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A Fluid-Dynamic Model for the Movement of Pedestrians

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Abstract. A fluid-dynamic description for the collective movement of pedestrians is developed on the basis of a Boltzmann-like gas-kinetic model. The differences between these pedestrian-specific equations and those for ordinary fluids are worked out; they concern, for example, the mechanism of relaxation to equilibrium, the role of "pressure," the special influence of internal friction, and the origin of "temperature." Some interesting results are derived that can be compared to real situations—for example, the development of walking lanes and of pedestrian jams, the propagation of waves, and behavior on a dance floor. Possible applications of the model to town- and traffic-planning are outlined.

1. Introduction

Previous publications on the behavior of pedestrians have been predominantly empirical (often in the sense of regression analyses), and were intended to facilitate planning of efficient traffic [16, 24, 34]. While there also exist theoretical approaches to pedestrian movement [2, 3, 7, 9, 12, 29, 32, 33], most theoretical work has been done in the related topic of automobile traffic (see, for example, [1, 6, 8, 26, 27, 28]). In particular, some Boltzmann-like (gas-kinetic) approaches have been developed [1, 26, 27].

The author has observed that footprints of pedestrian crowds in the snow and quick-motion pictures of pedestrians resemble fluid streamlines. It is the object of this paper to give a suitable explanation of the fluid-dynamic properties of pedestrian crowds. Henderson was the first to apply gas-kinetic and fluid-dynamic models to empirical data of pedestrian crowds [12, 13, 14, 15]. His work, however, began with the conventional theory for ordinary fluids, and assumed a *conservation of momentum and energy.* In contrast to Henderson's approach, this article develops a *special theory for pedestrians*—without making use of unrealistic conservation assumptions.

Our procedure is described as follows. Pedestrians will be distinguished into groups of different types μ of motion, normally representing different intended directions of walking. At a time t the pedestrians of each type μ of motion can be characterized by several quantities, such as their place $\vec{x}$, their velocity $\vec{v}_\mu$, and their intended velocity $\vec{v}_\mu^0$ (in other words, the velocity they *wish* to walk with). So, we find in a given area A a density $\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t)$ of pedestrians having a special type of motion μ, and showing approximately the quantities $\vec{x}$, $\vec{v}_\mu$, and $\vec{v}_\mu^0$ at time t. For the densities $\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t)$ equations of motion can be established (section 2). From these equations we shall derive coupled differential equations for the spatial density $\langle \rho_\mu \rangle$ of pedestrians, their mean velocity $\langle v_\mu \rangle$, and velocity variance $\langle (\delta v_{\mu,i})^2 \rangle$ (section 3). The resulting equations show many similarities to the equations for ordinary fluids, but they contain additional terms that take into account pedestrian intentions and interactions (sections 3.1, 4.1, 5.1, and 6). In section 4 we shall treat equilibrium situations and the propagation of density waves. In *non*equilibrium situations, however, the final adaptation time to local equilibrium gives rise to internal friction (viscosity) and other additional terms (section 5). Effects of interactions (that is, of avoidance maneuvers) between pedestrians will be discussed in section 6. These effects will lead to some conclusions applicable to town- and traffic-planning (section 7).

Readers who are not interested in the mathematical aspects may skip the formulas in the following. However, the mathematical results are important for analytical, computational or empirical evaluations.

2. Gas-kinetic equations

Pedestrians can be distinguished into different types μ of motion, for example, by their different intended directions $\vec{e}_\mu := \vec{v}_\mu^0 / \|\vec{v}_\mu^0\|$ of motion (normally two opposite directions; at crossings, four directions). More precisely, a pedestrian shall belong to a type μ of motion if he wants to walk with an intended velocity

$$\vec{v}^0 \in \mathcal{N}_\mu,$$

where

$$\mathcal{N}_\mu := \{\vec{v}_\mu^0\}$$

is one of several disjoint and complementary sets. A type μ of motion still contains pedestrians with a variety of intended velocities $\vec{v}_\mu^0$, but the advantage resulting from a suitable choice of the sets $\mathcal{N}_\mu$ is the ability to get approximately *unimodal* densities $\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t)$, and therefore to obtain appropriate mean value equations (see section 3). $\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t)$ describes the number N_μ of pedestrians of type μ, within an area $A = A(\vec{x})$ around place $\vec{x}$, having the approximate *intended* velocity $\vec{v}_\mu^0$, but the approximate *actual* velocity $\vec{v}_\mu$. Specifically, $\hat{\rho}_\mu$ is defined by

