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Cellular Automata as Algebraic Systems*

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Abstract. Infinite cellular automata have been studied mostly using empirical and statistical techniques, with some combinatorial analysis. Here we show how concepts of universal algebra such as subdirect decomposition and chains of varieties can be applied to their study. Cellular automata with ultimately periodic behavior are shown to correspond to varieties of groupoids. Relationships between these varieties are analyzed.

Introduction

A one-dimensional cellular automaton (CA) is determined by a quadruple $\mathcal{A} = (S, l, r, \sigma)$, where S is finite set (of states), l and r are natural numbers, and σ is a mapping of S^{l+r+1} into S, sometimes required to satisfy $\sigma(0, 0, \ldots, 0) = 0$ for some state 0 in S (a quiescent state). The automaton has a doubly infinite one-dimensional array of cells, named by the integers $\mathbb{Z}$, which are each initially (at time $t = 0$) in one of the states in S. The states of cells change in discrete time steps, $t = 1, 2, \ldots$. A configuration (or global state) of the automaton is an assignment of a state to each cell. If $\mathbf{s} : \mathbb{Z} \to S$ is a configuration, then $\mathbf{s}(i)$ naturally denotes the state of cell i. The state of cell n at time t of the automaton $\mathcal{A}$ with initial configuration $\mathbf{s}$ is denoted $\mathcal{A}_{t,n}(\mathbf{s})$. The state of a cell at time $t + 1$ is determined by the states of the cell, its l left neighbors, and its r right neighbors at time t according to σ:

$$\mathcal{A}_{t+1,n}(\mathbf{s}) = \sigma(\mathcal{A}_{t,n-l}(\mathbf{s}), \mathcal{A}_{t,n-l+1}(\mathbf{s}), \ldots, \mathcal{A}_{t,n+r}(\mathbf{s})).$$

The global state or configuration of $\mathcal{A}$ at time t is denoted $\mathcal{A}_t(\mathbf{s})$. The evolutionary behavior of the automaton is then observable from the sequence of global states $\mathcal{A}_0(\mathbf{s}) = \mathbf{s}, \mathcal{A}_1(\mathbf{s}), \mathcal{A}_2(\mathbf{s}), \ldots$, which can be displayed one underneath the other, forming an infinite two-dimensional array. In this way, patterns in the evolution become readily visible. Numerous computer

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simulations have evidenced that the evolution of CAs can be very complex even for small S, l, and r. Some CAs will evolve to constant or periodic patterns for all or almost all initial configurations, others will give fractal evolutionary diagrams, some give seemingly random patterns, and yet others appear to exhibit some locally organized behavior. Many empirical studies have been made of these behaviors [9], and there has been some combinatorial analysis (e.g., [6]), mainly for totalistic automata or when the automaton has a finite (circular) array of cells. An early algebraic and topological approach to the relationship between the local CA rule σ and the global behavior it induces on the configuration space $S^{\mathbb{Z}}$ is [4], but it does not explain the variety of behaviors observed. The current paper investigates how some of the basic concepts of universal algebra can be applied to determining the evolutionary behavior of cellular automata. General references for universal algebra are [7] and [3]. For the reader's convenience, the definitions and results from universal algebra relevant to this paper are summarized below.

The basic algebraic operations on groupoids—products, quotients, and subalgebras—will be shown to have direct visual interpretations for the evolution of the CA they define. Then cellular automata with eventually periodic behavior are proved to correspond to subvarieties of groupoids, and various relationships between these varieties are explored.

Cellular automata and groupoids

It has been shown (see [8] or, more directly, [1]) that $l = 1, r = 0$ is sufficient to simulate all other CA in the sense that if $\mathcal{A} = (S, m, n, \sigma)$ is any CA, then there is a CA $\mathcal{B} = (T, 1, 0, \tau)$, with $T \supseteq S$, and constants c and k such that for all $\mathbf{s} \in S^{\mathbb{Z}}$, $\mathcal{B}_{kt,n+ct}(\mathbf{s}) = \mathcal{A}_{t,n}(\mathbf{s})$. That is, from evolution diagrams for $\mathcal{B}$ one can read off the corresponding evolution diagrams for $\mathcal{A}$. But if $l = 1, r = 0$ then $\sigma : S^2 \rightarrow S$ is just a binary operation on S. A set with a single binary operation on it that is not required to satisfy any particular extra conditions is known in algebra as a *groupoid.* Groups and semigroups, with which the reader may be more familiar, are examples of special groupoids that satisfy extra conditions. The rest of this paper is therefore devoted to relating the properties of CAs to algebraic properties of groupoids. For example, we may ask whether there are special properties of the CA corresponding to groups, or what conditions the groupoid must satisfy to give specified evolutionary behavior of the corresponding CA. The condition that CAs have a quiescent state $0 \in S$ is natural for many purposes, and we will suppose henceforth that our CAs obey it. Thus we will actually study "zero-pointed groupoids," that is, groupoids with an element 0 such that $0 \cdot 0 = 0$, although many of the results will be valid for arbitrary groupoids.

