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Decision Procedures for Openness and Local Injectivity

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Abstract. Let F be a cellular automaton on the space of all sequences from a finite alphabet. From the local description of F a finite labeled digraph is constructed with a distinguished subdigraph. The surjectivity and openness of F are proved to correspond to properties of these digraphs. Similarly, another finite labeled digraph is constructed with a distinguished subset of its vertex set. The injectivity and local injectivity of F are proved to correspond to properties of this digraph and its subset. All the relevant properties can be checked in finitely many steps. Analogous results are also obtained for biinfinite sequences.

1. Introduction

Let S denote a finite set with N elements. We shall call S the *alphabet*, and each member of S a *symbol*. In the examples we will choose, typically $S = \{0, 1, \ldots, N-1\}$. Write Σ_+ for the set of infinite sequences $x = x_0 x_1 x_2 \ldots$ where $x_j \in S$ for $j = 0, 1, 2, \ldots$. We regard each x in Σ_+ as an infinite *word* or *configuration*. If we give S the discrete topology, we may then give Σ_+ the product topology since it may be regarded as the product of countably many copies of the discrete topological space S. In this topology a sequence of words x^i converges to the word x if and only if for each $j \geq 0$ there is an integer M such that, for $i > M$, the symbols in the jth position of x^i—that is, x^i_j—all agree with x_j. By the Tychonoff theorem Σ_+ is a compact space and is in fact metrizable.

The topology of Σ_+ plays an important role in some applications. For example, Σ_+ arises in the consideration of the Julia sets of certain polynomials in the complex plane where, by means of symbolic dynamics, the Julia set is shown to be homeomorphic with the configuration space Σ_+ (see [2]). The topology of the Julia set is entirely determined by the topology of the complex plane, and it is important to consider this topology when questions about the Julia set are reduced to questions about Σ_+.

Suppose that m is a nonnegative integer. A *local map* f of *order* m is a function $f : S^{m+1} \to S$. Any local map f determines a *global map* or *cellular* *automaton* $F : \Sigma_+ \to \Sigma_+$ of *order* m defined by $(Fx)_j = f(x_j, x_{j+1}, \dots, x_{j+m})$ for $j \geq 0$. An important example of a cellular automaton is the *shift map* $\sigma : \Sigma_+ \to \Sigma_+$ defined by $(\sigma x)_j = x_{j+1}$. It is easy to see that any cellular automaton is a continuous function that commutes with σ. The converse is also known, namely that any continuous map $F : \Sigma_+ \to \Sigma_+$ that commutes with the shift map σ is in fact a cellular automaton with order m for some m (see [5]).

A cellular automaton is determined by a table for the local map f. The question arises, however, whether properties of the global map F can be inferred from properties of the table for the local map.

Most past work in this area has concerned the slightly different situation where Σ denotes the set of doubly infinite sequences $x = \dots x_{-2}x_{-1}x_0x_1x_2 \dots$ and a local map f induces a global map $G : \Sigma \to \Sigma$ in the same manner as above. In particular, Amoroso and Patt [1] presented decision procedures for determining whether $G : \Sigma \to \Sigma$ was surjective or injective. Alternative procedures have been provided by [3] and [4], among others.

In this paper we modify the constructions of Amoroso and Patt to determine whether $F : \Sigma_+ \to \Sigma_+$ is surjective or injective. The modified constructions also permit us to answer other related questions that are not resolved by the Amoroso and Patt procedures. In particular, we are able to characterize when $F : \Sigma_+ \to \Sigma_+$ is an open map and when it is a locally injective map. These topological properties are of use in dealing with cellular automata that arise on the symbolic dynamics of a Julia set. Since the Julia set is more closely associated with Σ_+ than with Σ, the extension to Σ_+ is useful. The final section of this paper sketches the analogous results for the corresponding maps $G : \Sigma \to \Sigma$.

Here is an overview of our construction. If we are given the local map f for a cellular automaton $F : \Sigma_+ \to \Sigma_+$, we construct two finite labeled digraphs (directed graphs) $\mathcal{S}$ (for surjectivity) and $\mathcal{I}$ (for injectivity). Whether F is surjective can be decided by studying the digraph $\mathcal{S}$; the problem reduces to whether there is a walk from one distinguished vertex to another distinguished vertex. The same digraph can also be used to decide whether F is open; here the problem reduces to whether a certain subdigraph of $\mathcal{S}$ contains a directed cycle. Similarly, whether F is injective can be determined from $\mathcal{I}$, and we can use the same digraph to decide whether F is locally injective. In some ways, our constructions resemble de Bruijn graphs (see [6]).

The digraphs in question, while finite, may unfortunately be large. A crude upper bound for the number of vertices of $\mathcal{S}$ is 2 to the power N^{m+1}. A crude upper bound for the number of vertices of $\mathcal{I}$ is $N^{m+1} + (1/2)N^{2m+2}$.

2. Surjectivity

We construct the labeled digraph $\mathcal{S}$, called the *surjectivity digraph*, as follows. The vertices of $\mathcal{S}$ will be all subsets B of S^{m+1} such that all $x_0x_1 \dots x_m \in B$ have the same image under f. Equivalently, a subset $B \subset S^{m+1}$ is a vertex if and only if for any two elements $x_0x_1 \dots x_m$ and $y_0y_1 \dots y_m$ of B

we have $f(x_0x_1 \dots x_m) = f(y_0y_1 \dots y_m)$. Note that the empty set $\emptyset$ is a distinguished vertex of $\mathcal{S}$. Other distinguished vertices, for any $y \in S$, are $B_y = \{x_0x_1 \dots x_m \in S^{m+1} : f(x_0x_1 \dots x_m) = y\}$. All subsets of B_y are also vertices of $\mathcal{S}$, and no element of B_y lies in any other vertex of $\mathcal{S}$ other than a subset of B_y.

