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Dynamics of Multicellular Automata with Unbounded Memory

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Abstract. Cellular automata with unbounded memory, also known as mathematical neuron models or threshold automata, have been studied in the single-cell case by a number of authors. In this paper these automata are connected into multicelled networks and the dynamics of the resulting complex systems examined for certain neighborhood systems and interaction rules. Computer simulation and theoretical analysis are both presented. Some of the previously known properties of the dynamics of single cells persist in these systems, but many new properties appear. Most of these results pertain to networks of two or three cells with very simple forms of interaction between cells; however, there are also some implications for more general, larger systems.

1. Introduction

The dynamical behavior of a single neural automaton with memory has been investigated by a number of authors [1–10], and has been shown to be extremely rich. In this paper, we formulate a particular model of such a neural automaton in a way that permits assembling automata into multicelled systems, and investigate the computational and theoretical properties of the dynamics of such systems. While we treat only a few simple modes of interaction in these systems, arbitrary neighborhood schemes and interaction rules are possible. Although many of the known dynamical properties of the single-celled automaton persist in these larger systems, some quite interesting new features appear that require elucidation.

The precise structure of the single neural automaton that we use is given in section 2, where some of the previous results pertaining to its interesting dynamics are also mentioned. The reformulation that permits the assemblage of these automata into networks is given in section 3, and some of the computational simulation results are presented. The dynamics of one particular type of two-celled network is investigated in section 4, and the extension to larger linear networks is given in section 5. A three-celled network is given in section 6.

2. Cellular automata with memory

We use a form of the neuron model proposed by Caianiello [1, 2] and later used by Yamaguti and Hata [10], Cosnard and Goles [4], and others. If x_n represents the state of the neuron at time n, $x_n \in \{0, 1\}$, then the evolution equation is

$$x_{n+1} = \mathbb{U}\left(T - \sum_{i=1}^{n} K^i x_{n-i}\right), \tag{1}$$

where $T > 0$, $0 < K < 1$, and $\mathbb{U}(x) = 0$ if $x < 0$, $\mathbb{U}(x) = 1$ if $x \geq 0$. The sum can be viewed as *memory* in the form of a stored charge, in which case K can be interpreted as the rate of retention of the charge over one unit time interval. T is the *threshold*, with the cell "firing" (state 1) if the memory falls below the threshold, otherwise becoming quiescent (state 0).

We will refer to an automaton defined by equation (1) as a *cellular automaton with (unbounded) memory* (CAM); it is called by some a *mathematical neuron model* or *threshold automaton.*

Yamaguti and Hata showed that stable periodic orbits of all periods, as well as nonperiodic orbits, can occur in a CAM of the type just described [10]. More precisely, for fixed K they showed a one-to-one correspondence between disjoint subintervals in the interval $0 < T < 1/(1-K)$ and all reduced fractions q/p in the unit interval, where p can be interpreted as the period of an orbit. It follows that as T varies, with K fixed, there are exactly $\phi(p)$ disjoint subintervals in T for which there are orbits of period p, where ϕ is Euler's ϕ-function.

The results just mentioned, and others, can be illustrated by computer simulations of the evolution of a CAM. We have performed such simulations for single-celled CAMs of the type defined above (as well as multicelled CAMs to be discussed later) for a wide range of values of K and T. The evolution of the CAM is calculated to steady state, then examined for the existence of a fixed point or an n-cycle in the dynamics of the automaton. The periods found in such a series of calculations can then be displayed by color or shading in a *KT-diagram.* Figure 1 shows the KT-diagram obtained when 400 generations of a single CAM are calculated for 64,000 different pairs of values of K and T. K varies from 0 at the top to 1 at the bottom and T varies from 0 at the left to 5 at the right. White indicates a fixed point and the degree of shading indicates the observed period (reduced modulo 3). It is easy to see the single region of period-2 dynamics ($\phi(2) = 1$), two regions each of period-3 or -4 dynamics ($\phi(3) = \phi(4) = 2$), and four regions of period-5 dynamics ($\phi(5) = 4$). The large fixed-point region at the upper right satisfies $T > 1/(1-K)$ and is called the *saturation region* since under this condition the cell state eventually is always 1 and the sum in equation (1) approaches its maximum limiting value of $1/(1-K)$.