The operation of the groupoid A defining a cellular automaton $\mathcal{A}$ will be denoted by multiplication, or just juxtaposition. If $\mathbf{s}$ is an initial configuration of $\mathcal{A}$ then

$$\mathcal{A}_{1,i}(\mathbf{s}) = \mathbf{s}(i-1)\mathbf{s}(i) \quad \text{for all } i \in \mathbb{Z} \tag{1}$$

and, in general,

$$\mathcal{A}_{t+1,i}(\mathbf{s}) = \mathcal{A}_{t,i-1}(\mathbf{s})\mathcal{A}_{t,i}(\mathbf{s}) \quad \text{for all } t \in \mathbb{N} \text{ and all } i \in \mathbb{Z}. \tag{2}$$

Another way to express these relationships is in terms of a (right) shift operator, Ξ, and a global version of the groupoid operation. If $\mathbf{s}$ is a configuration, then the configuration $\Xi(\mathbf{s})$ is defined by $(\Xi(\mathbf{s}))(i) = \mathbf{s}(i-1)$. If $\mathbf{s}$ and $\mathbf{t}$ are configurations, then $\mathbf{s} \cdot \mathbf{t}$ is the configuration defined by $(\mathbf{s} \cdot \mathbf{t})(i) = \mathbf{s}(i) \cdot \mathbf{t}(i)$. We note that shift commutes with this globalized multiplication (see [4]), that is,

$$\Xi(\mathbf{s} \cdot \mathbf{t}) = \Xi(\mathbf{s}) \cdot \Xi(\mathbf{t}). \tag{3}$$

In this notation, if $\mathbf{s}$ is the current configuration of a CA then the configuration at the next time instant is $\mathbf{s} \cdot \Xi(\mathbf{s})$, or

$$\mathcal{A}_{t+1}(\mathbf{s}) = \mathcal{A}_t(\mathbf{s}) \cdot \Xi(\mathcal{A}_t(\mathbf{s})). \tag{4}$$

There are several algebraic operations that can be applied to groupoids and therefore, as we will see in the next section, to cellular automata. The most fundamental are the direct product, quotients (homomorphic images), and taking subalgebras. These operations are defined for algebras in general and groupoids in particular in a similar way as for groups or rings. If $(A, \cdot_A)$ and $(B, \cdot_B)$ are groupoids, their direct product $(A \times B, \cdot)$ is the groupoid on the cartesian product $A \times B$ with the groupoid operation defined by $(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot_A a_2, b_1 \cdot_B b_2)$. A mapping $\theta : A \to B$ is a groupoid homomorphism if for all $a_1, a_2 \in A$, $\theta(a_1 \cdot_A a_2) = \theta(a_1) \cdot_B \theta(a_2)$. Such a θ is an isomorphism if it is onto and one-one. A homomorphism from A into itself is called an endomorphism. A groupoid B is a quotient of a groupoid A if there is a homomorphism from A onto B. B is isomorphic to A if there is an isomorphism between them. Refer to [7] or [3] for more information.

Algebraic operations on cellular automata

The following definitions can be made for any l, r, but we will only be interested in $l = 1, r = 0$, so we assume these values henceforth and omit them from the notation.

A cellular automaton $\mathcal{B} = (B, \tau)$ is a *subautomaton* of a CA $\mathcal{A} = (A, \sigma)$ if and only if $B \subseteq A$ and for all $\mathbf{b} \in B^{\mathbb{Z}}$ and for all $t \in \mathbb{N}$ and $n \in \mathbb{Z}$, $\mathcal{A}_{t,n}(\mathbf{b}) = \mathcal{B}_{t,n}(\mathbf{b})$. That is, the evolution of $\mathcal{A}$ is identical with that of $\mathcal{B}$ for all initial configurations containing only states in B.

A cellular automaton $\mathcal{B} = (B, \tau)$ is a *quotient automaton* of a CA $\mathcal{A} = (A, \sigma)$ if and only if there is a mapping θ of A onto B such that for all $t \in \mathbb{N}$, $n \in \mathbb{Z}$, and $\mathbf{s} \in A^{\mathbb{Z}}$,

$$\theta(\mathcal{A}_{t,n}(\mathbf{s})) = \mathcal{B}_{t,n}(\theta(\mathbf{s}))$$

where $\theta(\mathbf{s})$ denotes the vector obtained from $\mathbf{s}$ by applying θ to each component.

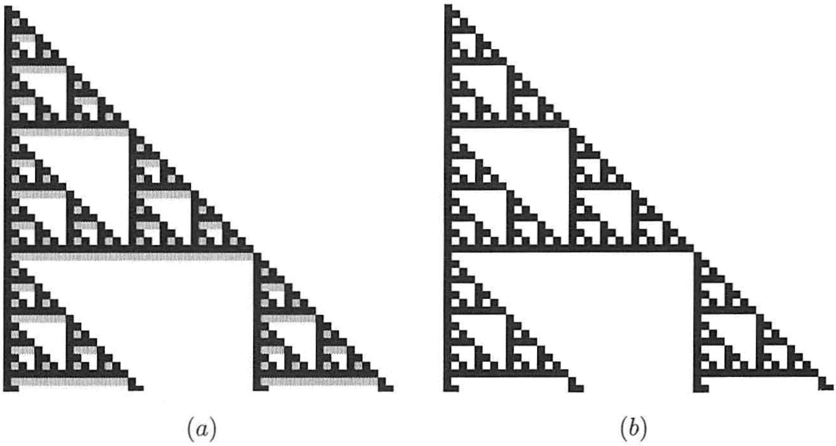


Figure 1: (a) Evolution of a three-state CA from an initial configuration of one cell in state 1 for 50 generations. (b) Evolution of a two-state quotient CA of the automaton in (a) under the mapping grey $\mapsto$ white, black $\mapsto$ black, white $\mapsto$ white. The rule for this CA is $\sigma(x, y) = x + y \pmod 2$.