If B is a vertex of $\mathcal{S}$ and $y \in S$ then there will be an arc in $\mathcal{S}$ from B to C labeled by y where $C = \{x_0x_1 \dots x_m : f(x_0x_1 \dots x_m) = y$ and there exists $x_{-1} \in S$ such that $x_{-1}x_0x_1 \dots x_{m-1} \in B\}$. All arcs of $\mathcal{S}$ arise in this manner. Note that each vertex has exactly N outgoing arcs, one for each label y. It is possible that several of these arcs go to the vertex $\emptyset$. Moreover, each of the N arcs outgoing from $\emptyset$ terminates at $\emptyset$.

A *walk* in $\mathcal{S}$ is a sequence $\langle A_0 \;\; y_1 \;\; A_1 \;\; y_2 \;\; A_2 \dots y_k \;\; A_k\rangle$ where $A_0, \dots, A_k$ are vertices of $\mathcal{S}$; $y_1, \dots, y_k$ are symbols in S; and for each j the arc outgoing from A_{j-1} and labeled by y_j goes to A_j. Observe that, given an initial vertex A_0 and any sequence of labels $y_1, \dots, y_k$, a walk starting at A_0 is uniquely determined. We shall also allow for infinite walks.

The following result relates walks in $\mathcal{S}$ to finite strings of symbols in the image of F.

Lemma 2.1. *Let A_0 be a vertex of $\mathcal{S}$ and let $y_0 = f(x_0 \dots x_m)$ for all $x_0 \dots x_m$ in A_0. Let $y_1, \dots, y_k$ be a sequence of symbols that determine a walk $\langle A_0 \;\; y_1 \;\; A_1 \;\; y_2 \;\; A_2 \dots y_k \;\; A_k\rangle$. There exists $x_0x_1 \dots x_{k+m}$ such that $x_j \dots x_{j+m} \in A_j$ for $j = 0, \dots, k$ if and only if $A_k \neq \emptyset$. If $A_k \neq \emptyset$ then any $x \in \Sigma_+$ that begins with such $x_0x_1 \dots x_{k+m}$ satisfies that $F(x)$ begins with $y_0y_1 \dots y_k$. If $A_k = \emptyset$ then there exists no $x = x_0x_1 \dots$ satisfying both that $x_0 \dots x_m \in A_0$ and that $F(x)$ begins with $y_0y_1 \dots y_k$.*

Proof. Suppose that $x_0x_1 \dots x_{k+m}$ exists; then clearly $x_k \dots x_{k+m}$ lies in A_k, so $A_k \neq \emptyset$. Conversely, if $A_k \neq \emptyset$, choose any $x_k \dots x_{k+m}$ in A_k. Since there is an arc from A_{k-1} to A_k, it follows that there exists $x_{k-1} \in S$ such that $x_{k-1}x_k \dots x_{k-1+m} \in A_{k-1}$; moreover, since the arc was labeled y_k, it follows that $f(x_k \dots x_{k+m}) = y_k$. Since there is an arc labeled y_{k-1} from A_{k-2} to A_{k-1}, it follows that there exists $x_{k-2} \in S$ such that $x_{k-2}x_{k-1} \dots x_{k-2+m} \in A_{k-2}$ and $f(x_{k-1} \dots x_{k-1+m}) = y_{k-1}$. Repeating this argument, we obtain the desired $x_0x_1 \dots x_{k+m}$. The other assertions of the lemma follow easily. ∎

The most fundamental result concerning $\mathcal{S}$ is the following.

Theorem 2.2. *The cellular automaton F is surjective if and only if there is no walk in $\mathcal{S}$ from B_0 to $\emptyset$.*

It follows that one can determine whether F is surjective by doing a depth-first search in $\mathcal{S}$ starting at B_0 to decide whether $\emptyset$ can be reached. Here the choice of B_0 is somewhat arbitrary; it could be replaced by B_y for any $y \in S$.

Proof of Theorem 2.2. The proof will be broken into a series of lemmas.

Lemma 2.3. *F is surjective if and only if for each k and each finite string $y_0y_1\dots y_k$ of symbols from S there exists $x \in \Sigma_+$ such that $F(x)$ begins with $y_0y_1\dots y_k$.*

Proof. The "only if" portion is immediate. For the "if" implication, assume that $y = y_0y_1y_2\dots \in \Sigma_+$ is given, and we seek $x \in \Sigma_+$ such that $F(x) = y$. For each $k \geq 0$, by hypothesis we may choose $x^k \in \Sigma_+$ such that $F(x^k)$ begins with $y_0y_1\dots y_k$. The sequence $\{x^k\}$ lies in the compact topological space Σ_+ and hence has a subsequence $\{x^{k(i)}\}$ converging to some point of Σ_+, which we shall call x. Since F is continuous, it follows that the sequence $\{F(x^{k(i)})\}$ converges to $F(x)$. For any n and all sufficiently large i, $F(x^{k(i)})$ begins with $y_0y_1\dots y_n$; hence $F(x)$ begins with $y_0y_1\dots y_n$ by the topology of Σ_+. The lemma follows. ∎

Corollary 2.4. *F is surjective if and only if for each finite string $y_1y_2\dots y_k$ there exists $x \in \Sigma_+$ such that $F(x)$ begins with $0y_1y_2\dots y_k$.*

Proof. Since F commutes with the shift map σ, if x is as given then $F(\sigma(x))$ begins with $\sigma(0y_1y_2\dots) = y_1y_2\dots y_k$. ∎

Lemma 2.5. *Suppose that there is a walk in $\mathcal{S}$ from B_0 to $\emptyset$. Then F is not surjective.*

Proof. Let the given walk be $\langle A_0 \;\; y_1 \;\; A_1 \;\; y_2 \;\; A_2 \dots y_k \;\; A_k\rangle$ where $A_0 = B_0$, $A_k = \emptyset$, and each $y_j \in S$. We shall see that there is no $x \in \Sigma_+$ such that $F(x)$ begins with $0y_1y_2\dots y_k$.