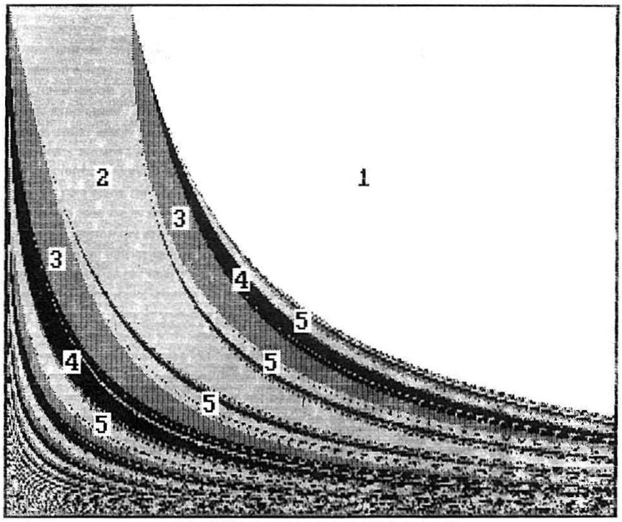

Figure 1: KT-diagram for a single-celled CAM, K in $[0,1]$, T in $[0,5]$. The numbers superimposed on the diagram are the observed periods of the dynamics of the automaton for the corresponding values of K and T.

3. Multicelled CAMs

In order to construct arrays of interacting CAM cells, we proceed as follows. We envision a single CAM cell, A, as consisting of two components, $A = (m, c)$. Here m is a real number that can be thought of as internal memory, analogous to the charge in a storage battery, with no direct external effect. $c \in \{0, 1\}$ is the "state" of the cell that is visible to all "neighboring" cells and may influence their behavior. The precise definitions of "neighborhood" and of the mechanism of mutual interaction among cells must be elaborated for each CAM network constructed.

In the single-cell case, with a cell's neighborhood consisting of the cell itself, one realization of such a CAM is as follows. If (m, c) denotes the CAM at time n and (m', c') the CAM at time $n+1$, we require

$$m' = F(m, c), \qquad c' = B(m'),$$

where $F : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$ and B is a Boolean-valued function $B : \mathbb{R} \to \{0, 1\}$. The simple special case in which

$$m' = Km + c, \qquad c' = \mathbb{U}(T - m') \tag{2}$$

can easily be shown to be equivalent to the single-celled CAM in equation (1) above.

This approach permits an easy extension to an arbitrary array of cells. Let $(\overline{m}, \overline{c})$ denote an N-celled CAM at time t and $(\overline{m}', \overline{c}')$ the same CAM at

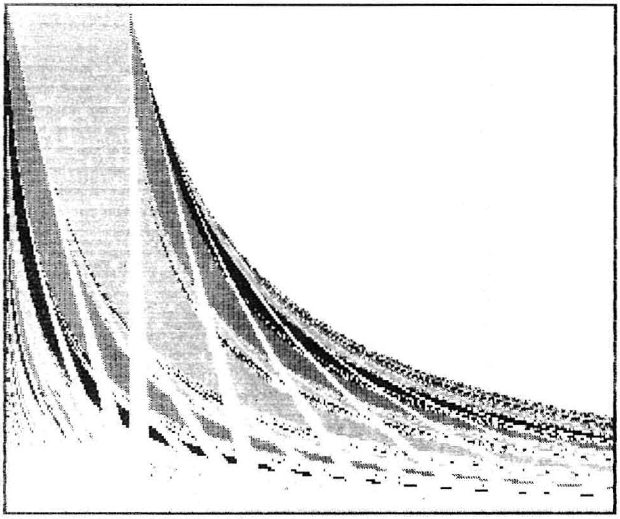

Figure 2: KT-diagram for a two-celled CAM (see equation (4)). The coordinates are the same as in figure 1.

time $t+1$, where $\overline{m} = (m_0, m_1, \ldots, m_{N-1})$ and $\overline{c} = (c_0, \ldots, c_{N-1})$ denote the arrays of cell attributes for the N cells making up the CAM network. Then we define

$$m_i' = F(m_i, \mathcal{N}_i(\overline{c})), \qquad c_i' = B(m_i'), \tag{3}$$

for real-valued F and Boolean-valued B, and where $\mathcal{N}_i(\overline{c})$ denotes the set of states of cells in the neighborhood of the ith cell.

A simple example of a two-celled CAM can be defined as follows. Denote the CAM by $[(m_0, c_0), (m_1, c_1)]$ at time t. The evolution of the CAM then proceeds in discrete time steps, with a prime denoting time $t+1$, by

$$\begin{aligned} m_0' &= K m_0 + c_1, \quad c_0' = \mathbb{U}(T - m_0'); \\ m_1' &= K m_1 + c_0, \quad c_1' = \mathbb{U}(T - m_0'). \end{aligned} \tag{4}$$

Computer simulation of the two-celled CAM just described results in the KT-diagram shown in figure 2, when initialized in the state $[(1,0),(1,.3)]$. Notice that the dynamics exhibits rather striking differences from the related single-celled CAM shown in figure 1, with numerous period-1, or fixed-point, regions superimposed on a background dynamics that appears to be identical to that for a single cell.

4. Dynamics of a two-celled CAM

An examination of the KT-diagram of a two-celled CAM (shown in figure 2) raises interesting questions concerning the precise description of the many regions of fixed-point dynamics. Some answers are provided in this section.