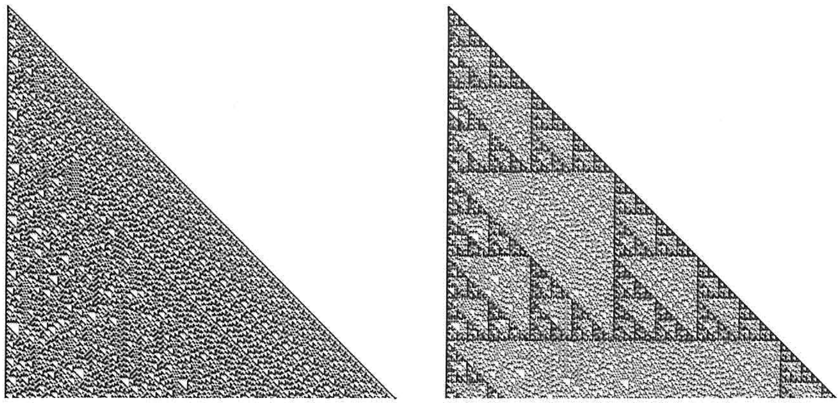
This means that the evolution diagram for $\mathcal{B}(\mathbf{s})$ can be obtained from the one for $\mathcal{A}(\mathbf{s})$ just by replacing every state s by $\theta(s)$, as illustrated in figure 1. If the mapping θ is also one-one, then $\mathcal{B}$ is *isomorphic* to $\mathcal{A}$. A cellular automaton $\mathcal{A} = (A, \sigma)$ is a *product automaton* of the cellular automata $\mathcal{B}$ and $\mathcal{C}$ if and only if $A = B \times C$, and for all $t, n \in \mathbb{N}$ and all $\mathbf{s} \in A^{\mathbb{Z}}$,

$$\mathcal{A}_{t,n}(\mathbf{s}) = (\mathcal{B}_{t,n}(\pi_1(\mathbf{s})), \mathcal{C}_{t,n}(\pi_2(\mathbf{s})))$$

where $\pi_1(\mathbf{s})$ is the vector obtained from $\mathbf{s}$ by replacing each state $a = (b, c)$ by b, and $\pi_2(\mathbf{s})$ is similarly obtained by replacing a with c. If $\mathcal{A}$ is isomorphic to a product of $\mathcal{B}$ and $\mathcal{C}$ then the evolution diagram for $\mathcal{A}$ is just the superposition of the evolution diagrams of $\mathcal{B}$ and $\mathcal{C}$ (imagine them drawn on transparencies), as illustrated in figure 2.

Theorem 1. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be cellular automata corresponding to groupoids A, B, and C, respectively. Then $\mathcal{A}$ is isomorphic to (a subautomaton of, a quotient automaton of) $\mathcal{B}$ if and only if A is isomorphic to (a subgroupoid of, a quotient groupoid of) B. $\mathcal{A}$ is the product automaton of $\mathcal{B}$ and $\mathcal{C}$ if and only if A is the groupoid product of B and C.*

Proof. The subgroupoid case is clear. For the quotient case, suppose $\mathcal{A}$ is a quotient automaton of $\mathcal{B}$. Let $\theta : A \to B$ be the quotient map. Then it will be shown that θ is a groupoid homomorphism, so that B is a quotient of A.



(*a*) (*b*)

Figure 2: (*a*) Evolution of the three-state automaton whose rule is given by $\sigma(x,y) = x^2 + y \pmod 3$. (*b*) Evolution of the product automaton of the CAs in figure 1(*b*) and figure 2(*a*). At this resolution, all non-zero values appear black.

Let a and b be any two elements of A. Let $\mathbf{s}$ be any initial configuration with $\mathbf{s}(0) = a$ and $\mathbf{s}(1) = b$. Then

$$\theta(\mathcal{A}_{1,1}(\mathbf{s})) = \mathcal{B}_{1,1}(\theta(\mathbf{s}))$$

from the property of being a quotient automaton and, using (1), this gives

$$\theta(ab) = \theta(a)\theta(b).$$

For the converse, if $\theta : A \to B$ is a groupoid homomorphism onto B, let $\mathbf{s}$ be any configuration of $\mathcal{A}$. Then for all $n \in \mathbb{Z}$ we have $\theta(\mathbf{s}(n-1)\mathbf{s}(n)) = \theta(\mathbf{s}(n-1))\theta(\mathbf{s}(n))$, which says

$$\theta(\mathcal{A}_{1,n}(\mathbf{s})) = \mathcal{B}_{0,n-1}(\theta(\mathbf{s}))\mathcal{B}_{0,n}(\theta(\mathbf{s})).$$

The right-hand side is just $\mathcal{B}_{1,n}(\theta(\mathbf{s}))$, that is, $\theta(\mathcal{A}_{1,n}(\mathbf{s})) = \mathcal{B}_{1,n}(\theta(\mathbf{s}))$ for all n. Since $\mathbf{s}$ was any configuration of $\mathcal{A}$, we can take it to be the configuration at time $t-1$ to establish the same equality for any time t instead of 1.