To prove this, suppose that such an $x = x_0x_1\dots$ existed. Then $x_0x_1\dots x_m \in B_0$ by the definition of B_0 since $F(x)$ begins with 0. Because the next symbol in $F(x)$ is y_1 it follows that $f(x_1x_2\dots x_{m+1}) = y_1$; but $x_1x_2\dots x_{m+1}$ extends $x_0x_1x_2\dots x_m \in A_0$, so by the definition of the arcs of $\mathcal{S}$ we see that $x_1x_2\dots x_{m+1} \in A_1$. In a similar manner, since $f(x_2\dots x_{m+2}) = y_2$ and $x_2\dots x_{m+2}$ extends $x_1x_2\dots x_{m+1}$ it follows that $x_2\dots x_{m+2} \in A_2$. If we continue in this manner we see that $x_kx_{k+1}\dots x_{k+m} \in A_k = \emptyset$, which is not possible. ∎

Lemma 2.6. *Suppose that there is no walk in $\mathcal{S}$ from B_0 to $\emptyset$. Then F is surjective.*

Proof. Suppose the finite string $y_1y_2\dots y_k$ is given. Let $A_0 = B_0$ and consider the walk $\langle A_0 \;\; y_1 \;\; A_1 \;\; y_2 \;\; A_2\dots y_k \;\; A_k\rangle$. There is no walk from B_0 to $\emptyset$, so $A_k \neq \emptyset$. By Lemma 2.1 we obtain symbols $x_0, x_1, \dots, x_{k+m}$ such that $x_j\dots x_{j+m} \in A_j$ for all j satisfying $0 \leq j \leq k$; and, moreover, $F(x_0x_1\dots x_{k+m}\dots)$ begins with $0y_1y_2\dots y_k$. By Corollary 2.4 it follows that F is surjective. ∎

The lemmas together complete the proof of Theorem 2.2. ∎

3. Openness

A continuous map F is *open* if and only if, whenever U is an open set, it follows that $F(U)$ is also open. In this paper the topological space of interest is Σ_+, and it is easy to see that a basis for the topology is all sets of form $x_0x_1 \dots x_k\Sigma_+$ where adjacency indicates concatenation and $x_0x_1 \dots x_k$ ranges over all finite strings of symbols. In other words, a set U is open if and only if it is a union of sets of form $x_0x_1 \dots x_k\Sigma_+$ for various, not necessarily finitely many, choices of k and of $x_0x_1 \dots x_k$.

Let $\mathcal{S}$ be the surjectivity digraph described above. A vertex B of $\mathcal{S}$ is called *restricted* if there exists a walk in $\mathcal{S}$ from B to $\emptyset$. Note that $\emptyset$ itself is a restricted vertex. The *restricted digraph* $\mathcal{R}$ of $\mathcal{S}$ is the labeled subdigraph of $\mathcal{S}$ generated by all restricted vertices; its vertices are precisely the restricted vertices and its arcs are all the arcs of $\mathcal{S}$ between two restricted vertices.

A *cycle* is a walk along directed arcs (with at least one arc) that starts and ends at the same vertex.

Theorem 3.1. *F is open if and only if the digraph $\mathcal{R}$ contains no cycle with a vertex other than $\emptyset$.*

Note that any cycle in $\mathcal{S}$ containing $\emptyset$ necessarily is a walk for which all vertices are $\emptyset$. Since there are N arcs from $\emptyset$ to itself, there could be many arcs involved in such a cycle.

Proof of Theorem 3.1. The proof of the theorem will involve several lemmas. The idea is first to reduce to the case of open sets of the form $x_0x_1 \dots x_m\Sigma_+$, and then to relate the vertex $\{x_0x_1 \dots x_m\}$ to the digraph $\mathcal{R}$.

Lemma 3.2. *F is open if and only if, for each string $x_0x_1 \dots x_m$ with $m+1$ symbols, the set $F(x_0x_1 \dots x_m\Sigma_+)$ is open.*

Proof. The "only if" implication is immediate. For the "if" implication we must show that if U is open then $F(U)$ is open. It suffices to prove this when U is a member of the basis for the topology, so we assume that $U = x_0x_1 \dots x_k\Sigma_+$ for some choice of k and symbols x_i. If $k = m$ then $F(U)$ is open by hypothesis. If $k < m$ then U equals the union of all sets $x_0x_1 \dots x_kx_{k+1} \dots x_m\Sigma_+$ where $x_{k+1} \dots x_m$ ranges over all possible sequences of symbols of length $m-k$. But then $F(U)$ is the union of all such $F(x_0x_1 \dots x_kx_{k+1} \dots x_m\Sigma_+)$, each of which is open by hypothesis, so that $F(U)$ is also open.