We first show that for certain initial conditions and for many of the possible values of the parameters K and T, the asymptotic dynamics of the two-celled CAM of equation (4) is identical to that of two isolated (i.e., uncoupled) cells governed by equation (1) or (2).

When necessary for clarity we denote the two-celled CAM at time t by $[(m_0^{(t)}, c_0^{(t)}), (m_1^{(t)}, c_1^{(t)})]$; that is, the cells are distinguished by the subscript and the time indicated by the superscript. Primes may also be used to indicate time $t+1$ as evolved from time t.

Theorem 1. *Let the two-celled CAM in equation (4) be initialized at $t = 0$ with $m_0 = \mu$, $m_1 = 0$, and $0 < \mu < 1$, and with $c_0 = c_1$ chosen arbitrarily from $\{0, 1\}$. Also let (K, T), $0 < K < 1$, $T > 0$, lie outside the region $\cup\mathcal{R}(p_n)$ where $\mathcal{R}(p_n) = \{(K, T) : p_n(k) \leq T < p_n(k) + \mu K^{n+1}$, p_n a polynomial of degree $n\}$, the union being taken over all polynomials p_n with coefficients from $\{0, 1\}$. Then each of the two cells has the same asymptotic behavior as the single-celled CAM of equation (1) or (2).*

Proof. By hypothesis, we have $c_0^{(0)}$ and $c_1^{(0)}$ both equal to the common value $c^{(0)}$, say, and for every $K \in (0, 1)$, $T \notin [p_0(K), \mu K + p_0(K)]$ for any choice $p_0(K)$ with Boolean coefficients. Thus $T \notin [c^{(0)}, \mu K + c^{(0)}]$. This implies that either

$$T < m_0^{(1)} = c^{(0)} \leq \mu K + c^{(0)} = m_1^{(1)},$$

and thus $c_0^{(1)} = c_1^{(1)} = 0$, or

$$m_0^{(1)} \leq m_1^{(1)} + c^{(0)} \leq T,$$

and thus $c_0^{(1)} = c_1^{(1)} = 1$. In either case we see that $c_0^{(1)} = c_1^{(1)}$, with common value $c^{(1)}$, say. Now suppose that $c_0^{(n)} = c_1^{(n)}$, with common value $c^{(n)}$, for $n = 0, 1, 2, \ldots, m-1$. Then

$$m_0^{(m)} = K^m\mu + c^{(0)}K^{m-1} + c^{(1)}K^{m-2} + \cdots + c^{(m-2)}K + c^{(m-1)}$$

and

$$m_1^{(m)} = c^{(0)}K^{m-1} + c^{(1)}K^{m-2} + \cdots + c^{(m-2)}K + c^{(m-1)}.$$

Also, we know by hypothesis that $T \notin [p_{m-1}(K), \mu K^m + p_{m-1}(K)]$, that is,

$$T \notin [c^{(0)}K^{m-1} + \cdots + c^{(m-1)}, \mu K^m + c^{(0)}K^{m-1} + \cdots + c^{(m-1)}].$$

This implies that either

$$T < m_0^{(m)} \leq m_1^{(m)},$$

and thus $c_0^{(m)} = c_1^{(m)} = 0$, or

$$m_0^{(m)} \leq m_1^{(m)} \leq T,$$

and thus $c_0^{(m)} = c_1^{(m)} = 1$. In either case we see by mathematical induction that $c_0^{(m)} = c_1^{(m)}$, for $m = 0, 1, 2, \ldots$, and thus the two cells always have the same state under the stated hypotheses.

It remains to be shown that the memory values of the two cells approach a common value. We know that

$$m_0^{(n)} = p_{n-1}(K)$$

and

$$m_1^{(n)} = \mu K^n + p_{n-1}(K)$$

for some polynomial p_{n-1}. Upon subtracting these we get

$$m_1^{(n)} - m_0^{(n)} = \mu K^n.$$

Since $K \in (0, 1)$, by hypothesis, this quantity approaches zero in the limit as $n \to \infty$. Hence we have shown that the memory levels in the two cells approach the same level. Thus, asymptotically, a reference of a cell to the state of a neighbor is identical to a reference to its own state, showing that the dynamics is that of the single-celled CAM of equation (1) or (2). ∎

If both cells of the two-celled CAM in equation (4) are initialized with the same values of m and c, then any reference by either cell to the other is equivalent to a self reference by that cell, and thus the two-celled CAM clearly can have any behavior possible for the single-celled CAM.

The result just established explains the single-celled behavior of the asymptotic dynamics of the two-celled CAM of equation (4) in the part of the KT-diagram outside the fixed-point (white) regions of figure 2. That figure also clearly demonstrates the empirical result that there are many overlapping regions of such period-1 dynamics. We now give a theoretical confirmation of the observed convergence of the dynamics to a stable fixed point in certain of these regions.