The proof for products is similar. ■

The real content of the theorem is just that the basic algebraic operations on groupoids have direct visual counterparts for the evolution diagrams of the CAs they define. This makes results from universal algebra meaningful for CAs. An ideal situation would be if there were a small number of finite groupoids from which all others could be obtained by the fundamental algebraic operations discussed above. Then we would only need to understand the behavior of the CAs corresponding to these "building blocks" to determine the behavior of all CAs. In fact, there is no such finite set of finite groupoids from which all others can be obtained by taking finite products, quotients, and subalgebras, because there are infinitely many finite groupoids that cannot be expressed this way in terms of more elementary groupoids. These are known as the subdirectly irreducible groupoids. It is a basic theorem of universal algebra that any finite groupoid is a subdirect product of finitely many subdirectly irreducible groupoids (which are quotients of the given groupoid). (Consult [7] or [3] for further information.) Although this is a better result than if we decomposed only by direct products (then any groupoid with a prime number of elements would be indecomposable), there are still too many subdirectly irreducible groupoids for this result to be really useful. It does demonstrate, however, that we need only determine the behavior of subdirectly irreducible CAs to understand the behavior of all CAs.

If we restrict our attention to special classes of groupoids then we may obtain better decomposition results. For example, the fundamental theorem of abelian groups says that any finite abelian group is a direct product of cyclic p-groups (see section II.2 in [5] for example). Thus, to understand abelian group CAs, we need only determine the behavior of CAs corresponding to the groups $\mathbb{Z}_{p^m}$ (p is a prime, m any positive integer). This can be achieved with some precision.

Let the elements of the cyclic group $\mathbb{Z}_n$ be denoted by $\{0, 1, 2, \ldots, n-1\}$, where 0 is the identity element and 1 is a generator. The behavior of the cellular automaton $\mathcal{A}$ corresponding to $\mathbb{Z}_n$ on any initial configuration can be determined from its behavior on the initial configuration $\hat{1}$ defined by

$$\hat{1}(i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, using $+$ for the abelian group operation, then in the notation of (3) and (4), for all configurations $\mathbf{s}$ and $\mathbf{t}$,

$$\begin{aligned} \mathcal{A}_1(\mathbf{s}+\mathbf{t}) &= (\mathbf{s}+\mathbf{t}) + \Xi(\mathbf{s}+\mathbf{t}) \quad \text{by (4)} \\ &= \mathbf{s}+\mathbf{t}+\Xi(\mathbf{s})+\Xi(\mathbf{t}) \quad \text{by (3)} \\ &= (\mathbf{s}+\Xi(\mathbf{s})) + (\mathbf{t}+\Xi(\mathbf{t})) \quad \text{by commutativity} \\ &= \mathcal{A}_1(\mathbf{s}) + \mathcal{A}_1(\mathbf{t}). \end{aligned} \tag{5}$$

Since any initial configuration $\mathbf{s}$ can be expressed as $\mathbf{s} = \sum_{i=-\infty}^{\infty} \mathbf{s}(i)\Xi^i(\hat{1})$, this shows that for any initial configuration $\mathbf{s}$, any $t \geq 1$, and any $i \in \mathbb{Z}$,

$$\mathcal{A}_{t,i}(\mathbf{s}) = \sum_{k=0}^{i} \mathbf{s}(k)\mathcal{A}_{t,i-k}(\hat{1}). \tag{6}$$

Furthermore, it is easily established by induction that $\mathcal{A}_{t,k}(\hat{1}) = \binom{t}{k} (\operatorname{mod} n)$, where the binomial coefficient $\binom{t}{k}$ is understood to be 0 if $k > t$ or $k < 0$. Combining this with (6) yields the following theorem.

Theorem 2. *Let $\mathcal{A}$ be the CA corresponding to the cyclic group $\mathbb{Z}_n$. Then for any initial configuration* $\mathbf{s}$ *and any* $t \geq 1$,

$$\mathcal{A}_{t,i}(\mathbf{s}) = \sum_{k=0}^{i} \mathbf{s}(k)\binom{t}{i-k} \pmod n.$$

This theorem can be used together with the fundamental theorem of abelian groups to obtain $\mathcal{A}_{t,i}$ for any abelian group A, by first expressing A as a direct product of $\mathbb{Z}_{p^k}$ for various p, k and then using the theorem on each factor.

While this decomposition theory is quite successful for abelian groups, such good results should not be expected for other types of groupoids because of the lack of analogues for the fundamental theorem of abelian groups, and the lack of (6). Even for nonabelian groups a complete analysis appears difficult.

