *two* neurons with the sigmoidal neuron activation function and has no external inputs and no time delay. It turns out that even such a simple network with different synaptic weight connections can display almost every kind of complex dynamical behavior encountered in the literature in one-dimensional discrete-time dynamical systems [40]. Here we concentrate on the existence of chaos. We treat the simple network as a one-parameter family of two-dimensional maps with the neuron gain as the parameter, and prove analytically the existence of period-doublings to chaos in the network with an excitatory neuron and an inhibitory neuron. Specifically, we prove that, for a certain class of connection weight matrices, the simple neural network is dynamically equivalent to a one-parameter full family of $S$-unimodal maps on the interval $[0, 1]$, which is well-known to become chaotic through the period-doubling route as the parameter varies [10, 13, 17].

We note that the discrete-time neural networks are similar to cellular automata [11, 26, 42] but with an infinite number of possible states. Very simple two-cell-state cellular automata with high spatial dimensions display complex behavior and complicated spatial-temporal patterns. Our result here shows, on complementary, that very low spatial dimensional neural networks with continuum state space exhibit chaotic dynamics, which makes them most suitable as "building blocks" to study and produce high-dimensional chaos in rather large neural networks [40].

The rest of the paper is organized as follows. In section 2, we define the simple neural network and show that, when the connection weight matrix is of rank one, the network behaves dynamically like a one-dimensional map. In section 3, we describe the mechanism of period-doublings to chaos in a full family of $S$-unimodal maps. We present our analytical and experimental results on the existence of chaos in section 4. Finally we give some concluding remarks in section 5.

## 2. A simple neural network

The simple neural network is, as shown in figure 1, a fully connected network of two neurons. States of the two neurons are denoted as $x$ and $y$, respectively,

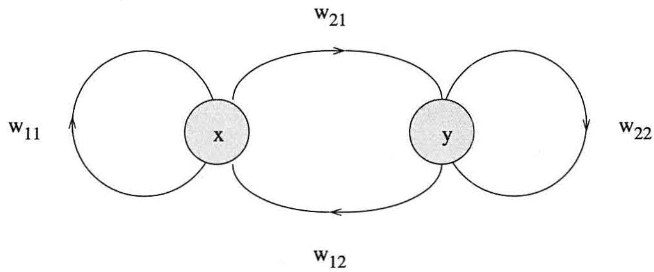


Figure 1: A simple neural network.

whose values range in the interval $I = [0, 1]$, and a state of the network is denoted as a vector $(x, y)$ in the state space $I^2 = [0, 1] \times [0, 1]$. The connectivity weights of the network form a $2\times 2$ real-valued matrix $W = [w_{ij}]$. The network updates its state in discrete time, $t = 0, 1, 2, \ldots$, and in a parallel fashion[1] according to

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = \begin{pmatrix} \sigma_\mu(w_{11}x(t) + w_{12}y(t)) \\ \sigma_\mu(w_{21}x(t) + w_{22}y(t)) \end{pmatrix} \equiv F_\mu \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{1}$$

where

$$\sigma_\mu(z) = \frac{1}{1 + e^{-\mu z}}$$

is a neuron activation function of sigmoid type with a parameter $\mu > 0$, and

$$F_\mu = \sigma_\mu \cdot W \tag{2}$$

is the network map with $\sigma_\mu(x, y) = (\sigma_\mu(x), \sigma_\mu(y))$.

The parameter $\mu$, also called *neuron gain*, plays a very important role in our study. First it controls the maximal slope of the function $\sigma_\mu(z)$: the larger the value of $\mu$ is, the closer the function $\sigma_\mu(z)$ approximates the step function. Second, it serves a purpose (at least mathematically) that a change of $\mu$ causes a change of all connectivity weights $w_{ij}$ and therefore affects the dynamic behavior of the network.

From the dynamical system point of view [13, 17], for any given matrix $W$, the network in (1) defines a one-parameter family of two-dimensional maps $F_\mu$. So the dynamics of the network can be studied from the perspective of iterations of the map $F_\mu$ on $I^2$.

The first thing we can claim about the network is the existence of fixed points. Because the state space $I^2$ is a convex compact subset of $\mathbb{R}^2$ and the map $F_\mu$ is continuous for any given $\mu$ and $W$, $F_\mu$ always has some fixed point in $I^2$, according to the Brouwer fixed point theorem (see [30]). Another property of $F_\mu$ that can easily be seen from (2) is that $F_\mu$ is a diffeomorphism

[1] This differs from asynchronous state-updating mechanisms described in [3, 6, 21].

(i.e., a differentiable map with a differentiable inverse) if and only if the weight matrix $W$ is non-singular.

In the rest of this paper, we shall be interested in the network whose weight matrix takes the form

$$W = \begin{bmatrix} a & ka \\ b & kb \end{bmatrix} \tag{3}$$

for some non-zero $a, b, k \in \mathbb{R}$. The singularity of $W$ may be considered as relating to the fact that all neurons in the brain do not necessarily function independently. Some (and maybe many) of them are only auxiliary; their roles in neural computation may be viewed as being "cooperative" in one perspective and "redundant" in another. We like to show the network map $F_\mu$ with such $W$ behaves like a one-dimensional map. To this end, we need some technical concepts and results.

Assume that $F : X \to X$ and $G : Y \to Y$ are two maps with $X' \subseteq X$ and $Y' \subseteq Y$ being two respective invariant subsets; that is, $F(X') \subseteq X'$ and $G(Y') \subseteq G(Y')$. We say that $F$ on $X'$ is *topologically conjugate* to $G$ on $Y'$ if there exists a homeomorphism (i.e., a one-to-one and continuous map with a continuous inverse) $H : X' \to Y'$ such that $G = H \cdot F \cdot H^{-1}$. The homeomorphism $H$ is called a topological conjugacy of $F$ and $G$. It is known that if $F$ and $G$ are topologically conjugate then they have the same dynamical behavior, namely the same orbit structure and stability on the respective invariant sets.