Theorem 2. *Let the two-celled CAM of equation (4) be initialized to* $[(\mu, 1), (0.0, 1)]$ *with* $0 < \mu < 1$ *and with* $0 < K < 1$, $T > 0$. *Then the automaton converges to the fixed point* $[(1/(1-K), 0), (0.0, 1)]$ *in the regions defined by*

1. $1 + K + K^2 + \cdots + K^n \leq T < 1 + K + K^2 + \cdots + K^n + \mu K^{n+1}$, $n = 0, 1, 2, \ldots$,

or

2. $K^n \leq T < K^n + \mu K^{n+1}$, $n = 0, 1, 2, \ldots$.

Proof. We first consider regions of type (1). Let n be arbitrary, $n \geq 1$. Then, by the hypotheses,

$$m_0^{(1)} = K\mu + 1 \leq T, \quad \text{so} \quad c_0^{(1)} = \mathbb{U}(T - K\mu - 1) = 1,$$

and

$$m_1^{(1)} = K \cdot 0 + 1 \leq T, \quad \text{so} \quad c_1^{(1)} = \mathbb{U}(T - 1) = 1.$$

We now suppose that $c_0^{(j)} = c_1^{(j)} = 1$ for $j = 0, 1, \ldots, m-1$ where $m - 1 < n$. Then

$$m_0^{(m)} = K^m\mu + K^{m-1} + \cdots + K + 1 \leq T, \quad \text{so } c_0^{(m)} = 1,$$

and

$$m_1^{(m)} = 0 + K^{m-1} + \cdots + K + 1 \leq T, \quad \text{so } c_1^{(m)} = 1.$$

Therefore $c_0^{(j)} = c_1^{(j)} = 1$ for $j = 0, 1, 2, \ldots, n$, by mathematical induction. However,

$$m_0^{(n+1)} = K^{n+1}\mu + K^n + \cdots + K + 1 > T, \quad \text{so } c_0^{(n+1)} = 0,$$

and

$$m_1^{(n+1)} = 0 + K^n + \cdots + K + 1 \leq T, \quad \text{so } c_1^{(n+1)} = 1.$$

Now suppose that $c_0^{(j)} = 0$ and $c_1^{(j)} = 1$ for $j = n+1, n+2, \ldots, p-1$, for $p - 1 \geq n + 1$. Then

$$m_0^{(p)} = K^p\mu + K^{p-1} + \cdots + K + 1 > T, \quad \text{so } c_0^{(p)} = 0,$$

and

$$m_1^{(p)} = 0 + K^{p-1} + \cdots + K^{p-n-1} \leq T, \quad \text{so } c_1^{(p)} = 1.$$

Therefore $c_0^{(j)} = 0$ and $c_1^{(j)} = 1$ for $j = n+1, n+2, \ldots$, by mathematical induction. Hence

$$\lim_{j\to\infty} m_0^{(j)} = \lim_{j\to\infty} \left[K^j\mu + \sum_{i=0}^{j-1} K^i \right] = 1/(1-K),$$

and

$$\lim_{j\to\infty} m_1^{(j)} = \lim_{j\to\infty} (K^{j-1} + \cdots + K^{j-n-1}) = 0.$$

The proof for regions of type (2) is similar and is therefore omitted. ∎

Theorem 2 establishes the asymptotic fixed-point behavior of the two-celled CAM of equation (4) in all of the period-1 (white) regions that appear as strips running from the top to the bottom of the KT-diagram in figure 2. The extremely complicated nature of other possible conditions for period-1