Suppose now that $k > m$. Then $F(x_{k-m} \dots x_k\Sigma_+)$ is open by hypothesis. Let $y_j = f(x_j \dots x_{j+m})$ for $j = 0, \dots, k-m-1$. Since F has order m it is clear that $F(U) = F(x_0x_1 \dots x_kx_{k+1} \dots x_m\Sigma_+) = y_0y_1 \dots y_{k-m-1}F(x_{k-m} \dots x_k\Sigma_+)$ (where again adjacency indicates concatenation). But if V is open then V is a union of sets of the form $z_0z_1 \dots z_r\Sigma_+$; hence for any symbol y it follows that yV is a union of sets of the form $yz_0z_1 \dots z_r\Sigma_+$, and thus is also open. We conclude that $F(U)$ is open, and the proof is complete. ∎

Notation. If A is a vertex of $\mathcal{S}$, then $A\Sigma_+$ will denote all possible configurations arising by concatenating an element of A on the left with an element of Σ_+ on the right. Observe that $A\Sigma_+$ is automatically both open and closed; since Σ_+ is compact it follows that $A\Sigma_+$ is also compact.

Lemma 3.3. *Let A_0 be a nonempty vertex of $\mathcal{S}$ and let $y_0 = f(x_0x_1 \dots x_m)$ for any $x_0 \dots x_m$ in A_0. If A_0 is not a vertex of $\mathcal{R}$ then $F(A_0\Sigma_+) = y_0\Sigma_+$ and hence is open.*

Proof. It suffices to show that for any finite string $y_1 \dots y_k$ there exists an element x in $A_0\Sigma_+$ such that $F(x)$ begins with $y_0y_1 \dots y_k$. (This is by an argument like that for Lemma 2.3 using the fact that $A_0\Sigma_+$ is compact.) Consider the walk $\langle A_0 \quad y_1 \quad A_1 \quad y_2 \quad A_2 \dots y_k \quad A_k\rangle$. Since A_0 is not a vertex of $\mathcal{R}$ it follows that $A_k \neq \emptyset$; hence, by Lemma 2.1, we can find symbols $z_0z_1 \dots z_{k+m}$ such that $z_j \dots z_{j+m} \in A_j$ for $j = 0, \dots, k$. Since $z_0z_1 \dots z_m \in A_0$ it follows that any configuration z beginning with $z_0z_1 \dots z_{k+m}$ lies in $A_0\Sigma_+$, and by construction $F(z)$ begins with $y_0y_1 \dots y_k$. The lemma follows. ∎

Corollary 3.4. *Suppose that $\langle A_0 \quad y_1 \quad A_1 \quad y_2 \quad A_2 \dots y_k \quad A_k\rangle$ is a walk in $\mathcal{S}$. Let $f(x_0x_1 \dots x_m) = y_0$ for $x_0x_1 \dots x_m$ in A_0. Suppose finally that A_k is not a vertex of $\mathcal{R}$. Then $F(A_0\Sigma_+)$ contains the open set $y_0y_1 \dots y_k\Sigma_+$.*

Proof. By Lemma 3.3, given any element y of $y_k\Sigma_+$ there exists x in $A_k\Sigma_+$ such that $F(x) = y$. Suppose that x begins with $x_kx_{k+1} \dots x_{k+m} \in A_k$. Using the given walk we may choose x_j for $j = k-1, \dots, 0$ such that $x_j \dots x_{j+m}$ lies in A_j. Let $z = x_0x_1 \dots x_{k-1}x$. Then z lies in $A_0\Sigma_+$ and $F(z) = y_0y_1 \dots y_{k-1}y$, proving the result. ∎

Lemma 3.5. *Suppose that A_0 is a nonempty vertex of $\mathcal{S}$, and for some positive integer M every walk $\langle A_0 \quad y_1 \quad A_1 \quad y_2 \quad A_2 \dots y_M \quad A_M\rangle$ satisfies that either $A_M = \emptyset$ or A_M is not in $\mathcal{R}$. Then $F(A_0\Sigma_+)$ is open.*

Proof. Let $x = x_0x_1 \dots x_{M+m} \dots$ lie in $A_0\Sigma_+$ and let $F(x) = y = y_0y_1 \dots$. We must show that y lies in an open set contained entirely inside $F(A_0\Sigma_+)$. Consider the walk in $\mathcal{S}$ given by $\langle A_0 \quad y_1 \quad A_1 \quad y_2 \quad A_2 \dots y_M \quad A_M\rangle$. Then A_M is nonempty since it contains $x_M \dots x_{M+m}$, whence it follows that A_M is not in $\mathcal{R}$. By the corollary, $F(A_0\Sigma_+)$ contains the open set $y_0y_1 \dots y_M\Sigma_+$, which in turn contains the point y. ∎

We may now prove Theorem 3.1. First, suppose that $\mathcal{R}$ has no cycles except those involving the vertex $\emptyset$; we must show that F is open. Let $x_0x_1 \dots x_m$ be arbitrary and let A_0 denote the singleton set $\{x_0x_1 \dots x_m\}$. By Lemma 3.2 it suffices to show that $F(A_0\Sigma_+)$ is open. If A_0 is not a vertex of $\mathcal{R}$, the result follows by Lemma 3.3. If A_0 is a vertex of $\mathcal{R}$, let M denote the number of vertices in $\mathcal{R}$. Then any walk starting at A_0 with M steps ends either outside $\mathcal{R}$ or at $\emptyset$. (Otherwise two vertices of the walk would coincide and lie in $\mathcal{R}$, yielding a cycle of $\mathcal{R}$ not involving $\emptyset$.) By Lemma 3.5, $F(A_0\Sigma_+)$ is open.