$$\hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t) \equiv \hat{\rho}_\mu(\vec{x}, \vec{u}_\mu, t) := \frac{N_\mu(\mathcal{U}(\vec{x}) \times \mathcal{V}(\vec{u}_\mu), t)}{A \cdot V}, \qquad (1)$$

where N_μ is the number of pedestrians of type μ that are, at time t, in a *state*

$$(\vec{x}', \vec{u}'_\mu) \in \mathcal{U}(\vec{x}) \times \mathcal{V}(\vec{u}_\mu)$$

belonging to the neighborhood $\mathcal{U}(\vec{x}) \times \mathcal{V}(\vec{u}_\mu)$ of $\vec{x}$ and

$$\vec{u}_\mu := (\vec{v}_\mu, \vec{v}^0_\mu).$$

State $(\vec{x}, \vec{u}_\mu)$ is an abbreviation for the property

$$(\vec{x}, \vec{u}_\mu) := (\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu),$$

that an individual is at place $\vec{x}$ and wants to walk with the intended velocity $\vec{v}^0_\mu$, but in fact walks with velocity $\vec{v}_\mu$.

$$\mathcal{U}(\vec{x}) := \{\vec{x}^* \in \mathcal{M} : \|\vec{x}^* - \vec{x}\|_l \leq r\} \tag{2}$$

is a neighborhood around the place $\vec{x}$, and belongs to the domain $\mathcal{M}$, which represents all *accessible* (or public) places $\vec{x}$. $A = A(\vec{x})$ denotes the area of $\mathcal{U}(\vec{x})$. Similarly,

$$\mathcal{V}(\vec{u}_\mu) := \{\vec{u}^*_\mu = (\vec{v}^*_\mu, \vec{v}^{0*}_\mu) : \|\vec{u}^*_\mu - \vec{u}_\mu\|_l \leq s, \vec{v}^0_\mu \in \mathcal{N}_\mu\}$$

is a neighborhood of $\vec{u}_\mu := (\vec{v}_\mu, \vec{v}^0_\mu)$, with a volume $V = V(\vec{u}_\mu)$.

We shall now establish a set of *continuity equations*, which are similar to the construction of Alberti and Belli [1]:

$$\begin{aligned}\frac{d\hat{\rho}_\mu}{dt} &\equiv \frac{\partial\hat{\rho}_\mu}{\partial t} + \nabla_{\vec{x}}\left(\hat{\rho}_\mu\vec{v}_\mu\right) + \nabla_{\vec{v}_\mu}\left(\hat{\rho}_\mu\frac{\vec{f}_\mu}{m_\mu}\right) + \nabla_{\vec{v}^0_\mu}\left(\hat{\rho}_\mu\dot{\vec{v}}^0_\mu\right) \\ &:= \frac{\hat{\rho}^0_\mu - \hat{\rho}_\mu}{\tau_\mu} + \sum_\nu \hat{S}_{\mu\nu} + \sum_\nu \hat{C}_{\mu\nu} + \hat{q}_\mu.\end{aligned} \tag{3}$$

These equations can be interpreted as gas-kinetic equations (see chapters 2.4 and 2.7 of [18], and §3 of [19]). m_μ denotes the average mass of pedestrians belonging to type μ. Apart from special situations it will not depend on μ; in other words, $m_\mu \approx m_\nu$. The forces $\vec{f}_\mu := m_\mu\dot{\vec{v}}_\mu$ can often be neglected. However, they may be locally varying functions, depending on the attraction of the places $\vec{x}$. If a pedestrian does not change his type (direction) μ, the temporal change $\dot{\vec{v}}^0_\mu$ of the intended velocity $\vec{v}^0_\mu$ is normally a small quantity ($\dot{\vec{v}}^0_\mu \approx \vec{0}$), although $\vec{v}^0_\mu$ can in principal be a function of place $\vec{x}$ and time t.

According to (3), the change of the density $\hat{\rho}_\mu$ over time is given by four effects.

