# Sequences Generated by Neuronal Automata with Memory 

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#### Abstract

We study the sequences generated by neuronal recurrence equations of the form $x_{n}=1\left[\sum_{1 \leq i \leq k} a_{i} x_{n-i}-\theta\right]$, where $k$ is called the memory length. We show that, if there is a neuronal recurrence equation with memory length $k$ that generates sequences of periods $p_{1}, \ldots, p_{r}$, then there is a neuronal recurrence equation with memory length $k r$ that generates a sequence of period $l c m\left(p_{1}, \ldots, p_{r}\right) r$, where lcm denotes the least common multiple. As an application we show that, for any integer $k_{0}$, there is a neuronal equation with memory length $3 k, k \geq k_{0}$, that generates a sequence of period $O\left(k^{3}\right)$.


## 1. Introduction

Caianiello and De Luca [1] have suggested modeling the dynamic behavior of a single neuron with memory that does not interact with other neurons, using the recurrence equation

$$
\begin{equation*}
x_{n}=1\left[\sum_{1 \leq i \leq k} a_{i} x_{n-i}-\theta\right] \tag{1.1}
\end{equation*}
$$

where

- $x_{n}$ is a boolean variable representing the state of the neuron at time $t=n$;
- $k$ is the memory length, namely the state at time $t=n$, and depends on the states assumed by the neuron at the $k$ previous steps $t=$ $n-1, \ldots, n-k$;

[^0]- $a_{i}(i=1, \ldots, k)$ are real numbers called the weighting coefficients;
- $\theta$ is a real number called the threshold;
- $1[v]=0$ if $v<0$, and $1[v]=1$ if $v \geq 0$.

Such a recurrence sequence is entirely characterized by the memory length $k$, the threshold $\theta$, the weighting coefficients $a_{i}(i=1, \ldots, k)$, and the initial values $x_{0}, x_{1}, \ldots, x_{k-1}$. Hereafter we denote $x_{n}=f_{a, \theta, k}\left(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}\right)$.

The system obtained by interconnecting several neurons is called a neural network. Such networks were introduced by McCulloch and Pitts [7], and are quite powerful. Indeed, it can be shown that they can be used to simulate any Turing machine. More recently, neural networks have been studied extensively as tools for solving various problems such as classification, speech recognition, and image and signal processing [5]. In this paper, we are interested in the study of sequences generated by single neuronal equations.

Let $p$ and $T$ be two positive integers such that $p>0$ and $T \geq 0$. The sequence (1.1) is said to be of period $p$ and transient $T$ if and only if $x_{n+p}=x_{n}$ for $n \geq T$, and $x_{n+q} \neq x_{n}$ for $n<T$ or $q<p$.

It is generally admitted that the period and transient of sequences generated by a neuron are good measures of the complexity of the behavior of the neuron. In this interpretation a neuron with simple behavior corresponds to small periods and transients. The trivial case is a neuron that never changes its state, that is, $p=1$ and $T=0$. On the other hand, a neuron that generates sequences with very long periods and transients is supposed to be very complex.

In this paper we are interested in the longest period $L P(k)$ that can be generated by a neuronal equation with memory length $k$. In [3] it was conjectured that $L P(k) \leq 2 k$. This conjecture has been disproved in [8], where neuronal recurrence sequences with period $2 k+6$ have been exhibited. We show here that, for arbitrarily large values of $k$, there are neuronal recurrence equations that generate sequences of period $3(k-1)(k-2)(k-4)$.

## 2. The shuffle construction

For any function $f$ from $\{0,1\}^{k}$ to $\{0,1\}$, and for any positive integer $r$, let us define the shuffle image $s^{r}[f]$ of $f$ as follows:

$$
s^{r}[f]\left(x_{1}, x_{2}, \ldots, x_{r k}\right)=f\left(x_{1}, x_{r+1}, \ldots, x_{(k-1) r+1}\right)
$$

With this definition, $s^{r}[f]$ is a function from $\{0,1\}^{r k}$ to $\{0,1\}$. A family $\Phi$ of functions is said to be stable with respect to the shuffle operation if $s^{r}[f] \in \Phi$ for all $f \in \Phi$. Many families of functions are stable. Examples of such stable families are polynomials and threshold functions.