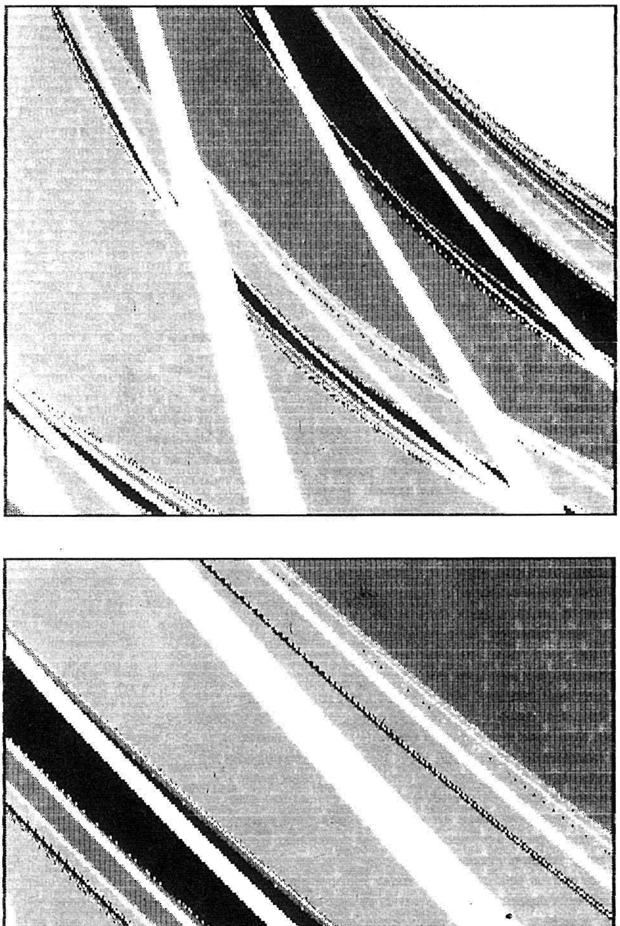

Figure 3: Successive enlargements of portions of figure 2.

dynamics can be seen in figure 3, which shows successive enlargements of a region of figure 2.

Since the regions just treated in Theorem 2 are included in those of Theorem 1 (*outside* the union of which the behavior is single-celled in nature), it can be conjectured that any two-celled CAM satisfying the conditions of Theorem 2 has a dynamics that approaches the dynamics of a single-celled CAM, or converges to a fixed point. It should be noted, however, that Theorems 1 and 2 are limited to a very specific initial configuration of the CAM. Other initial conditions may give quite different dynamics.

The results of this section have, as an application, the following rather startling consequence. Let us suppose that two isolated rooms are constructed and are to be heated by thermostatically controlled heaters. Suppose that the specific mode of operation of the heaters is such that they are activated at each unit time interval if the temperature of the room is below the threshold setting T of the thermostats, which is the same for each of the two rooms, and that if activated, each of the heaters imparts one unit of heat into the room. Further, suppose that the ambient temperature is zero, that the initial heat content of one room is μ and that of the other is zero, and that cooling takes place according to Newton's law of cooling, with a heat retention factor of K during one time period. Finally, suppose that the wiring is crossed so that the thermostat in each room controls the heater in the other room. The behavior of this dynamical system is identical to the two-celled CAM of this section, so Theorems 1 and 2 apply. Thus, one concludes that, for pairs K, T not in the white regions of figure 2, the two rooms eventually behave as if the wiring was correctly installed, but in the white regions one room will never be heated while the other will be heated to the greatest possible degree.

5. Extension to larger arrays of cells

We first consider an arbitrary linear array of CAM cells of the type introduced in section 3, with the neighborhood of a cell consisting of the single cell next on the right, more specifically described below. As before, we use the two-component model of a CAM cell and consider an arbitrary number N of cells denoted, at time t, by $A_i = (m_i, c_i)$. The cells are assembled into a linear network with the evolution of the network described by

$$m_i' = Km_i + c_{i+1}, \quad c_i' = \mathbb{U}(T - m_i'), \quad i = 0, 1, 2, \ldots, N-1, \tag{5}$$

where a prime again denotes time $t+1$ and where "wraparound" is invoked at the ends of the array, that is, the cell index is reduced modulo N so that $A_N = A_0$. In the following, by *synchronous* behavior of a multicelled CAM we mean any time all of the cells in the CAM have the same values of memory m and state c. If the cells have the same dynamic behavior, but shifted in time, the behavior is said to be *asynchronous*. The next result extends one obtained in the previous section with a two-celled CAM.

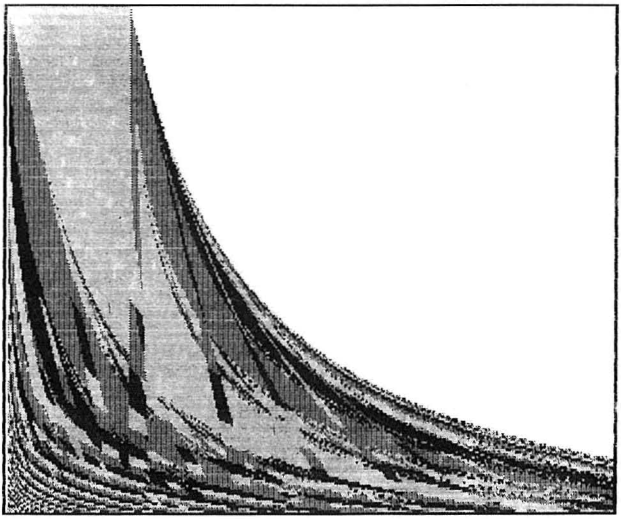

Figure 4: KT-diagram of a three-celled CAM (see equation (5)) initialized at $[(1, 0.2), (1, 0), (1, 0)]$.