The following lemma shows an example of how a two-dimensional map is topologically conjugate to a one-dimensional map.

**Lemma 1.** *Consider a map $F : X_1 \times X_2 \to X_1 \times X_2$, defined by*

$$F(x_1, x_2) = (f_1(x_1), f_2(x_1))$$

*and hence*

$$F^n(x_1, x_2) = (f_1^n(x_1), f_2(f_1^{n-1}(x_1))$$

*where both component functions $f_1$ and $f_2$ of $F$ depend* only *on the variable $x_1$. If $f_2$ is a homeomorphism from $X_1$ to $f_2(X_1)$, then $F$ on $X$ is topologically conjugate to $f_1$ on $X_1$, where $X = F(X_1 \times X_2) = \{(f_1(x_1), f_2(x_1)) \mid x_1 \in X_1\}$.*

**Proof.** Certainly, $X$ is invariant under $F$. Define a homeomorphism $h : X \to X_1$ by

$$h(f_1(x_1), f_2(x_1)) = x_1 \quad \text{and} \quad h^{-1}(x_1) = (f_1(x_1), f_2(x_1)).$$

Then, for any $x_1 \in X_1$,

$$\begin{aligned} f_1(x_1) &= h(f_1(f_1(x_1)), f_2(f_1(x_1))) \\ &= h(F(f_1(x_1), f_2(x_1))) \\ &= h(F(h^{-1}(x_1))), \end{aligned}$$

and the lemma follows. ■

We note that, by induction, Lemma 1 can be generalized to high-dimensional maps of the similar property, especially to those for feed-forward networks or cascades [20].

**Lemma 2.** *Consider the network map $F_\mu : I^2 \to I^2$ in (1). If the weight matrix $W$ takes the form in (3), then $F_\mu$ on $F_\mu(I^2)$ is topologically conjugate to a one-dimensional map $g_\mu$ on an interval $[s,t]$ that is defined by*

$$g_\mu(x) = \sigma_\mu(ax) + k\sigma_\mu(bx),$$

*where $s = \min\{x + ky \,|\, x, y \in I\}$ and $t = \max\{x + ky \,|\, x, y \in I\}$.*

**Proof.** Rewrite $W$ as a product $W = LU$ with

$$L = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad U^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}.$$

Let $X = \{(x + ky, y) \,|\, x, y \in I\}$ and consider the map $G_\mu : X \to X$, which is the topological conjugate of $F_\mu$ under the conjugacy $U$; that is,

$$G_\mu \begin{pmatrix} x \\ y \end{pmatrix} = U \cdot F_\mu \cdot U^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sigma_\mu(ax) + k\sigma_\mu(bx) \\ \sigma_\mu(bx) \end{pmatrix} \equiv \begin{pmatrix} g_\mu(x) \\ \bar{g}_\mu(x) \end{pmatrix}.$$

Then both component maps $g_\mu$ and $\bar{g}_\mu$ of $G_\mu$ depend only on the variable $x$. Projecting the domain $X$ of $G_\mu$ to the first component, we see that $g_\mu$ is a map from $[s,t]$ to $[s,t]$. It follows from Lemma 1 that $G_\mu$ on $G_\mu(X)$ is topologically conjugate to $g_\mu$ on $[s,t]$, as $\bar{g}_\mu$ is a diffeomorphism. Since $U : F_\mu(I^2) \to G_\mu(X)$ is onto, $F_\mu$ is topologically conjugate to $G_\mu$ and hence topologically conjugate to $g_\mu$. ■

The analysis thus far allows us to concentrate the further study on the dynamics of $g_\mu$. For the one-dimensional maps, we have rather rich results at hand on their dynamics. The most well-studied families of one-dimensional maps are *one-parameter full families of $S$-unimodal maps on the interval* $[0,1]$. One prototypical example of such a family is the family of quadratic maps [10, 13, 17]

$$f_\mu(x) = \mu x(1 - x), \quad 0 < \mu < 4.$$

## 3. Full families of $S$-unimodal maps

A map $f$ of the interval [0,1] into itself is *unimodal* if (i) $f(0) = f(1) = 0$ and (ii) $f$ has a unique critical point $c$ with $0 < c < 1$. Hence, a unimodal map $f$ is increasing on the interval $[0,c)$ and decreasing on $(c,1]$, and $f$ is differentiable except possibly at $c$. A unimodal map $f$ is called *$S$-unimodal* if in addition (iii) $f$ is $C^3$, that is, the third derivative $f'''$ of $f$ exists and is continuous, and (iv) the Schwarzian derivative $Sf = f'''/f' - 3/2(f''/f')^2 < 0$

for all $x \in [0,1]\backslash\{c\}$. An $S$-unimodal map is known [10, 17] to have at most one stable periodic orbit and, if the critical point is not attracted to a stable periodic orbit, the map has no stable periodic orbit. In the following we will substitute (iv) by (iv$'$) the Schwarzian derivative $Sf < 0$ for all $x \in [0,1]\backslash\{c,1\}$; that is, we allow $Sf = 0$ at $x = 1$. If $f(1) = 0$, and thus the endpoint 1 cannot be in any periodic orbit or any attractor of the map $f$, this tiny relaxation does not change any asymptotic behavior of an $S$-unimodal map.

Let $f_\mu : [0,1] \mapsto [0,1]$ be a one-parameter family of $S$-unimodal maps with $\mu \in (\mu', \mu'')$. The family $f_\mu$ is called a *full* family if (i) every map $f_\mu$ is once differentiable with respect to $\mu$, (ii) $\lim_{\mu\to\mu'} f_\mu(x) = 0$ for all $x \in [0,1]$, and (iii) $\lim_{\mu\to\mu''} f_\mu(f_\mu(c)) = 0$[2].