Let us assume that the threshold function defined by $x^{\prime}=f_{a, \theta, k}\left(x_{n-k}\right.$, $\left.x_{n-k+1}, \ldots, x_{n-1}\right)$ generates $r$ sequences $\left(x_{m}^{(j)} ; m \geq 0\right)$ of periods $p_{j}$,
$j=1, \ldots, r$. Let us consider the sequence

$$
\begin{gathered}
\left(y_{n} ; n \geq 0\right)=\left(x_{0}^{(1)}, x_{0}^{(2)}, \ldots, x_{0}^{(r)}, x_{1}^{(1)}, x_{1}^{(2)}, \ldots, x_{1}^{(r)}, \ldots,\right. \\
\\
\left.x_{m}^{(1)}, x_{m}^{(2)}, \ldots, x_{m}^{(r)}, \ldots\right)
\end{gathered}
$$

obtained by shuffling the $r$ sequences $\left(x_{m}^{(j)} ; m \geq 0\right)$. Clearly, the sequence $\left(y_{n} ; n \geq 0\right)$ is of period $\operatorname{lcm}\left(p_{1}, \ldots, p_{r}\right) r$. Moreover, for $n=(m-1) r+j-1$ and $m \geq 1$,

$$
\begin{aligned}
y_{n} & =x_{m}^{(j)} \\
& =f_{a, \theta, k}\left(x_{m-k}^{(j)}, x_{m-k+1}^{(j)}, \ldots, x_{m-1}^{(j)}\right) \\
& =f_{a, \theta, k}\left(y_{n-r k}, \ldots, y_{n-2 r}, y_{n-r}\right) \\
& =f_{a^{\prime}, \theta, k^{\prime}}\left(y_{n-k^{\prime}}, y_{n-k^{\prime}+1}, \ldots, y_{n-k}, y_{n-k+1}, y_{n-k+2}, \ldots, y_{n-1}\right)
\end{aligned}
$$

where $k^{\prime}=r k$ and $a^{\prime}=\left(0, \ldots, 0, a_{1}, 0, \ldots, 0, a_{2}, \ldots, 0, \ldots, 0, a_{k}\right)$. This shows that, if there is a neuronal recurrence equation with memory length $k$ that generates sequences of periods $p_{1}, \ldots, p_{r}$, then there is a neuronal recurrence equation with memory length $k r$ that generates a sequence of period $l c m\left(p_{1}, \ldots, p_{\tau}\right) r$, where $l c m$ denotes the least common multiple.

The shuffle construction is very instructive. Indeed, let us consider a family of neurons, characterized by a family of functions, that is stable with respect to the shuffle operation. The property above says that the asymptotic complexity of this family of neurons can be measured by studying either the longest period generated by a neuron of the family, or the lcm of the periods of the sequences generated by a neuron of the family.

Example. Let us consider a neuronal recurrence equation that generates the periodic sequence ( $a, a^{\prime}, a^{\prime \prime}, a, a^{\prime}, a^{\prime \prime}, \ldots$ ) of period 3 , and the periodic sequence ( $b, b^{\prime}, b, b^{\prime}, \ldots$ ) of period 2 . The sequence obtained by shuffling these two sequences is the sequence ( $\left.a, b, a^{\prime}, b^{\prime}, a^{\prime \prime}, b, a, b^{\prime}, a^{\prime}, b, a^{\prime \prime}, b^{\prime}, \ldots\right)$, which is of period $12=2 \mathrm{lcm}(2,3)$. Moreover, if the two starting sequences are generated by a recurrence equation with memory length $k$, then the shuffle sequence is generated by an equation of memory length $2 k$.

## 3. A family of neuronal equations with long periods

In this section we show that, for any integer $k_{0}$, it is possible to construct three neuronal equations with memory length $k \geq k_{0}$ that exhibit recurrence sequences with periods $k-1, k-2$, and $k-4$. We then combine these three neuronal equations into a single neuronal equation with memory length $3 k$, which generates a sequence of period $3(k-1)(k-2)(k-4)$.