Conversely, suppose that $\mathcal{R}$ has a cycle $\langle A_0 \;\; y_1 \;\; A_1 \;\; y_2 \;\; A_2 \ldots y_k \;\; A_k\rangle$ where $A_0 = A_k \neq \emptyset$. Let $y = y_k y_1 y_2 \ldots y_k y_1 y_2 \ldots y_k y_1 y_2 \ldots \in \Sigma_+$. We show that $y \in F(A_0\Sigma_+)$ yet y has no open neighborhood completely contained in $F(A_0\Sigma_+)$. To see that $y \in F(A_0\Sigma_+)$ observe that, since $A_0 = A_k$ and the walk to A_k has label y_k, each member of A_0 gets mapped by f to y_k. For any r consider the walk $\langle C_0 \;\; z_1 \;\; C_1 \;\; z_2 \ldots z_r \;\; C_r\rangle$ where $C_j = A_{j\,(\mathrm{mod}\,k)}$ and $z_j = y_{j\,(\mathrm{mod}\,k)}$. Since $C_r \neq \emptyset$ by Lemma 2.1, we may choose $x_0, \ldots, x_{r+m}$ such that $f(x_j \ldots x_{j+m}) = z_j$ and $x_j \ldots x_{j+m} \in C_j$. Using compactness of $A_0\Sigma_+$ yields $x \in A_0\Sigma_+$ such that $F(x) = y$.

If y had a neighborhood completely contained in $F(A_0\Sigma_+)$, there would exist a basis element of the form $u_0 u_1 \ldots u_s \Sigma_+$ completely contained in $F(A_0\Sigma_+)$. Clearly we may choose $u_0 \ldots u_s$ to have the form $y_k y_1 y_2 \ldots y_k y_1 \ldots y_k$ by choosing a slightly smaller basis element if necessary. But we shall show that there exists an element v of $u_0 u_1 \ldots u_s \Sigma_+$ not in $F(A_0\Sigma_+)$, giving a contradiction. To find v, we observe that by the definition of $\mathcal{R}$ there is a walk from A_0 to $\emptyset$; call it $\langle A_0 \;\; p_1 \;\; C_1 \;\; p_2 \;\; C_2 \ldots p_t \;\; C_t\rangle$ where $C_t = \emptyset$. Let v begin with $u_0 u_1 \ldots u_s p_1 p_2 \ldots p_t$. The sequence of labels $u_1 u_2 \ldots u_s p_1 p_2 \ldots p_t$ determines a walk starting at A_0 that by construction ends at $\emptyset$. By Lemma 2.1 there exists no $x \in A_0\Sigma_+$ such that $F(x)$ begins with $u_0 u_1 \ldots u_s p_1 p_2 \ldots p_t$. Hence $v \notin F(A_0\Sigma_+)$. ∎

4. Relationships between openness and surjectivity

In this section we see that if F is open then it is surjective; but the converse is false since we will exhibit an example that is surjective but not open.

Proposition 4.1. *If F is open, then F is surjective.*

Proof. Suppose that F has order m. Let $U = x_0 x_1 \ldots x_m \Sigma_+$ for some sequence $x_0 \ldots x_m$. Since U is open, it follows that $F(U)$ is open and hence contains a basic open set $y_0 y_1 \ldots y_s \Sigma_+$. Suppose z is in Σ_+. Then $y_0 y_1 \ldots y_s z$ lies in $F(U)$ and hence has form $F(x_0 x_1 \ldots x_m x_{m+1} \ldots)$. Since F commutes with the shift map, it follows that $F(x_{s+1} x_{s+2} \ldots) = z$, proving surjectivity since z was arbitrary. ∎

Example. Suppose the alphabet $S = \{0, 1\}$ and the local rule $f : S^3 \to S$ is given by

$$\begin{array}{ll} f(000) = 1 & f(100) = 0 \\ f(001) = 0 & f(101) = 1 \\ f(010) = 0 & f(110) = 1 \\ f(011) = 0 & f(111) = 1 \end{array}$$

Then $B_0 = \{001, 010, 011, 100\}$ and $B_1 = \{000, 101, 110, 111\}$. It is easy to see that in $\mathcal{S}$ the two arcs from B_0 go to B_0 and B_1; and similarly the arcs from B_1 go to B_0 and B_1. Hence there is no walk from B_0 to $\emptyset$ and, by Theorem 2.2, F is surjective.

On the other hand there is a cycle $\langle\{001\}\ 0\ \{010, 011\}\ 0\ \{100\}\ 1\ \{000\}\ 0\ \{001\}\rangle$, and all these vertices lie in $\mathfrak{R}$ since there is a walk $\langle\{001\}\ 1\ \emptyset\rangle$. Hence, by Theorem 3.1, F is not open.

5. Injectivity

In order to decide questions of injectivity we introduce a different labeled digraph $\mathfrak{I}$. If the local rule f for F has order m, the vertices of $\mathfrak{I}$ are of two types: either *singleton* sets $\{x_0x_1 \dots x_m\}$ consisting of a single element, or *doubleton* sets $\{x_0x_1 \dots x_m, y_0y_1 \dots y_m\}$ with two distinct elements such that $f(x_0x_1 \dots x_m) = f(y_0y_1 \dots y_m)$. There are no other vertices of $\mathfrak{I}$. Given a singleton set $\{x_0x_1 \dots x_m\}$ for any $x_{m+1} \in S$ there will be an arc from $\{x_0x_1 \dots x_m\}$ to $\{x_1 \dots x_mx_{m+1}\}$ labeled by $f(x_1 \dots x_mx_{m+1})$. If there are two distinct symbols x_{m+1} and y_{m+1} such that $f(x_1 \dots x_mx_{m+1}) = f(x_1 \dots x_my_{m+1})$, then there will also be an arc from $\{x_0x_1 \dots x_m\}$ to $\{x_1 \dots x_mx_{m+1}, x_1 \dots x_my_{m+1}\}$ labeled by $f(x_1 \dots x_mx_{m+1})$. There are no other arcs outgoing from a singleton vertex.