It is well known that the dynamics of a full family of maps $f_\mu$ involves a sequence of period-doubling bifurcations, by means of which the maps become chaotic as the parameter $\mu$ varies from $\mu'$ to $\mu''$. Specifically, there are two sequences of parameter values $\mu_{2^n}, \tilde{\mu}_{2^n}, n = 0,1,2,\ldots$, with $\mu_1 < \mu_2 < \cdots < \mu_{2^n} < \cdots < \tilde{\mu}_{2^n} < \cdots < \tilde{\mu}_2 < \tilde{\mu}_1$ so that [41]

- $f_{\mu_{2^n}}$ has periodic points of period $2^n$. The values $\mu_{2^n}$ represent the bifurcation points at which periodic orbits of periods $2^n$ bifurcate to periodic orbits of periods $2^{n+1}$ of $f_\mu$.

- The values $\tilde{\mu}_{2^n}$ correspond to the bifurcation points at which periodic orbits of periods $2^n \cdot 3$ bifurcate, and $f_{\tilde{\mu}_{2^n}}^{1+2^{n+1}}(c)$ are unstable periodic points of periods $2^n$. At $\mu = \tilde{\mu}_{2^n}$ the maps $f_\mu$ are chaotic and have strange attractors, as they are topologically conjugate to chaotic piecewise linear maps $x \mapsto s - 1 - s|x|$, $s = 2^{-2^{n+1}}$.

- Embedded between each pair $\tilde{\mu}_{2^n}$ and $\tilde{\mu}_{2^{n-1}}$, there are ranges for $\mu$ corresponding to periods of form $2^n \cdot k$ in which $f_\mu^{2^n \cdot k}$ is a full family on some subinterval of $[0,1]$. The interval structures within these pairs are exactly the same as that of the outer one $(\mu_1, \tilde{\mu}_1)$. Overall, the order in which periods of stable periodic orbits appear as the parameter varies is given in the Sarkovskii ordering: $1, 2, 2^2, \ldots, 2^3 \cdot 5, 2^3 \cdot 3, \ldots, 2^2 \cdot 5, 2^2 \cdot 3, \ldots, 2 \cdot 5, 2 \cdot 3, \ldots, 7, 5, 3$.

- The sequences $\mu_{2^n}$ and $\tilde{\mu}_{2^n}$ converge [5, 15], from below and above respectively, to a parameter value $\mu_*$ ($\mu_* = 3.5699456\ldots$ for the quadratic family [15, 32]), at the *universal rate*

$$\delta = \lim_{n\to\infty} \frac{\mu_n - \mu_{n+1}}{\mu_{n+1} - \mu_{n+2}} = \lim_{n\to\infty} \frac{\tilde{\mu}_n - \tilde{\mu}_{n+1}}{\tilde{\mu}_{n+1} - \tilde{\mu}_{n+2}} = 4.669\cdots$$

  The $\delta$ is also known as the Feigenbaum number, which has been studied analytically by Sullivan [38] using quasiconformal homeomorphisms.

[2] To avoid introducing the kneading sequence of a unimodal map we use this definition, which appears to be stronger than the one in [13] but weaker than the one in [17].

We refer the reader to references [5, 10, 13, 17, 29, 41] for details on the mechanism of such period-doublings to chaos and the existence of one-dimensional strange attractors.

## 4. Analytical and experimental results

In this section we want to show that, when the gain parameter $\mu$ varies, the dynamical behavior of the two-neuron network with some weight matrix is that of a full family of $S$-unimodal maps. So there exist a period-doubling route to chaos and possible strange attractors within the simple network.

**Theorem**. Consider the network maps $F_\mu$ in (1) with weight matrix

$$W = \begin{bmatrix} a & -a \\ b & -b \end{bmatrix}.$$

If (i) $b < a < 0$ with $d = b/a \geq 2$, or (ii) $a > b > 0$ with $d = b/a \leq 1/2$, then the family of maps $F_\mu$ on $F_\mu(I^2)$ is topologically conjugate to a full family of $S$-unimodal maps $f_\mu$ on $[0, 1]$ with $\mu \in (0, \infty)$.

**Proof**. First we notice that the maps $F_\mu$ in cases (i) and (ii) have the same dynamics since they differ only by renaming the two neurons. So in the following we only consider case (i).

By Lemma 2, $F_\mu$ on $F_\mu(I^2)$ is topologically conjugate to a one-dimensional map $g_\mu : [-1, 1] \to [-1, 1]$, defined by $g_\mu(x) = \sigma_\mu(ax) - \sigma_\mu(bx)$. Since $g_\mu(x) \to 0$ exponentially as $x \to \pm\infty$, most "interesting" dynamical behavior of $g_\mu$ will occur in the interval $[-1, 1]$ as $\mu$ become large enough. However, for convenience, we will consider the extension of $g_\mu$ defined on the extended real line $\mathbb{R}^\infty = \mathbb{R} \cup \{\pm\infty\}$ with $g_\mu(\pm\infty) = 0$.

The maps $g_\mu$ on $\mathbb{R}$ have the following properties: for any $\mu > 0$,

(a) $g_\mu$ is an odd function: $g_\mu(-x) = \sigma_\mu(-ax) - \sigma_\mu(-bx) = (1 - \sigma_\mu(ax)) - (1 - \sigma_\mu(bx)) = -g_\mu(x)$.

(b) $g_\mu(0) = 0$, $g_\mu(x) > 0$ for $x > 0$, $g_\mu(x) < 0$ for $x < 0$, and $g_\mu(x) \to 0$ as $x \to \pm\infty$.