Lemma 1. Consider a neuronal recurrence equation with memory length $k=2 s+1$ that satisfies the six following conditions:

| time $t$ | $x_{0}, x_{1}, \ldots, x_{n-1}$ | $\sum_{1 \leq i \leq k} a_{i} x_{n-i}$ | condition |
| :---: | :---: | :---: | :---: |
| $t=k-1$ | $10 \ldots 0010 \ldots 001$ | $a_{1}+a_{s}+a_{k}$ | (a) |
| $t=k$ | $10 \ldots 0010 \ldots 0010$ | $a_{2}+a_{s+1}$ | (b) |
| $\cdots$ |  | $a_{i}+a_{i+s-1}$ | (b) |
| $t=k+s-2$ | $10 \ldots 0010 \ldots 0010 \ldots 0$ | $a_{s}+a_{k-2}$ | (b) |
| $t=k+s-1$ | $10 \ldots 0010 \ldots 0010 \ldots 00$ | $a_{s+1}+a_{k-1}$ | (c) |
| $t=k+s$ | $10 \ldots 0010 \ldots 0010 \ldots 001$ | $a_{1}+a_{s+2}+a_{k}$ | (d) |
| $t=k+s+1$ | $10 \ldots 0010 \ldots 0010 \ldots 0010$ | $a_{2}+a_{s+3}$ | (e) |
| $\cdots$ |  | $a_{i}+a_{i+s+1}$ | (e) |
| $t=2 k-4$ | $10 \ldots 0010 \ldots 0010 \ldots 0010 \ldots 0$ | $a_{s-2}+a_{k-2}$ | (e) |
| $t=2 k-3$ | $10 \ldots 0010 \ldots 0010 \ldots 0010 \ldots 00$ | $a_{s-1}+a_{k-1}$ | (f) |
| $t=2 k-2$ | $10 \ldots 0010 \ldots 0010 \ldots 0010 \ldots 001$. | $a_{1}+a_{s}+a_{k}$ | (a) |

Table 1: A sequence of period $k-1$.
(a) $a_{1}+a_{s}+a_{k}<\theta$
(b) $a_{i}+a_{i+s-1}<\theta$ for $i=2, \ldots, s$
(c) $a_{s+1}+a_{k-1} \geq \theta$
(d) $a_{1}+a_{s+2}+a_{k}<\theta$
(e) $a_{i}+a_{i+s+1}<\theta$ for $i=2, \ldots, s-2$
(f) $a_{s-1}+a_{k-1} \geq \theta$

The corresponding neuronal equation generates a sequence of period $k-1$.
Proof. A sequence of period $k-1$ is exhibited in table 1. In this table,

- the first column indicates the time step $t=n-1$;
- the second column shows the states $x_{0}, \ldots, x_{n-1}$ assumed at times $t=$ $0, \ldots, n-1$;
- the third column gives the sum $\sum_{1 \leq i \leq k} a_{i} x_{n-i}$;
- the fourth column shows the condition that is used to derive $x_{n}$.
- the variables $x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}$ are underlined.

For instance, at time $t=k-1, \sum_{1 \leq i \leq k} a_{i} x_{k-i}=a_{1}+a_{s}+a_{k}$. Thus, from condition (a), $x_{k}=0$.

It is important to note that the conditions of Lemma 1 characterize all the neuronal equations with memory length $k$ such that the configuration of the first row of table 1 generates a periodic sequence of period $k-1$.

Lemma 2. Consider a neuronal recurrence equation with memory length $k$ that satisfies the eight following conditions:
(a) $a_{2}+a_{6}+a_{k}<\theta$
(b) $a_{i}+a_{i+4}<\theta$ for $i=3, \ldots, k-7$