Theorem 3. *(M. Smith) Let the N-celled CAM of equation (5) be initialized with $m_i = \mu_i \geq 0$ and $c_0 = c_1 = \cdots = c_{N-1}$, $c_i \in \{0, 1\}$. Also let (K, T), $0 < K < 1$, $T > 0$, lie outside the region $\cup \mathcal{R}(p_n)$, where $\mathcal{R}(p_n) = \{(K, T) : p_n(x) + K^{n+1}\mu_{\min} \leq T < p_n(x) + K^{n+1}\mu_{\max}$, p_n a polynomial of degree $n\}$, the union being taken over all polynomials p_n with coefficients from $\{0, 1\}$. Here $\mu_{\min} = \min\{\mu_0, \ldots, \mu_{N-1}\}$ and $\mu_{\max} = \max\{\mu_0, \ldots, \mu_{N-1}\}$. Then each of the N cells approaches, asymptotically and in synchronization, the same behavior as the single-celled CAM of equation (1) or (2).*

Proof. The proof is essentially an extension of the proof of Theorem 1 and so is omitted. ∎

The KT-diagram for the CAM of equation (5), with $N = 3$, is shown in figure 4. There are no fixed-point regions (other than the saturation region $T > 1/(1 - K)$), however there are noticeable strips outside which the dynamics is that of a single-celled CAM. This CAM will be studied further in section 6.

Two-dimensional cellular automata with memory have also been investigated empirically by computer simulation. Figure 5 shows two of the KT-diagrams that have been obtained for these automata.

6. Synchrony and asynchrony in the dynamics of a linear three-celled CAM

A three-celled CAM, connected in a circle and evolving according to equation (5), has already been considered for certain initial conditions and shown

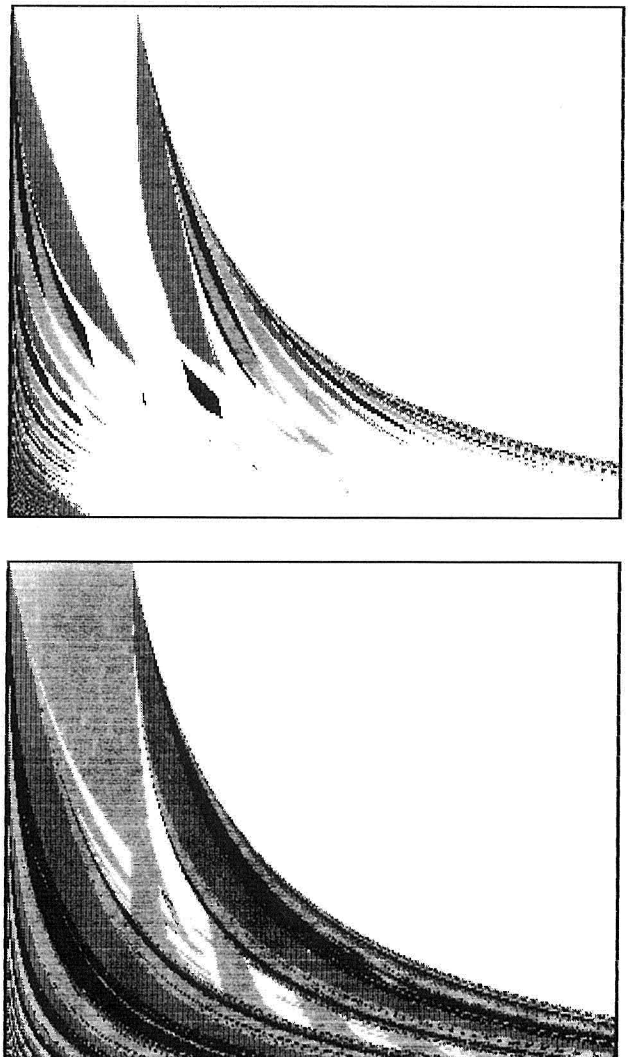

Figure 5: KT-diagrams of two-dimensional CAMs. On the left is the diagram of a 3×2 CAM and on the right that of a 5×4 CAM.

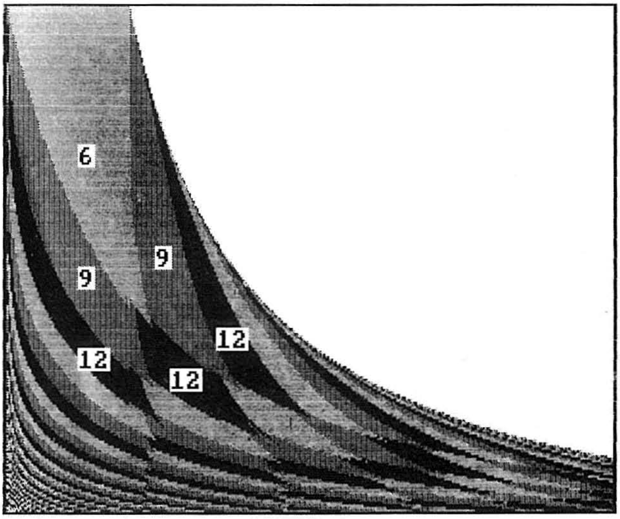

Figure 6: KT-diagram for a three-celled CAM (see equation (5)) initialized at $[(1,0),(0,0),(0,0)]$.

to lead to synchronous behavior, with each of the cells acting as a single-celled CAM in much of the KT-plane. However, other initial conditions have been found to result in convergence to a steady-state dynamics, which appears to be devoid of (synchronous) single-cell behavior throughout the unsaturated portion of the KT-plane.