(c) $g_\mu$ has only two critical points $c > 0$ and $-c$ in $(-\infty, \infty)$. As $b < a < 0$, the functions $\sigma_\mu(ax)$ and $\sigma_\mu(bx)$ are concave-up and $\sigma_\mu(ax) > \sigma_\mu(bx)$ for all $x > 0$. Since $\sigma_\mu(ax) = \sigma_\mu(bx) = 1/2$ at $x = 0$ and both $\sigma_\mu(ax), \sigma_\mu(bx) \to 1$ as $x \to \infty$, $g_\mu(x)$ has only one critical point $c$ in $(0, \infty)$. Since $g_\mu$ is an odd function, it has another critical point $-c$ in $(-\infty, 0)$.

(d) $g_\mu$ is $C^3$ and has negative Schwarzian derivative $Sg_\mu(x)$ for all $x \geq 0$. First, $g_\mu$ restricted to $[0, \infty)$ is a composition of two maps, $g_\mu(x) = \phi(\psi(x))$, where $\psi : [0, \infty) \to [1, \infty)$, $\phi : [1, \infty) \to [0, \infty)$,

$$\psi(x) = e^{-\mu a x} \quad \text{and} \quad \phi(x) = \frac{1}{1+x} - \frac{1}{1+x^d}, \quad d = b/a.$$

According to a property of the Schwarzian derivative [13, 17],

$$S(\phi \cdot \psi) \;=\; S\phi(\psi)\,\psi'^2 + S\psi, \tag{4}$$

if $S\psi < 0$ and $S\phi < 0$, then $Sg_\mu < 0$. It is easy to show $S\psi < 0$:

$$S\psi(x) = \frac{(-\mu a)^3 e^{-\mu a x}}{-\mu a\, e^{-\mu a x}} - \frac{3}{2}\left(\frac{(-\mu a)^2 e^{-\mu a x}}{-\mu a\, e^{-\mu a x}}\right)^2 = -\frac{1}{2}(\mu a)^2 < 0.$$

But it needs some effort to carry out $S\phi(x)$ for $x \geq 1$:

$$S\phi(x) \;=\; \frac{\phi'''(x)}{\phi'(x)} - \frac{3}{2}\left(\frac{\phi''(x)}{\phi'(x)}\right)^2 \;=\; -\frac{\phi_1(x)}{\phi_2(x)}$$

where

$$\begin{aligned}
\phi'(x) &= \frac{-1}{(1+x)^2} + \frac{dx^{d-1}}{(1+x^d)^2}\\
\phi''(x) &= \frac{2}{(1+x)^3} + \frac{d(d-1)x^{d-2}}{(1+x^d)^2} - \frac{2d^2x^{2d-2}}{(1+x^d)^3}\\
\phi'''(x) &= \frac{-6}{(1+x)^4} + \frac{d(d-1)(d-2)x^{d-3}}{(1+x^d)^2}\\
&\quad - \frac{6d^2(d-1)x^{2d-3}}{(1+x^d)^3} + \frac{6d^3x^{3d-3}}{(1+x^d)^4}\\
\phi_1(x) &= 2d(d^2-3d+2)x^{3d+3} + 4d(d^2-1)x^{3d+2}\\
&\quad + 2d(d^2+3d+2)x^{3d+1} + d^2(d^2-1)x^{2d+4}\\
&\quad + 4d(d^3-2d^2-d+2)x^{2d+3}\\
&\quad + 4d(d^3-2d^2-d+2)x^{2d+1}\\
&\quad + 2d(3d^3-8d^2-3d-8)x^{2d+2} + d^2(d-1)x^{2d}\\
&\quad + 2d(d^2+3d+2)x^{d+3} + 4d(d^2-1)x^{d+2}\\
&\quad + 2d(d^2-3d+2)x^{d+1}\\
\phi_2(x) &= [2d^2x^{2d+6} + 8d(d-1)x^{2d+5} + 2x^{4d+4} - 4dx^{3d+5}]\\
&\quad + [8d(d-1)x^{3d+4} - 4dx^{3d+3}]\\
&\quad + [4(3d^2-4d+3)x^{2d+4} - 4dx^{d+5}]\\
&\quad + [8d(d-1)x^{2d+3} - 8(d-1)x^{d+4}]\\
&\quad + [2d^2x^{2d+2} - dx^{d+3}] + 2x^4
\end{aligned}$$

By the given condition $d \geq 2$, it is straightforward to check that $\phi_1(x) > 0$ and $\phi_2(x) > 0$ (each term within [] of $\phi_2(x)$ is positive). So $S\phi(x) < 0$ for all $x \geq 1$.

From properties (a) and (b), $g_\mu : \mathbb{R}^\infty \to \mathbb{R}^\infty$ can be decomposed into two maps $g_\mu|_{[-\infty,0]}$ and $g_\mu|_{[0,\infty]}$, which are topologically conjugate to each other under a conjugacy $h(x) = -x$. Therefore, for the dynamics, we can just consider $g_\mu$ defined on the interval $[0,\infty]$.

We define a topological conjugate $f_\mu$ of $g_\mu$,

$$f_\mu(x) = \begin{cases} h \cdot g_\mu \cdot h^{-1}(x) & 0 \le x < 1 \\ 0 & x = 1 \end{cases}$$

where $h : [0, \infty] \mapsto [0, 1]$ is given by $h(x) = x/(1 + x)$, with its inverse $h^{-1}(x) = x/(1 - x)$. We claim that the maps $f_\mu$ have the property of being a full family of $S$-unimodal maps on the interval $[0, 1]$ with $\mu \in (0, \infty)$. Let's check the conditions one by one.

(i) $f_\mu(0) = f_\mu(1) = 0$ for all $\mu$. Trivial.