Suppose $\{x_0x_1 \dots x_m, y_0y_1 \dots y_m\}$ is a doubleton vertex. If there exist x_{m+1} and y_{m+1} in S such that $f(x_1 \dots x_mx_{m+1}) = f(y_1 \dots y_my_{m+1})$, then there is an arc from $\{x_0x_1 \dots x_m, y_0y_1 \dots y_m\}$ to $\{x_1 \dots x_mx_{m+1}, y_1 \dots y_my_{m+1}\}$ labeled by $f(x_1 \dots x_mx_{m+1})$, whether the latter set is a singleton or a doubleton vertex. There are no other arcs in $\mathfrak{I}$.

Note that there may be several arcs with the same label outgoing from a given vertex, or there may be no arcs with a given label outgoing from a vertex. It is possible that some doubleton vertices have no outgoing arcs at all, but every singleton has at least N outgoing arcs.

We next define a subset $\mathfrak{T}$ of the set of vertices of $\mathfrak{I}$, called the *terminating vertices*. $\mathfrak{T}$ is the smallest set of vertices A of $\mathfrak{I}$ satisfying the following properties:

1. If a vertex A has no outgoing arcs in $\mathfrak{I}$, then $A \in \mathfrak{T}$.
2. If each outgoing arc from a vertex A goes to a vertex in $\mathfrak{T}$, then $A \in \mathfrak{T}$ also.

Given $\mathfrak{I}$ there is clearly a finite recursive procedure to determine $\mathfrak{T}$. Note that $\mathfrak{T}$ consists of all vertices A at which no walks of arbitrarily long length start. More precisely, we have the following.

Lemma 5.1. *Let A be a vertex of $\mathfrak{I}$. Then $A \notin \mathfrak{T}$ if and only if there exists an infinite walk in $\mathfrak{I}$ starting at A.*

Proof. If there is an infinite walk starting at A, then clearly no vertex on the walk fails to have outgoing arcs, so no vertex can lie in $\mathfrak{T}$. Conversely, if $A \notin \mathfrak{T}$ then there is an arc from A to some $A_1 \notin \mathfrak{T}$; then there is an arc from A_1 to some $A_2 \notin \mathfrak{T}$. Continuing in this manner we can construct an infinite walk starting at A. ■

Lemma 5.2. *Suppose that A is a vertex of $\mathfrak{I}$ that is not in $\mathfrak{T}$. Let $y_0 = f(x_0x_1 \ldots x_m)$ for all elements $x_0x_1 \ldots x_m$ of A, and let $y_1, y_2, \ldots$ be the successive labels of an infinite walk in $\mathfrak{I}$ starting at A. Then for each $x_0x_1 \ldots x_m$ in A there exist symbols $x_{m+1}, x_{m+2}, \ldots$ such that $F(x_0x_1 \ldots x_m x_{m+1} \ldots) = y_0y_1y_2 \ldots$.*

Proof. Let the infinite walk be $\langle A \;\; y_1 \;\; A_1 \;\; y_2 \;\; A_2 \ldots \rangle$. For each $x_0x_1 \ldots x_m \in A$ there exists x_{m+1} such that $x_1 \ldots x_m x_{m+1} \in A_1$, whence $f(x_1 \ldots x_m x_{m+1}) = y_1$. But $A_1 \notin \mathfrak{T}$ since there is an infinite walk starting at A_1, so there exists x_{m+2} such that $x_2x_3 \ldots x_{m+2} \in A_2$ and $f(x_2x_3 \ldots x_{m+2}) = y_2$. If we iterate this argument we find the desired symbols. ∎

Theorem 5.3. *F is injective if and only if $\mathfrak{T}$ contains every doubleton vertex of $\mathfrak{I}$.*

Proof. Suppose that A is a doubleton vertex that is not in $\mathfrak{T}$. Then by Lemma 5.2 both the elements of A extend to yield preimages of some $y_0y_1 \ldots \in \Sigma_+$, where $y_1, y_2, \ldots$ are the labels of some infinite walk starting at A and $y_0 = f(x_0x_1 \ldots x_m)$ for all $x_0x_1 \ldots x_m \in A$. Thus F cannot be injective.

Conversely, assume that $\mathfrak{T}$ contains every doubleton vertex of $\mathfrak{I}$. We show that F is injective. Suppose that $F(x) = F(z) = y$ where $x = x_0x_1 \ldots$, $y = y_0y_1 \ldots$, and $z = z_0z_1 \ldots$. If $x \neq z$ then, without loss of generality, by using the shift map we may assume $x_0 \neq z_0$. Then $\{x_0x_1 \ldots x_m, z_0z_1 \ldots z_m\}$ is a doubleton vertex of $\mathfrak{I}$ and there exists an infinite walk in $\mathfrak{I}$ starting at A. By Lemma 5.1 $A \notin \mathfrak{T}$, contradicting that $\mathfrak{T}$ contains every doubleton vertex of $\mathfrak{I}$. ∎

Corollary 5.4. *F is injective if and only if no infinite walk in $\mathfrak{I}$ contains a doubleton vertex.*

6. Local injectivity

Let X and Y be topological spaces. A function $F : X \to Y$ is *locally injective* if each $x \in X$ lies in a neighborhood U such that the restriction $F \mid U$ is injective.