An interesting example of this behavior appears in figure 6, which shows the KT-diagram of a three-celled CAM initialized in the state $[(1,0),(0,0),(0,0)]$. Only periods of length $6, 9, 12, \ldots, 3n, \ldots$ are observed to occur. Further analysis of the steady state reveals period-6 dynamics of the form 000111 ($0^3 1^3$) in the state of each cell of the CAM. If we let $0^m 1^n$ denote a period consisting of m consecutive 0s followed by n consecutive 1s, we find period-9 dynamics of the forms $0^5 1^4$ and $0^4 1^5$, period-12 dynamics of the forms $0^7 1^5$, $0^6 1^6$, and $0^5 1^7$, and so forth.

The following lemma is useful in explaining some of the features of this three-celled CAM.

Lemma. *Let A be a three-celled CAM that evolves according to equation (5) with wraparound. Then the following types of behavior of A cannot occur:*

1. $c_i = 1$, $c_i' = c_{i+1} = 0$, *with any* $T > 0$,
2. $c_i' = c_{i+1} = 1$, *with* $c_i \in \{0, 1\}$ *and* $T < 1$.

Proof. (1) The stated conditions, together with equation (5), require that $m_i < T$ and $K m_i > T$, an obvious contradiction since $K < 1$.

(2) Under the stated conditions we find that $Km_i + 1 < T$, again an obvious contradiction since $T < 1$, $K > 0$, and $m_i \geq 0$. ∎

As pointed out above, no periods shorter than 6 were observed to occur in the computer simulations of this CAM. The following theorem shows that such periods are not possible.

Theorem 4. *The three-celled CAM of equation (5), with $T < 1$, has no asynchronous periodic behavior with period less than 6.*

Proof. We suppose that dynamics of periods less than 4 have already been examined and consider the possible period-4 dynamics. Of the $2^4 = 16$ possible cycles of length 4 for a particular cell, say cell 0, only three—1000, 1100, and 1110—need be considered, since all others are either translations of one of these or have a shorter period of 1 or 2. We will analyze the two cases in detail; one of which we will show cannot occur, and the other we will show can occur only with all three cells acting in synchronization. We first consider the case 1100. The conditions of the initialization, periodicity of period 4, and wraparound immediately give the state values shown in the table below at time $t = 0$, where two periods have been shown. For convenience, we denote $c_i^{(j)}$ by cij and an unspecified entry by a dot.

	Time →							
Cell 0	1	1	0	0	1	1	0	0
Cell 1	0	$c11$	$c12$	$c13$	0	$c11$	$c12$	$c13$
Cell 2	0	$c21$	$c22$	·	0	$c21$	$c22$	·
Cell 0	1	1	0	0	1	1	0	0

By the lemma, $c11 = 1$, $c13 = 0$, $c21 = 0$, and $c22 = 0$, as indicated in the following updated table.

1	1	0	0	1	1	0	0
0	1	$c12$	0	0	1	$c12$	0
0	0	0	·	0	0	0	·
1	1	0	0	1	1	0	0

But this implies that, to satisfy the lemma, $c12$ must be 1 when the sub-array

$$\begin{matrix} 1 & c12 \\ 0 & \end{matrix}$$

is considered, but cannot be 1 when the sub-array

$$\begin{matrix} c12 & 0 \\ 0 & \end{matrix}$$

is considered—obviously an impossibility. Thus the period-4 cycle 1100 cannot occur.

Now consider the cycle 1000. Direct application of the lemma immediately requires that the state table, in the form used above, be as follows.

1	0	0	0	1	0	0	0
1	$c11$	$c12$	0	1	$c11$	$c12$	0
$c20$	0	$c22$	0	$c20$	0	$c22$	0
1	0	0	0	1	0	0	0

There are now four cases for $c11, c12$. The case $1, 0$ gives a translation of 1100, which has already been considered above; the case $0, 1$ gives 1010, which has the shorter period 2 and thus need not be considered here. The case $1, 1$, together with the lemma, leads to the following table.

1	0	0	0	1	0	0	0
1	1	1	0	1	1	1	0
0	0	$c22$	0	0	0	$c22$	0
1	0	0	0	1	0	0	0

But this again leads to a contradiction, since the lemma requires $c22 = 1$ in

1	0
$c22$	

and $c22 = 0$ in

$c22$	0
0	

Finally, the case $0, 0$ leads to the following table.