(ii) Each $f_\mu$ has a critical point $c'_\mu = h^{-1}(c_\mu) = c_\mu/(1 + c_\mu)$ where $c_\mu$ is the critical point of $g_\mu$ in $(0, \infty)$. Since

$$\begin{aligned} f_\mu{}'(x) &= h'(g_\mu(h^{-1}(x)))\, g_\mu{}'(h^{-1}(x))\, h^{-1\prime}(x) \\ &= \frac{1}{(1 + g_\mu(h^{-1}(x)))^2}\, g_\mu{}'(h^{-1}(x))\, \frac{1}{(1 - x)^2}, \end{aligned}$$

$f'_\mu(x) = 0$ if and only if $g'_\mu(h^{-1}(x)) = 0$. Hence the existence and uniqueness of $c_\mu$ implies the existence and uniqueness of $c'_\mu$ in $(0, 1)$.

(iii) All $f_\mu$ are $C^3([0, 1])$. Trivial.

(iv) All $f_\mu$ have the negative Schwarzian derivatives. It is easy to check that $Sh = Sh^{-1} = 0$. Hence by the property of the Schwarzian derivative on composite maps given by (4),

$$\begin{aligned} Sf_\mu &= Sh(g_\mu \cdot h^{-1})(g_\mu \cdot h^{-1})'^2 + Sg_\mu(h^{-1})h^{-1\prime 2} + Sh^{-1} \\ &= Sg_\mu(h^{-1})h^{-1\prime 2}, \end{aligned}$$

$Sg_\mu < 0$ (except at $c$ and $\infty$) implies $Sf_\mu < 0$ (except at $c'$ and 1).

(v) All $f_\mu$ are once differentiable with respect to $\mu$. Trivial.

(vi) $\lim_{\mu \to 0} f_\mu(x) = 0$ for all $x \in [0, 1]$. Trivial.

(vii) Since all $f_\mu$ have the same function values $c^*$ at their critical points $c'_\mu$, that is, $c^* = f_\mu(c'_\mu)$ for all $\mu > 0$,

$$\lim_{\mu \to \infty} f_\mu(f_\mu(c'_\mu)) = \lim_{\mu \to \infty} f_\mu(c^*) = 0.$$

The proof of the theorem is now completed. ∎

**Remark.** The conditions on $a$ and $b$ in the theorem are only sufficient to carry out the technical part of the proof, $Sg_\mu < 0$. Simulations suggest that there are a lot of other cases for $a$ and $b$ in which the conclusion of the theorem is still true. In fact, computation of the Liapunov exponent [13, 17] of the map $f_\mu$ shows that the theorem also holds [40] for the cases where $a < b < 0$ with $1 < d = b/a < 2$ and $a > b > 0$ with $1/2 < d = b/a < 1$.

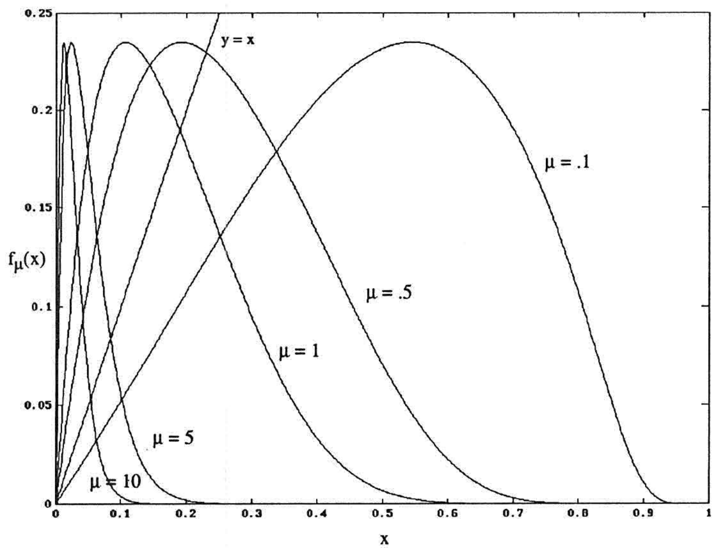


Figure 2: The full family of $S$-unimodal maps $f_\mu$ that are topologically conjugate to $F_\mu$ with weight matrix $w_{11} = -5, w_{12} = 5, w_{21} = -25, w_{22} = 25$.

To illustrate the chaotic behavior of the simple neural network, we perform some computer simulation on the network with weight matrix

$$W = \begin{bmatrix} -5 & 5 \\ -25 & 25 \end{bmatrix}. \tag{5}$$

The full family of $S$-unimodal maps $f_\mu$ for the maps $F_\mu$ is shown in figure 2 for some parameter values. To clarify the period-doubling route to chaos of the maps $F_\mu$, we draw bifurcation diagrams in figures 3 and 4 for the respective neuron states $x$ and $y$ as the parameter $\mu$ varies from 0 to 6. For each $\mu$ we draw a stable periodic orbit of $f_\mu$. We fix $\mu$ to be one of 600 equally spaced values in the interval $[0, 6]$. For such fixed $\mu$, we take 1000 iterations of the map $F_\mu$ with an initial state $(x(0), y(0)) = (.35, .55)$. On the vertical axis one point is plotted for each of the states $(x(900), y(900))$ to $(x(1000), y(1000))$, where $(x(t), y(t)) = F_\mu^t(x(0), y(0))$ for $t \geq 0$.

To see that some maps $F_\mu$ have strange attractors, we plot in figure 5 the last 1000 points in 10,000 iterates of $F_\mu$ with the initial point $(.35, .55)$ for some different parameter values. As we know from Lemma 2, state $y$ is a function of state $x$ when the network has a weight matrix of the form in (3). Hence the attractors in (a), (c), and (d) of figure 5 look like unions of several pieces of curves, which are essentially one-dimensional strange attractors [17].