| time $t$ | $x_{0}, x_{1}, \ldots, x_{n-1}$ | $\sum_{1 \leq i \leq k} a_{i} x_{n-i}$ | condition |
| :---: | :---: | :---: | :---: |
| $t=k-1$ | $10 \ldots 000100010$ | $a_{2}+a_{6}+a_{k}$ | (a) |
| $t=k$ | $10 \ldots 0001000100$ | $a_{3}+a_{7}$ | (b) |
|  |  | $a_{i}+a_{i+4}$ | (b) |
| $t=2 k-10$ | $10 \ldots 000100010 \ldots 0$ | $a_{k-7}+a_{k-3}$ | (b) |
| $t=2 k-9$ | $10 \ldots 000100010 \ldots 00$ | $a_{k-6}+a_{k-2}$ | (c) |
| $t=2 k-8$ | $10 \ldots 00 \underline{0} 00010 \ldots 001$ | $a_{1}+a_{k-5}+a_{k-1}$ | (d) |
| $t=2 k-7$ | $10 \ldots 000100010 \ldots 0010$ | $a_{2}+a_{k-4}+a_{k}$ | (e) |
| $t=2 k-6$ | $10 \ldots 000100010 \ldots 00100$ | $a_{3}+a_{k-3}$ | (f) |
| $t=2 k-5$ | $10 \ldots 000100010 \ldots 001000$ | $a_{4}+a_{k-2}$ | (g) |
| $t=2 k-4$ | $10 \ldots 000100010 \ldots 0010001$ | $a_{1}+a_{5}+a_{k-1}$ | (h) |
| $t=2 k-3$ | $10 \ldots 000100010 \ldots 00100010$ | $a_{2}+a_{6}+a_{k}$ | (a) |

Table 2: A sequence of period $k-2$.
(c) $a_{k-6}+a_{k-2} \geq \theta$
(d) $a_{1}+a_{k-5}+a_{k-1}<\theta$
(e) $a_{2}+a_{k-4}+a_{k}<\theta$
(f) $a_{3}+a_{k-3}<\theta$
(g) $a_{4}+a_{k-2} \geq \theta$
(h) $a_{1}+a_{5}+a_{k-1}<\theta$

The corresponding neuronal equation generates a sequence of period $k-2$.
Proof. A sequence of period $k-2$ is exhibited in table 2.
As for the proof of Lemma 1, the eight conditions of Lemma 2 characterize all the neuronal equations with memory length $k$ such that the first row of table 2 generates a periodic sequence of period $k-2$.

Lemma 3. Consider a neuronal recurrence equation with memory length $k$ that satisfies the three following conditions:
(a) $a_{i}+a_{i+k-4}<\theta$ for $i=1, \ldots, 4$
(b) $a_{i}<\theta$ for $i=5, \ldots, k-5$
(c) $a_{k-4} \geq \theta$

The corresponding neuronal equation generates a sequence of period $k-4$.
Proof. A sequence of period $k-4$ is exhibited in table 3 .
As for the proofs of Lemmas 1 and 2, the three conditions of Lemma 3 characterize all the neuronal equations with memory length $k$ such that the configuration of the first row of table 3 generates a periodic sequence of period $k-4$.

We are now going to show that, for any integer $k_{0}$, there is an integer $k \geq k_{0}$ such that $\operatorname{lcm}(k-1, k-2, k-4)=(k-1)(k-2)(k-4)$.

| time $t$ | $x_{0}, x_{1}, \ldots, x_{n-1}$ | $\sum_{1 \leq i \leq k} a_{i} x_{n-i}$ | condition |
| :---: | :---: | :---: | :---: |
| $t=k-1$ | $00010 \ldots 000001$ | $a_{1}+a_{k-3}$ | (a) |
| $t=k$ | $00010 \ldots 0000010$ | $a_{2}+a_{k-2}$ | (a) |
| $t=k+1$ | $00010 \ldots 00000100$ | $a_{3}+a_{k-1}$ | (a) |
| $t=k+2$ | $00010 \ldots 000001000$ | $a_{4}+a_{k}$ | (a) |
| $t=k+3$ | $00010 \ldots 0000010000$ | $a_{5}$ | (b) |
| $\cdots$ | $\cdots$ | $a_{i}$ | (b) |
| $t=2 k-7$ | $00010 \ldots 0000010000 \ldots 0$ | $a_{k-5}$ | (b) |
| $t=2 k-6$ | $00010 \ldots 0000010000 \ldots 00$ | $a_{k-4}$ | (c) |
| $t=2 k-5$ | $00010 \ldots 00 \underline{00010000 \ldots 001}$ | $a_{1}+a_{k-3}$ | (a) |

Table 3: A sequence of period $k-4$.

Lemma 4. If $k-2 \neq 2(\bmod 3)$ and $(k-2) \neq 0(\bmod 2)$, then $k-1, k-2$, and $k-4$ are relatively prime, hence $\operatorname{lcm}(k-1, k-2, k-4)=(k-1)(k-2)(k-4)$.