- First, by the tendency of the pedestrians to reach their intended velocity $\vec{v}^0_\mu$ [1, 9]. This causes $\hat{\rho}_\mu$ to approach

$$\hat{\rho}^0_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu, t) := \delta(\vec{v}_\mu - \vec{v}^0_\mu)\rho^0_\mu(\vec{x}, \vec{v}^0_\mu, t) \tag{4}$$

(the equilibrium density in the absence of disturbances), with a relaxation time

$$\tau_\mu \equiv \frac{m_\mu}{\gamma_\mu}$$

(see [9]). ρ^0_μ is the density of pedestrians with intended velocity $\vec{v}^0_\mu$ but arbitrary actual velocity $\vec{v}_\mu$. The Dirac delta function is denoted by $\delta(.)$, which is different from zero only when its argument (.) vanishes.

- Second, by the interaction of pedestrians, which can be modeled by a Boltzmann-like *stosszahlansatz* [10, 18, 19]: If we take into account that the interactions are of short range (in comparison with r, see (2)), we have

$$\begin{aligned} \hat{S}_{\mu\nu} = & \iiint \hat{\sigma}_{\mu\nu}(\vec{u}^*_\mu, \vec{u}^*_\nu; \vec{u}_\mu, \vec{u}_\nu; \vec{x}, t)\hat{\rho}_\mu(\vec{x}, \vec{u}^*_\mu, t)\hat{\rho}_\nu(\vec{x}, \vec{u}^*_\nu, t) d^4\vec{u}_\nu\, d^4\vec{u}^*_\mu\, d^4\vec{u}^*_\nu \\ & - \iiint \hat{\sigma}_{\mu\nu}(\vec{u}_\mu, \vec{u}_\nu; \vec{u}^*_\mu, \vec{u}^*_\nu; \vec{x}, t)\hat{\rho}_\mu(\vec{x}, \vec{u}_\mu, t)\hat{\rho}_\nu(\vec{x}, \vec{u}_\nu, t) d^4\vec{u}_\nu\, d^4\vec{u}^*_\mu\, d^4\vec{u}^*_\nu . \end{aligned} \tag{5}$$

This term describes *pair interactions* between pedestrians of types μ and ν, occuring with a total rate proportional to the densities $\hat{\rho}_\mu$ and $\hat{\rho}_\nu$ of both interacting types of motion. The *relative rate* for pedestrians of types μ and ν to change their states from $(\vec{x}, \vec{u}_\mu), (\vec{x}, \vec{u}_\nu)$ to $(\vec{x}, \vec{u}^*_\mu), (\vec{x}, \vec{u}^*_\nu)$, due to interactions, is given by $\hat{\sigma}_{\mu\nu}(\vec{u}_\mu, \vec{u}_\nu; \vec{u}^*_\mu, \vec{u}^*_\nu; \vec{x}, t)$. Assuming that only the actual velocities $\vec{v}_\mu, \vec{v}_\nu$ and not the intended velocities $\vec{v}^0_\mu, \vec{v}^0_\nu$ are affected by interactions, we obtain

$$\begin{aligned} &\hat{\sigma}_{\mu\nu}(\vec{u}^1_\mu, \vec{u}^1_\nu; \vec{u}^2_\mu, \vec{u}^2_\nu; \vec{x}, t) = \\ &\qquad \sigma_{\mu\nu}(\vec{v}^1_\mu, \vec{v}^1_\nu; \vec{v}^2_\mu, \vec{v}^2_\nu)\delta(\vec{v}^{0,2}_\mu - \vec{v}^{0,1}_\mu)\delta(\vec{v}^{0,2}_\nu - \vec{v}^{0,1}_\nu). \end{aligned}$$

This results in

$$\begin{aligned} & S_{\mu\nu}(\vec{x}, \vec{v}_\mu, t) \\ & := \int \hat{S}_{\mu\nu}(\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu, t) d^2\vec{v}^0_\mu \\ & = \iint \sigma_{\mu\nu}(\vec{v}^*_\mu, \vec{v}^*_\nu; \vec{v}_\mu, \vec{v}_\nu)\rho_\mu(\vec{x}, \vec{v}^*_\mu, t)\rho_\nu(\vec{x}, \vec{v}^*_\nu, t) d^2\vec{v}_\nu\, d^2\vec{v}^*_\mu\, d^2\vec{v}^*_\nu \\ & \quad - \iint \sigma_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu, \vec{v}^*_\nu)\rho_\mu(\vec{x}, \vec{v}_\mu, t)\rho_\nu(\vec{x}, \vec{v}_\nu, t) d^2\vec{v}_\nu\, d^2\vec{v}^*_\mu\, d^2\vec{v}^*_\nu \qquad (6) \\ & = \iint \sigma^*_{\mu\nu}(\vec{v}^*_\mu, \vec{v}^*_\nu; \vec{v}_\mu)\rho_\mu(\vec{x}, \vec{v}^*_\mu, t)\rho_\nu(\vec{x}, \vec{v}^*_\nu, t) d^2\vec{v}^*_\mu\, d^2\vec{v}^*_\nu \\ & \quad - \iint \sigma^*_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu)\rho_\mu(\vec{x}, \vec{v}_\mu, t)\rho_\nu(\vec{x}, \vec{v}_\nu, t) d^2\vec{v}_\nu\, d^2\vec{v}^*_\mu , \qquad (7) \end{aligned}$$