Another interesting phenomenon we found in simulations is that the route of period-doublings to chaos in the simple neural network persists under some small perturbation of the weight matrices. This suggests that the network

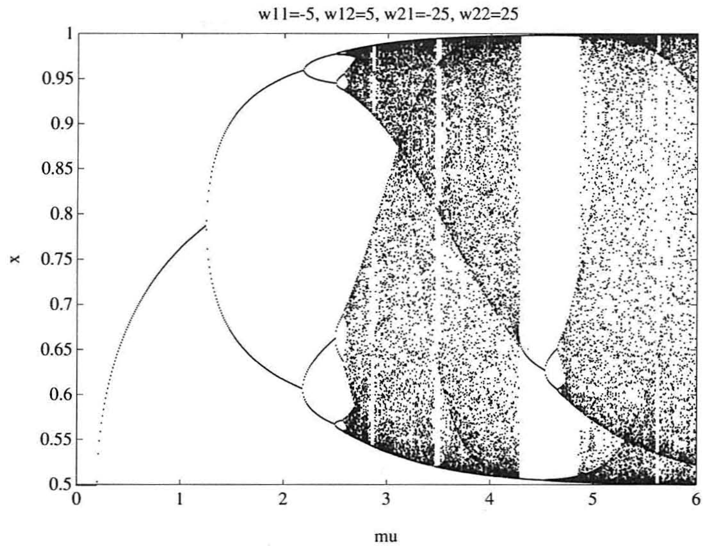


Figure 3: A bifurcation diagram in $x$ for the network with weight matrix $w_{11} = -5, w_{12} = 5, w_{21} = -25, w_{22} = 25$.

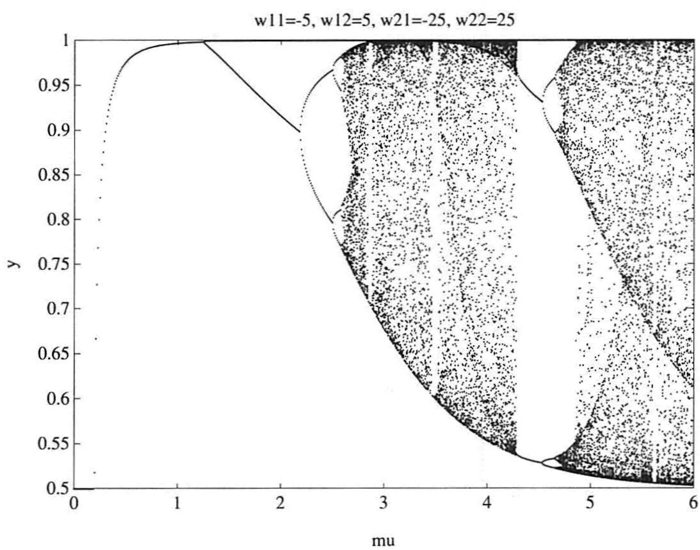


Figure 4: A bifurcation diagram in $y$ for the network with weight matrix $w_{11} = -5, w_{12} = 5, w_{21} = -25, w_{22} = 25$.

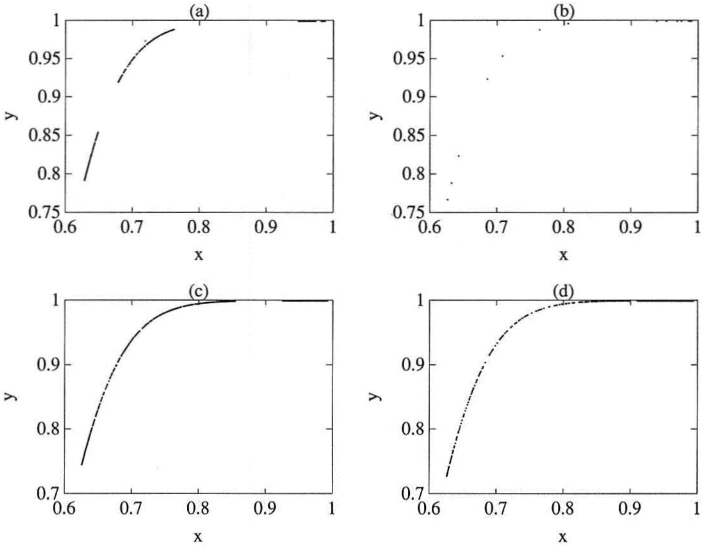


Figure 5: The attractors in network $F_\mu$ with weight matrix $w_{11} = -5, w_{12} = 5, w_{21} = -25, w_{22} = 25$. (a) $\mu = 3.25$; (b) $\mu = 3.5$; (c) $\mu = 3.75$; and (d) $\mu = 4$.

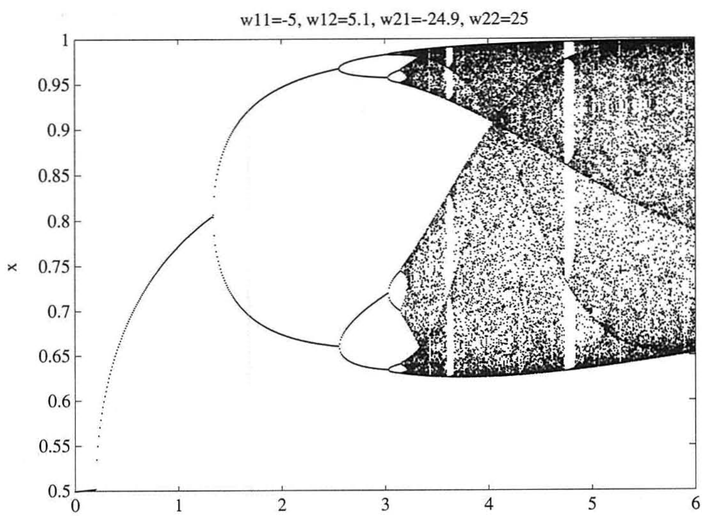


Figure 6: A bifurcation diagram in $x$ for the network with weight matrix $w_{11} = -5, w_{12} = 5.1, w_{21} = -24.9, w_{22} = 25$.