Varieties and periodic behavior

Algebraic operations are at the heart of one of the most successful approaches to the classification of algebras in general, which is by varieties. A variety is a class of algebras (with a fixed number of operations) that is closed under the formation of isomorphic copies of quotients, subalgebras, and arbitrary products. A fundamental result of Birkhoff [2] says that varieties can equivalently be defined as those classes of algebras that satisfy a (possibly infinite) set of identities (see below for a formal definition of satisfaction of an identity). Familiar examples of varieties include groups and rings, but not fields (which are not closed under products).

It might be hoped that common identities such as commutativity and associativity holding for the groupoid of a CA would lead to characteristically recognizable behaviors. In fact, the more usual identities do not appear to give recognizable behavior, but identities can be found to characterize certain CA behaviors. That is, there are identities such that a CA has certain (shift-periodic) behavior if and only if the groupoid of the CA satisfies the identities.

A groupoid identity is an equation $s = t$ where s and t are groupoid terms in the variables $x_1, x_2, \ldots, x_n$. A groupoid A satisfies the identity $s = t$ if for every n-tuple $(a_1, a_2, \ldots, a_n)$ of elements from A, the expressions $s[x_1/a_1, \ldots, x_n/a_n]$ and $t[x_1/a_1, \ldots, x_n/a_n]$ evaluate to the same element of A, where x_i/a_i denotes, as usual, the replacement of x_i by a_i at all occurrences of x_i. A more algebraic definition of satisfaction of an identity can be given in terms of free algebras and homomorphisms.

The free groupoid $F(X)$ on a set X is just the set of all groupoid terms involving only elements of X, with the groupoid operation applied to two terms s and t defined to yield the term $s \cdot t$. $F(X)$ has the property that any mapping from X into any other groupoid G can be extended uniquely to a homomorphism from $F(X)$ to G. With this terminology, a groupoid G will satisfy an identity $s = t$ written on the variables in X if and only if, for any mapping of X into G, the unique extension $\phi : F(X) \to G$ of the mapping has $\phi(s) = \phi(t)$. This equivalent definition of satisfaction will be used in the proofs below.

A cellular automaton $\mathcal{A}$ is *periodic* if, for all initial configurations, the rows of its evolution diagram eventually start repeating periodically. More precisely, $\mathcal{A}$ is k-periodic (starting at generation g) if there exists $g \in \mathbb{N}$ such that for all $\mathbf{s} \in A^{\mathbb{Z}}$ and all $t \geq g$,

$$\mathcal{A}_{t+k,n}(\mathbf{s}) = \mathcal{A}_{t,n}(\mathbf{s}).$$

Period k is defined to include periods less than k so that the groupoids corresponding to all k-periodic CAs will form a variety. A similar remark applies to g.

Theorem 3. *For each $g, k \in \mathbb{N}$, the groupoids defining k-periodic cellular automata after g generations form a variety, $\mathbf{V}_{g,k}$. If $g \leq h$ then $\mathbf{V}_{g,k} \subseteq \mathbf{V}_{h,k}$, and if k divides p then $\mathbf{V}_{g,k} \subseteq \mathbf{V}_{g,p}$.*

Proof. Let $\mathcal{A}$ be a CA. Let $\mathbf{s}$ be an initial configuration of $\mathcal{A}$. Define inductively

$$s_i^0 = \mathbf{s}(i), \quad s_i^{k+1} = s_{i-1}^k s_i^k. \tag{7}$$

Then it follows from (1) and (2) that for all $i \in \mathbb{Z}$ and all $t \in \mathbb{N}$, $\mathcal{A}_{t,i}(\mathbf{s}) = s_i^t$ is the value of cell i of $\mathcal{A}$ in generation t. A visual aid to this formulation of the CA evolution is presented below, with the entry in row i under c_j denoting the state of cell j in generation i.

$$\begin{array}{ccccc}
c_{-1} & c_0 & c_1 & c_2 & c_3 \\
c_{-2}c_{-1} & c_{-1}c_0 & c_0c_1 & c_1c_2 & c_2c_3 \\
(c_{-3}c_{-2})(c_{-2}c_{-1}) & (c_{-2}c_{-1})(c_{-1}c_0) & (c_{-1}c_0)(c_0c_1) & (c_0c_1)(c_1c_2) & (c_1c_2)(c_2c_3)
\end{array}$$

According to the definition, $\mathcal{A}$ is k-periodic starting at generation g if and only if

$$s_i^{t+k} = s_i^t \quad \text{for all } i \in \mathbb{Z} \text{ and all } t \geq g. \tag{8}$$

For example, referring to the diagram above, to obtain period 2 starting at generation 0 all the values in the third row have to be the same as those immediately above them in the first row; that is, $(c_{i-2}c_{i-1})(c_{i-1}c_i) = c_i$ for all i. Since the c_j can assume any values in the state set, this requires that the groupoid satisfy the identity $(xy)(yz) = z$. The values in the fourth row (not shown) also need to equal those in the second, and so on. It is fairly easy to see that if the groupoid of the CA satisfies the identity $(xy)(yz) = z$, then these further conditions are also satisfied. The general case is now examined.