Proposition 6.1. *Suppose that $F : \Sigma_+ \to \Sigma_+$ is a cellular automaton of order m. Then F is locally injective if and only if for each string $x_0x_1 \ldots x_m$ the restriction $F \mid x_0x_1 \ldots x_m\Sigma_+$ is injective.*

Proof. By the definition of local injectivity the "if" portion is immediate since any point of Σ_+ lies in an open set of the form $x_0x_1 \ldots x_m\Sigma_+$. Conversely, suppose that F is locally injective and $x = x_0x_1 \ldots x_m \ldots$. We show that $F \mid x_0x_1 \ldots x_m\Sigma_+$ is injective.

Since F is locally injective, for each $x \in \Sigma_+$ there exists a basic open set $U = x_0x_1 \ldots x_k\Sigma_+$ containing x (with k depending on x) on which F is injective. The set of such sets U forms an open cover of Σ_+, so by compactness there exists a finite subcover. Choosing the largest of the k's that appear in the finite subcover, we obtain k such that for each $x_0 \ldots x_k$ it is true that $F \mid x_0x_1 \ldots x_k\Sigma_+$ is injective. If $k = m$ we are done. If $k < m$ then U contains the smaller open set $x_0x_1 \ldots x_m\Sigma_+$ on which F will also be injective. Thus we reduce to the case where $k > m$.

Suppose that $F \mid x_0x_1 \ldots x_m\Sigma_+$ were not injective and $k > m$. Then there would exist distinct y and z in Σ_+ such that $F(x_0x_1 \ldots x_my) = F(x_0x_1 \ldots x_mz)$. Choose any symbols $x_{-1}, x_{-2}, \ldots, x_{-k+m}$ in S and let $Y = x_{-k+m} \ldots x_{-1}x_0 \ldots x_my$ while $Z = x_{-k+m} \ldots x_{-1}x_0 \ldots x_mz$. Then $F(Y) = F(Z)$ since these agree by hypothesis after the first $k-m$ sites, and they agree on the first $k-m$ sites since Y and Z agree on the first $k+1$ sites and F has order m. This contradicts that $F \mid x_{-k+m} \ldots x_{-1}x_0 \ldots x_m\Sigma_+$ must be injective. ∎

Theorem 6.2. *Let F have order m. Then $F \mid x_0 \ldots x_m\Sigma_+$ is injective if and only if there is no walk in $\mathfrak{I}$ from the singleton vertex $\{x_0 \ldots x_m\}$ to any nonterminating doubleton vertex.*

Corollary 6.3. *F is locally injective if and only if, whenever there exists a walk in $\mathfrak{I}$ from a singleton vertex to a doubleton vertex A, then A is terminating.*

Corollary 6.4. *F is locally injective if and only if no infinite walk in $\mathfrak{I}$ starting from a singleton vertex contains a doubleton vertex.*

It follows that one can decide whether F is locally injective by performing a depth-first search from each singleton vertex to see whether any nonterminating doubleton vertex can be reached.

Proof of Theorem 6.2. For the "only if" implication, assume that $F \mid x_0 \ldots x_m\Sigma_+$ is injective. Let $A_0 = \{x_0 \ldots x_m\}$, let $f(x_0 \ldots x_m) = y_0$, and assume that $\langle A_0 \; y_1 \; A_1 \; y_2 \ldots y_k \; A_k\rangle$ is a walk, where A_k is a nonterminating doubleton vertex. Let the two distinct elements of A_k be $w_k \ldots w_{k+m}$ and $v_k \ldots v_{k+m}$. By the definition of the arcs of $\mathfrak{I}$ there exist $w_0 \ldots w_{k-1}$ and $v_0 \ldots v_{k-1}$ such that $w_j \ldots w_{j+m}$ and $v_j \ldots v_{j+m}$ lie in A_j for $j = 0, \ldots, k$. (To see this, one proceeds inductively. Since there is an arc from A_{k-1} to A_k, one can select v_{k-1} and w_{k-1} such that $v_{k-1} \ldots v_{k+m-1}$ and $w_{k-1} \ldots w_{k+m-1}$ lie in A_{k-1}. Repeating this argument we obtain the remaining w_j and v_j.)

Since A_k is a nonterminating vertex, by Lemma 5.1 there exists an infinite walk starting at A_k; hence by Lemma 5.2 there exist $w_{k+m+1}w_{k+m+2} \ldots$ and $v_{k+m+1}v_{k+m+2} \ldots$ such that $F(w_kw_{k+1} \ldots) = F(v_kv_{k+1} \ldots)$. But then $F(w_0w_1 \ldots) = F(v_0v_1 \ldots)$ by the construction of the w_j and the v_j. Since $w_0 \ldots w_m$ and $v_0 \ldots v_m$ lie in A_0, they both must equal $x_0 \ldots x_m$. Hence we have exhibited two distinct elements of $x_0 \ldots x_m\Sigma_+$ that map to the same element under F. This contradicts the hypothesis.