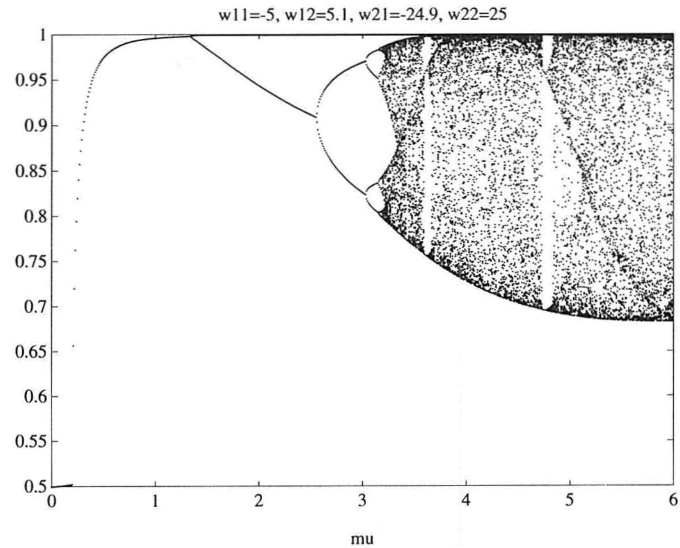


Figure 7: A bifurcation diagram in $y$ for the network with weight matrix $w_{11} = -5, w_{12} = 5.1, w_{21} = -24.9, w_{22} = 25$.

and also the full family of the $S$-unimodal maps $f_\mu$ have some kind of "structural stability" or "robustness" [13, 17, 40]. In figures 6 and 7, we plot bifurcation diagrams for the network with weight matrix

$$W = \begin{bmatrix} -5 & 5.1 \\ -24.9 & 25 \end{bmatrix}.$$

As we can see, a similar route of period-doublings to chaos occurs as that for the network with the matrix in (5).

## 5. Final remarks

(1) A biological implication of our results is that, given an inhibitory neuron $x$ and an excitatory neuron $y$ as shown, the simple network in (1) can still have interesting, even chaotic dynamics. Therefore, with properly chosen weight matrices, such a simple network can be used as a generator for oscillations of *any* period and even chaos, and as a "building block" for chaotic neural networks of more neurons [40]. For example, by adding one more neuron, we are able to perturb the one-dimensional maps in the theorem to two-dimensional maps that behave like the Hènon map [13, 17, 22].

(2) From our analytical result and numerical simulations, we see that the chaotic dynamics we have studied in this paper is not a rare phenomenon

that can happen in the network. In other words, if we randomly generate a weight matrix, we will have a positive probability of having one that yields chaotic dynamics in the network.

(3) From our analysis, we can see that lowering neuron gain can prevent the network from being chaotic and even from being oscillatory. But the network with very small gain may not have enough periodic orbits (including fixed points) and hence may not be very useful in practice, for example, when used as a model of associative memories [28] and oscillation generators [7].

(4) It can be shown [40] that another often used sigmoidal function $\tanh_\mu(x) = \tanh(\mu x)$ on the interval $[-1, 1]$ is not topologically conjugate to the sigmoid function $\sigma_\mu(x)$ used in this paper. But from our works [7, 40] on these networks, we see that the networks using $\tanh_\mu$ have the same period-doubling route to chaos as those using $\sigma_\mu$.

(5) In an autonomous neural network, its weight matrix determines the dynamics of the whole network, as the neuron gain of a neuron can be distributed to all its incoming weights. So study of how the various existing learning rules (like Hebbian and backpropagation rules) change the weights and what kinds of possible routes of bifurcations to chaos can result is definitely a very interesting subject in dynamics analysis of the neural networks. Maas, Verschure, and Molenaar [39] and Kolen and Pollack [25] have done some experiments along this direction. One of our future studies is to consider the simple neural network as a family of maps with multiple parameters (weights $w_{ij}$), to keep track of multiple parameter bifurcations caused by various learning rules, and to perform a thorough analysis on the dynamics of the network.

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## References

[1] K. L. Babcock and R. M. Westervelt, "Complex Dynamics in Simple Neural Circuits," pages 23–28 in *Neural Networks for Computing, AIP Conference Proceedings*, volume 151, Snowbird, UT, 1986.

[2] A. Babloyantz and A. Destexhe, "Chaos in Neural Networks," pages 31–39 in *IEEE International Conference on Neural Networks*, volume IV, San Diego, CA, 1989.

[3] J. Barhen and S. Gulati, "'Chaotic Relaxation' in Concurrently Asynchronous Neurodynamics," pages 619–626 in *IEEE International Conference on Neural Networks*, volume I, San Diego, CA, 1989.

[4] G. Basti and A. Perrone, "On the Cognitive Function of Deterministic Chaos in Neural Networks," pages 657–663 in *IEEE International Conference on Neural Networks*, volume I, San Diego, CA, 1989.

[5] T. Bedford and J. Swift, editors, *New Directions in Dynamical Systems*, volume 127 of *London Mathematical Society Lecture Note Series* (Cambridge, Cambridge Unversity Press, 1988).

[6] E. K. Blum, "Mathematical Aspects of Outer-Product Asynchronous Content-Addressable Memories," *Biological Cybernetics*, **62** (1990) 337–348.

[7] E. K. Blum and Xin Wang, "Stability of Fixed-Points and Periodic Orbits, and Bifurcations in Analog Neural Networks," University of Southern California, November 1990 (submitted for publication).

[8] M. Y. Choi and B. A. Huberman, "Dynamic Behavior of Nonlinear Networks," *Physical Review A*, **50** (1983) 1204–1206.

[9] M. S. Cohen and W. H. Julian, "Neural Networks with a Hopf Bifurcation: Slowly Modulated Waves," pages 161–168 in *IEEE International Conference on Neural Networks*, volume II, San Diego, CA, 1989.