We will define identities on the variables $X = \{x_0, x_1, x_2, \ldots\}$ whose satisfaction by a groupoid will be shown to be equivalent to the periodicity of the corresponding CA. To specify the identities, let Λ be the left shift operator on words on X. That is, if w is a word on X, then $\Lambda(w)$ is the word obtained from w by simultaneously replacing all occurrences of x_i by x_{i+1} for all i. (Let $\Lambda_0 : X \to X : x_i \mapsto x_{i+1}$. Then Λ is the (unique) extension of Λ_0 to an endomorphism of the free groupoid on X.)

Define groupoid terms α_i inductively by

$$\alpha_0 = x_0, \quad \alpha_{i+1} = \Lambda(\alpha_i) \cdot \alpha_i \tag{9}$$

for $i \in \mathbb{N}$. Thus α_1 is $x_1 x_0$, α_2 is $(x_2 x_1)(x_1 x_0)$, and so forth. Then the variety $\mathbf{V}_{g,k}$ of CAs that have period k after g generations is defined by

$$\alpha_{g+k} = \alpha_g. \tag{10}$$

For suppose the groupoid A satisfies the identity (10). We must show that (8) is satisfied. Choose $i \in \mathbb{Z}$ and $t \in \mathbb{N}$. Let $f_0 : X \to A$ be the map defined by $f_0(x_j) = s^t_{i-j}$, and let f be the extension of this map to a homomorphism from the free groupoid on X to A. Then, according to the definitions (7) and (9), $f(\alpha_m) = s_i^{m+t}$ for all m. Since A was assumed to satisfy (10), it follows that $f(\alpha_{g+k}) = f(\alpha_g)$, which gives, by the previous sentence, $s_i^{g+k+t} = s_i^{g+t}$. This is equivalent to (8) since $t \in \mathbb{N}$ was arbitrary.

Conversely, suppose $\mathcal{A}$ has period k after g generations. Then (8) holds for every initial configuration, and we must show that A satisfies the identity (10). Let $a_0, a_1, \ldots, a_{g+k}$ be any elements of A. Let $h_0 : X \to A$ be any map such that $h_0(x_i) = a_i$ for $i \le g + k$. Let h be the extension of h_0 to a homomorphism of the free groupoid on X into A. We must show that $h(\alpha_{g+k}) = h(\alpha_g)$. Let $\mathbf{s}$ be an initial configuration of $\mathcal{A}$ with $\mathbf{s}(i) = a_{g+k-i}, i = 0, 1, \ldots, g+k$. Define s_i^t for this $\mathbf{s}$ as in (3). Then $h(\alpha_t) = s^t_{g+k}$ follows from (7) and the definition of h. By (8), $s^{g+k}_{g+k} = s^g_{g+k}$, which gives $f(\alpha_{g+k}) = h(\alpha_g)$ as required. Thus these CAs are characterized by the identity (10), which shows they form a variety.

Finally, it is clear from the definition that if a CA has period k after g generations, then it also has period k after t generations for all $t > g$, and that it also has period mk for all $m = 1, 2, 3, \ldots$ (after g generations). ∎

Corollary 3.1. A product automaton of periodic automata is periodic. In fact, if $\mathcal{A} \in \mathbf{V}_{g,k}$ and $\mathcal{B} \in \mathbf{V}_{h,p}$, then $\mathcal{A} \times \mathcal{B} \in \mathbf{V}_{m,c}$ where $m = \max(g, h)$ and $c = \operatorname{lcm}(k, p)$.

Proof. From Theorem 3 we have $\mathbf{V}_{g,k} \subseteq \mathbf{V}_{m,k} \subseteq \mathbf{V}_{m,c}$ and $\mathbf{V}_{h,p} \subseteq \mathbf{V}_{m,p} \subseteq \mathbf{V}_{m,c}$ since both k and p divide c. The result follows at once since varieties ($\mathbf{V}_{m,c}$ in particular) are closed under products. ∎

Corollary 3.2. For all natural numbers n and k there exists a natural number $\mu_k(n)$ such that if a CA with at most n states is k-periodic, then it is k-periodic after $\mu_k(n)$ generations; that is, $\mathbf{V}_{\mu_k(n),k}$ contains all cellular automata of period k with at most n states. Also, for every n there exists $\pi(n)$ such that if a CA on n states is periodic, then it has period at most $\pi(n)$.