with

$$\sigma^*_{\mu\nu}(.,.;.) := \int \sigma_{\mu\nu}(.,.;.,\vec{v}) d^2\vec{v}. \tag{8}$$

Equation (6) is similar to (5), and can be interpreted analogously. The explicit form of $\sigma^*_{\mu\nu}$ will be based on a *microscopic model* for the interactions, and is discussed in section 6.

- Third, by pedestrians changing their type μ to another type ν, for example when turning to the right or left at a crossing, or when turning back (change of intended direction). This can be modeled by

$$\hat{C}_{\mu\nu}(\vec{x}, \vec{u}_\mu, t) = \int \hat{\sigma}_\mu^{\nu\mu}(\vec{u}_\nu; \vec{u}_\mu; \vec{x}, t)\hat{\rho}_\nu(\vec{x}, \vec{u}_\nu, t)d^4\vec{u}_\nu$$
$$- \int \hat{\sigma}_\mu^{\mu\nu}(\vec{u}_\mu; \vec{u}_\nu; \vec{x}, t)\hat{\rho}_\mu(\vec{x}, \vec{u}_\mu, t)d^4\vec{u}_\nu,$$

with a transition rate proportional to the density of the changing type of motion.

If we assume that for the moment of change both the intended velocity $\vec{v}_\mu^0$ and the actual velocity $\vec{v}_\mu$ remain the same (but of course not thereafter) we have

$$\hat{\sigma}_\mu^{1,2}(\vec{u}_1; \vec{u}_2; \vec{x}, t) = \hat{\sigma}_\mu^{1,2}(\vec{v}_1^0; \vec{v}_2^0; \vec{x}, t)\delta(\vec{v}_2 - \vec{v}_1).$$

This results in

$$C_{\mu\nu} := \int \hat{C}_{\mu\nu}(\vec{x}, \vec{u}_\mu, t)d^2\vec{v}_\mu^0$$
$$= \sigma_\mu^{\nu\mu}(\vec{x}, \vec{v}_\mu, t)\rho_\nu(\vec{x}, \vec{v}_\mu, t) - \sigma_\mu^{\mu\nu}(\vec{x}, \vec{v}_\mu, t)\rho_\mu(\vec{x}, \vec{v}_\mu, t), \tag{9}$$

where

$$\sigma_\mu^{1,2}(\vec{x}, \vec{v}_\mu, t) := \iint \hat{\sigma}_\mu^{1,2}(\vec{v}_1^0; \vec{v}_2^0; \vec{x}, t)\frac{\hat{\rho}_1(\vec{x}, \vec{v}_\mu, \vec{v}_1^0, t)}{\rho_1(\vec{x}, \vec{v}_\mu, t)} d^2\vec{v}_1^0\, d^2\vec{v}_2^0$$

and

$$\rho_\mu(\vec{x}, \vec{v}_\mu, t) := \int \hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t)d^2\vec{v}_\mu^0. \tag{10}$$

- Fourth, by the density gain $\hat{q}_\mu^+(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t)$ or density loss $\hat{q}_\mu^-(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t)$ per time unit. This gain or loss is caused by pedestrians who enter or leave the system $\mathcal{M}$ at a marginal place $\vec{x} \in \partial\mathcal{M}$ (for example, a house), with the intended velocity $\vec{v}_\mu^0 := \vec{v}^0 \in \mathcal{N}_\mu$ and the actual velocity $\vec{v}_\mu$:

$$\hat{q}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t) := \hat{q}_\mu^+(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t) - \hat{q}_\mu^-(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t). \tag{11}$$