Proof. If an integer $q$ divides $k-1$ and $k-2$, then it divides the difference $(k-1)-(k-2)=1$, and hence $q=1$. If an integer $q$ divides $k-1$ and $k-4$, then it divides the difference $(k-1)-(k-4)=3$, and hence $q=1$ or $q=3$. If $q=3$, then $k-1=0(\bmod 3)$, hence $k-2=2(\bmod 3)$, and this contradicts the hypothesis. This shows that $k-1$ and $k-4$ are relatively prime. If an integer $q$ divides $k-2$ and $k-4$, then it divides the difference $(k-2)-(k-4)=2$, and hence $q=1$ or $q=2$. Since from the hypothesis 2 does not divide $k-2$, it follows that $q=1$. This shows that $k-1, k-2$, and $k-4$ are relatively prime, hence $l c m(k-1, k-2, k-4)=(k-1)(k-2)(k-4)$.

We are now ready to establish the main result of this paper.
Proposition 1. For any integer $k_{0}$, there is a neuronal recurrence equation with memory length $3 k\left(k \geq k_{0}\right)$ that generates a sequence of period $3(k-1)(k-2)(k-4)$.

Proof. From the preceding lemma, and from the shuffle construction, we just need to exhibit a weighting vector $a=\left(a_{1}, \ldots, a_{k}\right)$ and a threshold $\theta$ such that all the conditions of Lemmas 1,2 , and 3 are satisfied. Let us consider the threshold function defined as follows:

- $\theta=k$;
- $a_{1}=a_{3}=a_{k-8}=a_{s-2}=a_{s-4}=-1$;
- $a_{2}=-2 ; a_{4}=a_{k-6}=4 ;$
- $a_{k-4}=k ; a_{k-2}=(k-4) ; a_{k-1}=(k-1)$.
- $a_{i}=1$ otherwise.

Hereafter we assume that $k$ is sufficiently large to ensure that all the indices that appear in the above definition are different. Let us first note that all the conditions of Lemmas 1, 2, and 3 involve at most three weighting coefficients.

We assume that $k \geq 9$. As a consequence, from the definition of the weighting coefficients, all sums of weights that do not involve $a_{k-4}, a_{k-2}$, or $a_{k-1}$ are less than or equal to $k$. This property is verified by all the conditions of Lemmas 1,2 , and 3. Now, we just need to check the conditions that involve $a_{k-4}, a_{k-2}$, or $a_{k-1}$. The corresponding sums are the following:

In Lemma 1: $k=2 s+1$ (we assume that $s<k-4$ ).
(b) $a_{s-2}+a_{k-4}=k-1$
$a_{s}+a_{k-2}=k-3$
(c) $a_{s+1}+a_{k-1}=k$
(e) $a_{s-4}+a_{k-4}=k-1$
$a_{s-2}+a_{k-2}=k-5$
(f) $a_{s-1}+a_{k-1}=k$

In Lemma 2 :
(b) $a_{k-8}+a_{k-4}=k-1$
(c) $a_{k-6}+a_{k-2}=k$
(d) $a_{1}+a_{k-5}+a_{k-1}=k-1$
(e) $a_{2}+a_{k-4}+a_{k}=k-1$
(f) $a_{4}+a_{k-2}=k$
(g) $a_{1}+a_{5}+a_{k-1}=k-1$

In Lemma 3:
(a) $a_{2}+a_{k-2}=k-6$
(b) $a_{3}+a_{k-1}=k-2$
(c) $a_{k-4}=k$

This ends the proof of the proposition.

## 4. Conclusion

We have introduced here a shuffle construction that establishes an equivalence between neuronal recurrence equations that generate sequences with large periods and neuronal recurrence equations that generate sequences whose periods have a large least common multiple. This approach has enabled us to exhibit a neuronal recurrence equation with memory length $3 k$ and period $3(k-1)(k-2)(k-4)$.

Note that structural constructions are very general and powerful tools for the study of the sequences generated by recurrence equations. In [9] another structural construction was introduced to establish strong relationships between the sequences generated by parallel and sequential iterations. Other results on neuronal recurrence sequences may be found in $[2,3,4,6,8]$.

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