Proof. This follows immediately from the facts that for given k, $g \leq h$ implies $\mathbf{V}_{g,k} \subseteq \mathbf{V}_{h,k}$ (the varieties form an ascending chain), and that there are only finitely many CAs on a given finite state set. ∎

It is perhaps surprising how fast the varietal chains stabilize, at least for small state sets. A computer enumeration showed $\mu_1(3) = 4$, $\mu_2(3) = 5$, and $\mu_1(4) \geq 6$. In fact, as might be expected, there are relatively few CAs that are periodic (for all initial configurations). Of the 2,352 non-isomorphic CAs on three states satisfying $0 \cdot 0 = 0$, 191 are 1-periodic and 256 are 2-periodic (see table 1 below). Also, $\pi(3) = 2$. Thus, there are only 256 periodic CAs with three states (and $l = 1, r = 0$). Let $\mathcal{C}_3$ denote the set of three-element groupoids. The computer enumeration showed that $\mathcal{C}_3 \cap \mathbf{V}_{g,2k+1} = \mathcal{C}_3 \cap \mathbf{V}_{g,1}$ for all $g \leq 4$. Since $\mu_1(3) = 4$, we may conclude this holds for all g. Similarly, since we find $\mathcal{C}_3 \cap \mathbf{V}_{g,2k} = \mathcal{C}_3 \cap \mathbf{V}_{g,2}$ for all $g \leq 5$, and $\mu_2(3) = 5$, we conclude the equality holds for all g. Moreover, since $\pi(3) = 2$, we have the next theorem.

Theorem 4. *For all positive integers k and g, $\mathcal{C}_3 \cap \mathbf{V}_{g,2k+1} = \mathcal{C}_3 \cap \mathbf{V}_{g,1}$ and $\mathcal{C}_3 \cap \mathbf{V}_{g,2k} = \mathcal{C}_3 \cap \mathbf{V}_{g,2}$.*

Since periodic three-state CAs have period at most 2 ($\pi(3) = 2$), and $\mu_2(3) = 5$, any three-state periodic CA has period (at most) 2 starting by generation 5. From the proof of Theorem 3 (see equation (10) and following), such CAs are characterized by satisfying the identity $\alpha_7 = \alpha_5$.

Theorem 5. *A three-state cellular automaton with $0 \cdot 0 = 0$ is periodic if and only if its groupoid satisfies the identity $\alpha_7 = \alpha_5$; explicitly this identity is*

$$\begin{aligned}
(\{[((st \cdot tu)(tu \cdot uv))((tu \cdot uv)(uv \cdot vw))]& \\
\cdot [((tu \cdot uv)(uv \cdot vw))((uv \cdot vw)(vw \cdot wx))]\}& \\
\cdot \{[((tu \cdot uv)(uv \cdot vw))((uv \cdot vw)(vw \cdot wx))]& \\
\cdot [((uv \cdot vw)(vw \cdot wx))((vw \cdot wx)(wx \cdot xy))]\})& \\
\cdot (\{[((tu \cdot uv)(uv \cdot vw))((uv \cdot vw)(vw \cdot wx))]& \\
\cdot [((uv \cdot vw)(vw \cdot wx))((vw \cdot wx)(wx \cdot xy))]\}& \\
\cdot \{[((uv \cdot vw)(vw \cdot wx))((vw \cdot wx)(wx \cdot xy))]& \\
\cdot [((vw \cdot wx)(wx \cdot xy))((wx \cdot xy)(xy \cdot yz))]\})& \\
= [((uv \cdot vw)(vw \cdot wx))((vw \cdot wx)(wx \cdot xy))]& \\
\cdot [((vw \cdot wx)(wx \cdot xy))((wx \cdot xy)(xy \cdot yz))].&
\end{aligned}$$

Shift-periodic CAs

When using $l = 1, r = 0$ to simulate CAs with other neighborhoods, a shift factor is introduced (see the second section). Therefore, to investigate periodic CAs in general, the definition in the last section should be broadened as follows. An $(l = 1, r = 0)$ cellular automaton $\mathcal{A}$ is *shift-periodic* of period k and shift factor c, starting at generation g, if, for all initial configurations $\mathbf{s}$ of $\mathcal{A}$,

$$\mathcal{A}_{t+k,n+c}(\mathbf{s}) = \mathcal{A}_{t,n}(\mathbf{s}) \quad \text{for all } t \geq g \text{ and all } n \in \mathbb{Z}.$$

For shift-periodic CAs there is this generalization of Theorem 3, whose proof is similar to that of Theorem 3.

Theorem 6. *For each $g, k, c \in \mathbb{N}$, the groupoids defining cellular automata of period k with shift factor c starting in generation g form a variety $\mathbf{V}_{g,k,c}$ defined by the identity*

$$\alpha_{g+k} = \Lambda^c(\alpha_g)$$

where α_i is as in Theorem 3 and Λ^c is the c-fold composition of the left shift operator.

If $A = (A, \cdot)$ is a groupoid, let $A^{\mathrm{op}} = (A, \circ)$, where $a \circ b = b \cdot a$. The next result shows that in analyzing $\mathbf{V}_{g,k,c}$ it is usually only necessary to consider $c \leq \lceil \frac{g+k}{2} \rceil$.