Conversely, for the "if" implication, suppose that $F \mid x_0 \ldots x_m\Sigma_+$ were not injective, so that there exist distinct $u = u_0u_1 \ldots$ and $v = v_0v_1 \ldots$ in

$x_0 \ldots x_m\Sigma_+$ such that $F(u) = F(v)$. Then these lead to an infinite walk in $\mathcal{J}$ starting at $A_0 = \{x_0x_1 \ldots x_m\}$. To see this, let $A_j = \{u_j \ldots u_{j+m}, v_j \ldots v_{j+m}\}$. Possibly A_j is a singleton. Since $F(u) = F(v)$, it is clear that $f(u_j \ldots u_{j+m}) = f(v_j \ldots v_{j+m})$, and we shall call this common value y_j. Then $\langle A_0 \quad y_1 \quad A_1 \quad y_2 \quad A_2 \ldots \rangle$ is an infinite walk in $\mathcal{J}$ starting at A_0. By Lemma 5.1, A_0 cannot be terminating. This completes the proof. ∎

7. The biinfinite cases

Suppose that $f : S^{m+1} \to S$ is a local map of order m. The results in the preceding sections have all concerned the global map $F : \Sigma_+ \to \Sigma_+$ given by $(Fx)_j = f(x_j, \ldots, x_{j+m})$ for $j \geq 0$. However, if Σ denotes the set of all doubly infinite sequences $\ldots x_{-1}x_0x_1 \ldots$ of symbols from S, then the same formula yields a map $G : \Sigma \to \Sigma$ given by $(Gx)_j = f(x_j, \ldots, x_{j+m})$ for all j, and the question arises whether global properties of these maps G can be inferred from properties of the local maps f. In this section we state the results. Proofs are similar to those of the corresponding theorems earlier in this paper and are omitted.

Let Σ_- denote the set of all left-infinite sequences $\ldots x_{-2}x_{-1}x_0$ of symbols from S. A basis for the topology for Σ consists of all sets $\Sigma_-y_ky_{k+1} \ldots y_{k+r}\Sigma_+$ where the notation indicates all doubly infinite sequences whose kth value is y_k, whose $(k+1)$st value is $y_{k+1}, \ldots$, whose $(k+r)$th value is y_{k+r}, but whose other values are unrestricted in S; and where k and r range over integer values with r nonnegative.

Let $\mathcal{S}$ denote the surjectivity digraph of section 2.

Theorem 7.1. *The map $G : \Sigma \to \Sigma$ is surjective if and only if there exists no walk in $\mathcal{S}$ from B_0 to $\emptyset$.*

From Theorem 2.2 we see that $G : \Sigma \to \Sigma$ is surjective if and only if $F : \Sigma_+ \to \Sigma_+$ is surjective.

Given the local rule f, we define another digraph $\mathcal{S}'$ called the *left surjectivity digraph* as follows. The vertices of $\mathcal{S}'$ are the same as the vertices of $\mathcal{S}$. If B is a vertex of $\mathcal{S}'$ and $y \in S$, then there will be an arc in $\mathcal{S}'$ from B to C labeled by y where $C = \{x_0x_1 \ldots x_m : f(x_0x_1 \ldots x_m) = y$ and there exists $x_{m+1} \in S$ such that $x_1x_2 \ldots x_{m+1} \in B\}$. Thus the definition of $\mathcal{S}'$ is completely analogous to that of $\mathcal{S}$ but refers to extensions of words in the opposite direction. Define a vertex B of $\mathcal{S}'$ to be *left-restricted* if there exists a walk in $\mathcal{S}'$ from B to $\emptyset$, and let $\mathcal{R}'$ denote the labeled subdigraph of $\mathcal{S}'$ generated by all left-restricted vertices.

Theorem 7.2. *The map $G : \Sigma \to \Sigma$ is open if and only if neither $\mathcal{R}$ nor $\mathcal{R}'$ contains a cycle with a vertex other than $\emptyset$.*

Let $\mathcal{J}$ denote the injectivity digraph of f. Analogous to $\mathcal{T}$ we define a subset $\mathcal{T}'$ of the set of vertices of $\mathcal{J}$, called the *left-terminating* vertices. $\mathcal{T}'$ is the smallest set of vertices A of $\mathcal{J}$ satisfying the following properties:

1. If a vertex A has no incoming arcs in $\mathcal{I}$, then $A \in \mathcal{T}'$.
2. If each incoming arc from a vertex A comes from a vertex of $\mathcal{T}'$, then $A \in \mathcal{T}'$ also.

Theorem 7.3. *The map $G : \Sigma \to \Sigma$ is injective if and only if each doubleton vertex of $\mathcal{I}$ lies in either $\mathcal{T}$ or $\mathcal{T}'$ or both.*

A *doubly infinite walk* in a digraph is a walk whose vertices are indexed by all integers, not just the nonnegative integers.

Corollary 7.4. *G is injective if and only if no doubleton vertex of $\mathcal{I}$ lies on any doubly infinite walk.*

Theorem 7.5. *The map $G : \Sigma \to \Sigma$ is locally injective if and only if, for each string $x_0x_1 \ldots x_m$, the restriction $G \mid \Sigma_- x_0x_1 \ldots x_m \Sigma_+$ is injective.*

Theorem 7.6. *The map $G \mid \Sigma_- x_0x_1 \ldots x_m \Sigma_+$ is injective if and only if both (1) there is no walk in $\mathcal{I}$ from the singleton vertex $\{x_0x_1 \ldots x_m\}$ to any doubleton vertex not in $\mathcal{T}$, and (2) there is no walk in $\mathcal{I}$ to $\{x_0x_1 \ldots x_m\}$ from any doubleton vertex not in $\mathcal{T}'$.*

Corollary 7.7. *G is locally injective if and only if no doubly infinite walk in $\mathcal{I}$ that contains a singleton vertex also contains a doubleton vertex.*

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