[10] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, volume 1 of *Progress in Physics* (Boston, Birkhaüser, 1981).

[11] J. P. Crutchfield and K. Kaneko, "Phenomenology of Spatio-Temporal Chaos," in *Directions in Chaos*, edited by B.-L. Hao (Singapore, World Scientific, 1987).

[12] H. Degn, A. V. Holden, and L. F. Olsen, editors, *Chaos In Biological Systems*, volume 138 of *NATO ASI Series A: Life Science* (New York and London, Plenum, 1987).

[13] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems* (Redwood City, CA, Benjamin/Cummimgs, 1986).

[14] G. B. Ermentrout, "Period-Doublings and Possible Chaos in Neural Networks," *SIAM Journal on Applied Mathematics*, **44** (1984) 80–95.

[15] M. J. Feigenbaum, "Universal Behavior in Nonlinear Systems," *Los Alamos Science*, **1** (1980) 4–27.

[16] S. Grossberg, "Nonlinear Neural Networks: Principles, Mechanisms, and Architectures," *Neural Networks*, **1**(1) (1988) 17–61.

[17] J. Guckenheimer and P. J. Holmes, *Non-linear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*, volume 42 of *Applied Mathematical Science* (New York, Springer Verlag, 1983).

[18] M. R. Guevara, L. Galss, M. C. Mackey, and A. Shrier, "Chaos in Neurobiology," *IEEE Transactions on Systems, Man, and Cybernetics*, SMC-13(5) (1983) 790–797.

[19] E. Harth, "Order and Chaos in Neural Systems: An Introduction to the Dynamics of Higher Brain Functions," *IEEE Transactions on Systems, Man, and Cybernetics*, SMC-13(5) (1983) 782–789.

[20] M. W. Hirsch, "Convergent Activation Dynamics in Continuous Time Networks," *Neural Networks*, **2** (1989) 331–349.

[21] J. J. Hopfield, "Neural Networks and Physical Systems with Collective Computational Abilities," *Proceedings of the National Academy of Sciences*, **79** (1982) 2554–2558.

[22] P. J. Holmes and D. Whitley, "Bifurcations of One- and Two-Dimensional Maps," *Philosophical Transactions of the Royal Society of London*, **A311** (1984) 43–102.

[23] K. Hornik, M. Stinchcombe, and H. White, "Multilayer Feedforward Networks Are Universal Approximators," *Neural Networks*, **2**(5) (1989) 359–366.

[24] K. E. Kürten and J. W. Clark, "Chaos in Neural Systems," *Physics Letters*, **A114** (1986) 413.

[25] J. F. Kolen and J. B. Pollack, "Backpropagation is Sensitive to Initial Conditions," Technical Research Report, 90-JK-BPSIC, Laboratory for Artificial Intelligence Research, Ohio State University, 1990.

[26] C. Langton, "Computation at the Edge of Chaos: Phase Transition and Emergent Computation," *Physica*, **D42** (1990) 12–37.

[27] C. M. Marcus and R. M. Westervelt, "Dynamics of Analog Neural Networks with Time Delay," in *Advances in Neural Information Processing Systems*, volume 1 (San Mateo, CA, Morgan Kauffman, 1988).

[28] C. M. Marcus and R. M. Westervelt, "Dynamics of Iterated-Map Neural Networks," *Physical Review A*, **40** (1989) 501–504.

[29] T. Nowicki, "On Some Dynamical Properties of $S$-Unimodal Maps on an Interval," *Fundamenta Mathematicae*, **126** (1985) 27–43.

[30] J. M. Ortega and W. C. Rheinboldt, *Iterative Solutions of Nonlinear Equations in Several Variables* (New York, Academic Press, 1970).

[31] F. J. Pineda, "Dynamics and Architecture in Neural Computation," *Journal of Complexity*, **4** (1988) 216–245.

[32] S. N. Rasband, *Chaotic Dynamics of Nonlinear Systems* (New York, John Wiley, 1990).

[33] S. Renals and R. Rohwer, "A Study of Network Dynamics," *Journal of Statistical Physics*, **58** (1990) 825–848.

[34] U. Riedal, R. Kükn, and J. L. van Hemmen, "Temporal Sequences and Chaos in Neural Nets," *Physical Review A*, **38** (1988) 1105–1108.

[35] D. E. Rumelhart, J. L. McCllelland, and The PDP Research Group, *Parallel Distributed Processing: Experiments in the Microstructure of Cognition* (Cambridge, MA, MIT Press, 1986).

[36] C. A. Skarda and W. J. Freeman, "How Brains Make Chaos in Order to Make Sense of the World," *Behavioral and Brain Science*, **10** (1987) 161–195.

[37] H. Sompolinsky, A. Crisanti, and H. J. Sommers, "Chaos in Random Neural Networks," *Physical Review Letters*, **61** (1988) 259–262.

[38] D. Sullivan, "Quasiconformal Homeomorphisms in Dynamics, Topology and Geometry," pages 1216–1228 in *Proceedings of the International Congress of Mathematics*, volumes 1 and 2 (Providence, RI, American Mathematical Society, 1987).

[39] H. L. J. van der Maas, P. F. M. J. Verschure, and P. C. M. Molenaar, "A Note on Chaotic Behavior in Simple Neural Networks," *Neural Networks*, **3** (1990) 119–122.

[40] Xin Wang, "Discrete-Time Neural Networks as Dynamical Systems," (Doctoral dissertation, University of Southern California, 1991).

[41] D. Whitley, "Discrete Dynamical Systems in Dimensions One and Two," *Bulletin of the London Mathematical Society*, **15** (1983) 177–217.

[42] S. Wolfram, editor, *Theory and Application of Cellular Automata* (Singapore, World Scientific, 1986).