Theorem 7. 1. *If $c > g + k$ then $\mathbf{V}_{g,k,c}$ is empty if $g = 0$ and otherwise consists precisely of all groupoids satisfying $\alpha_g = 0$.*

2. *Let $\mathcal{C}_n$ denote the class of n-element groupoids and let $c \leq \lceil \frac{g+k}{2} \rceil$. Then*

$$A \in \mathcal{C}_n \cap \mathbf{V}_{g,k,c} \qquad \textit{iff} \qquad A^{\mathrm{op}} \in \mathcal{C}_n \cap \mathbf{V}_{g,k,\lceil \frac{g+k}{2} \rceil - c}.$$

Proof. To prove (1), note that α_{g+k} has variables $x_0, x_1, \ldots, x_{g+k}$ and $\Lambda^c(\alpha_g)$ has variables $x_c, x_{c+1}, \ldots, x_{c+g}$, so that if $c > g+k$ then they have no variables in common. Also our groupoids satisfy $0 \cdot 0 = 0$. The result follows. For (2), observe that $x_i \mapsto x_{g+k-i}$ induces an involution on the free groupoid on $\{x_0, x_1, \ldots, x_{g+k}\}$, which are the variables occurring in α_{g+k}. ∎

The results on varietal chains for periodic CAs generalize as follows. The proofs are similar to the periodic case.

Theorem 8. 1. *If $g \leq h$ then $\mathbf{V}_{g,k,c} \subseteq \mathbf{V}_{h,k,c}$. Thus for each $k, c, n \in \mathbb{N}$ there exists a number $\mu_{k,c}(n)$ such that if a cellular automaton with n states is shift-periodic of period k with shift factor c then this starts at generation $\mu_{k,c}(n)$.*

2. *If $a = \gcd(k, c)$ and there exists m such that $pa = mk$ and $da = mc$ then $\mathbf{V}_{g,k,c} \subseteq \mathbf{V}_{g,p,d}$.*

Shift factor 0:

$g \backslash k$	1	2	3	4
0	1	4	1	4
1	35	47	35	47
2	143	178	143	178
3	182	236	182	236
4	191	255	191	255
5	191	256	191	256
6	191	256	191	256

Shift factor 1:

$g \backslash k$	1	2	3	4
0	1	0	0	0
1	35	11	1	1
2	143	52	34	34
3	182	81	54	54
4	191	94	60	60
5	191	96	60	60
6	191	97	60	60
7	191	97	60	60

Shift factor 2:

$g \backslash k$	1	2	3	4	5	6
0	0	1	0	0	0	0
1	1	35	1	11	1	1
2	34	178	34	52	34	34
3	54	236	54	81	54	54
4	60	255	60	94	60	60
5	60	256	60	96	60	60
6	60	256	60	97	60	60
7	60	256	60	97	60	60

Table 1: Numbers of non-isomorphic cellular automata in $\boldsymbol{V}_{g,k,c}$. The entries in the last two columns for shift factors 1 and 2 (and all higher k) reflect the groupoids satisfying $\alpha_g = 0$ (see Theorem 7).

Again by computer enumeration we have obtained $\mu_{1,1}(3) = 4$, $\mu_{2,1}(3) = 6$, and $\mu_{k,1}(3) = 4$ for $k \geq 3$.

The complete results of the computer enumeration for three-state CAs are presented in table 1. Each entry gives the number of non-isomorphic zero-pointed groupoids ($l = 1, r = 0$ CAs with a quiescent state) of order three with the given period k starting in the given generation g at each shift amount.

From these enumerations and an examination of the identities involved, it appears very likely that no further models arise for larger shift factors. If this is so then there are only 293 non-isomorphic one-dimensional CAs with three states that give stable (or shift-periodic) evolution for all initial configurations, from the total of 2,352. The figure of 293 is obtained from the 256 groupoids occurring with period 2 (no shift or shift 2) plus the 37 out of the 97 occurring with period 2 for shift 1 that are not already in the 256.

For four states, the numbers are much larger and we do not have complete results. For example, there are 508,144 non-isomorphic zero-pointed four-element groupoids of period 1 (shift 0) starting in generation 3.

Although the numbers of periodic CAs get much larger for more states, so of course do the total numbers of non-isomorphic CAs with that number of states. Asymptotically, the fraction of periodic CAs can be expected to be vanishingly small, as the freedom for non-periodicity increases. A thorough investigation of asymptotics and other issues left open by the above results awaits another paper.

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References

[1] J. Albert and K. Culik II, "A Simple Universal Cellular Automaton and Its One-way and Totalistic Version," *Complex Systems*, **1** (1987) 1–16.

[2] G. Birkhoff, "On the Structure of Abstract Algebras," *Proceedings of the Cambridge Philosophical Society*, **29** (1935) 433–454.

[3] G. Grätzer, *Universal Algebra*, 2nd edition (New York, Springer-Verlag, 1979).

[4] G. Hedlund, "Endomorphisms and Automorphisms of the Shift Dynamical System," *Mathematical Systems Theory*, **3** (1969) 320–339.

[5] T. Hungerford, *Algebra*, Graduate Texts in Mathematics 73 (New York, Springer-Verlag, 1974).

[6] E. Jen, "Global Properties of Cellular Automata," *Journal of Statistical Physics*, **43** (1986) 219–242.

[7] R. McKenzie, G. McNulty, and W. Taylor, *Algebras, Lattices, Varieties* (Monterey, Wadsworth & Brooks/Cole, 1987).

[8] A. R. Smith III, "Simple Computation Universal Cellular Spaces," *Journal of the Association for Computing Machinery*, **18** (1971) 339–353.

[9] S. Wolfram (ed.), *Theory and Applications of Cellular Automata* (Singapore, World Scientific, 1986).