1	0	0	0	1	0	0	0
1	0	0	0	1	0	0	0
1	0	$c22$	0	1	0	$c22$	0
1	0	0	0	1	0	0	0

But $c22$ must be 0 by the lemma. Thus the three cells must be synchronized in this case.

The proofs for the other n cycles, $n < 6$, are similar. ∎

For period 6, an example of a cycle can be shown, by calculation, to be the following when $K = 0.3$ and $T = 0.7$.

Time	$m1$	$c1$	$m2$	$c2$	$m3$	$c3$
0	1.3910	0	1.0113	0	0.1252	1
1	0.4173	1	1.3034	0	0.0375	1
2	0.1252	1	1.3910	0	1.0113	0
3	0.0375	1	0.4173	1	1.3034	0
4	1.0113	0	0.1252	1	1.3910	0
5	1.3034	0	0.0375	1	0.4173	1
6	1.3910	0	1.0113	0	0.1252	1

This period-6 cycle has each of the cells following the cycle 000111 in asynchronous fashion, with cell $i+1$ delayed by two time steps with respect to cell i. The following theorem shows that this is the only period-6 behavior of a three-celled CAM obeying equation (5), and also delimits the region of the KT-diagram where this behavior can occur.

Theorem 5. *Let A be a three-celled CAM evolving according to equation (5), with $T < \max\{1, 1/(1-K)\}$. Then A has no asynchronous periodic behavior of period 6 except with each cell having the cycle 000111. Furthermore, such period-6 behavior can occur only in the portion of the KT-plane where*

$$\frac{K+K^2+K^3}{1-K^6} \leq T < \frac{1+K^4+K^5}{1-K^6}.$$

Proof. The proof that 000111 is the only asynchronous period-6 cycle under the stated conditions follows from the lemma by the methods used in the proof of Theorem 4, and thus will not be given. The details show, however, that for this case the state dynamics are as given in the following table, in which the format is the same as that used previously.

Cell 0	1	1	1	0	0	0
Cell 1	0	0	1	1	1	0
Cell 2	1	0	0	0	1	1

Let m denote the memory value of cell 0 at the start of the cycle at time $= 0$. Then, by equation (5), we have the following inequalities during the six steps of the cycle.

$$\begin{aligned} m &\leq T \\ Km &\leq T \\ K^2m &\leq T \\ K^3m+1 &> T \\ K^4m+K+1 &> T \\ K^5m+K^2+K+1 &> T \end{aligned}$$

In the next step, periodicity gives the equality

$$K^6m+K^3+K^2+K=m,$$

from which m is found to be $m = (K+K^2+K^3)/(1-K^6)$. The inequalities give three lower bounds and three upper bounds on T and, after substituting the value just found for m, the greatest lower bound and the least upper bound yield the bounds stated in the theorem. ∎

Although the bounds just established for the 6-cycle of type 000111 agree reasonably well with the simulation results shown in the KT-diagram of figure 6, there is yet some discrepancy. Calculation of bounds of regions exhibiting other types of steady-state periodic dynamics of type 1^m0^n, by the

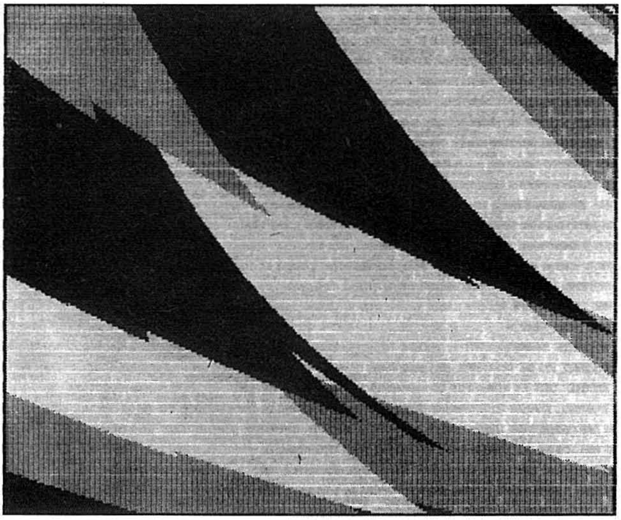

Figure 7: An enlargement of a portion of the KT-diagram of figure 6.

method used above, shows an overlap of parts of each pair of adjacent regions, within which the dynamics can converge to any one of two or more steady-state cycles, depending on the initial conditions used. This permits rather complex boundaries between regions of the KT-diagram with a particular type of steady state. This can be seen in figure 7, which shows an enlargement of a portion of figure 6.

7. Concluding remarks

Whether there exist other possible periodic solutions of equation (5) than those of the type 1^m0^n is an open question; likewise whether all periods must be multiples of three, as suggested by the simulations. Also, neighborhood systems $\mathcal{N}$ and forms of F and B in equation (3) that are different from the very limited cases dealt with here require much more study.

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