3. Macroscopic equations

For further discussion we need the following notations:

$$\langle\rho_\mu\rangle := \int \rho_\mu(\vec{x}, \vec{v}_\mu, t)d^2\vec{v}_\mu = \int \hat{\rho}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t)d^2\vec{v}_\mu\, d^2\vec{v}_\mu^0$$
$$\langle\psi_\mu(\vec{v}_\mu, \vec{v}_\mu^0)\rangle := \int \psi(\vec{v}_\mu, \vec{v}_\mu^0)\frac{\hat{\rho}(\vec{x}, \vec{v}_\mu, \vec{v}_\mu^0, t)}{\langle\rho_\mu\rangle} d^2\vec{v}_\mu\, d^2\vec{v}_\mu^0$$
$$\delta\vec{v}_\mu := \vec{v}_\mu - \langle\vec{v}_\mu\rangle$$
$$\delta\vec{v}_\mu^0 := \vec{v}_\mu^0 - \langle\vec{v}_\mu^0\rangle$$

$$\varrho_\mu(\vec{x}, \vec{v}_\mu, t) := m_\mu \rho_\mu(\vec{x}, \vec{v}_\mu, t) \tag{12}$$

$$p_{\mu,\alpha\beta} := \langle\varrho_\mu\rangle\langle\delta v_{\mu,\alpha}\delta v_{\mu,\beta}\rangle = \int \delta v_{\mu,\alpha}\delta v_{\mu,\beta}\varrho_\mu(\vec{x}, \vec{v}_\mu, t) d^2\vec{v}_\mu \tag{13}$$

$$\begin{aligned} \vec{j}_{\mu,i} &:= \langle\varrho_\mu\rangle\left\langle \delta\vec{v}_\mu \frac{(\delta v_{\mu,i})^2}{2} \right\rangle \\ &= \int \delta\vec{v}_\mu \frac{(\delta v_{\mu,i})^2}{2} \varrho_\mu(\vec{x}, \vec{v}_\mu, t) d^2\vec{v}_\mu \end{aligned} \tag{14}$$

$$\begin{aligned} \chi_{\mu\nu}(\psi_\mu(\vec{v})) &:= \iiint \psi_\mu(\vec{v})\sigma^*_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu) \frac{\rho_\mu(\vec{x}, \vec{v}_\mu, t)}{\langle\rho_\mu\rangle} \frac{\rho_\nu(\vec{x}, \vec{v}_\nu, t)}{\langle\rho_\nu\rangle} \\ &\quad \times d^2\vec{v}_\mu\, d^2\vec{v}_\nu\, d^2\vec{v}^*_\mu \end{aligned} \tag{15}$$

$$\begin{aligned} \chi^{1,2}_\mu(\psi_\mu(\vec{v}_1)) &:= \int \psi_\mu(\vec{v}_\mu)\sigma^{1,2}_\mu(\vec{x}, \vec{v}_\mu, t) \frac{\rho_1(\vec{x}, \vec{v}_\mu, t)}{\langle\rho_1\rangle} d^2\vec{v}_\mu \\ &= \int \psi_\mu(\vec{v}_1)\sigma^{1,2}_\mu(\vec{x}, \vec{v}_1, t) \frac{\rho_1(\vec{x}, \vec{v}_1, t)}{\langle\rho_1\rangle} d^2\vec{v}_1 \end{aligned}$$

$$\begin{aligned} q_\mu(\vec{x}, \vec{v}_\mu, t) &:= \int \hat{q}_\mu(\vec{x}, \vec{v}_\mu, \vec{v}^0_\mu, t) d^2\vec{v}_\mu\, d^2\vec{v}^0_\mu \\ Q_\mu\left(\frac{\psi_\mu(\vec{v}_\mu)}{m_\mu}\right) &:= \int \frac{\psi_\mu(\vec{v}_\mu)}{m_\mu} m_\mu q_\mu(\vec{x}, \vec{v}_\mu, t) d^2\vec{v}_\mu \end{aligned} \tag{16}$$

Here, $\psi_\mu(\vec{v}_\mu, \vec{v}^0_\mu)$ is an arbitrary function of $\vec{v}_\mu$ and $\vec{v}^0_\mu$.

As far as pedestrians of type μ are concerned, we are mostly interested in their density $\langle\rho_\mu\rangle$, their mean velocity $\langle\vec{v}_\mu\rangle$, and the variance $\langle(\delta v_{\mu,i})^2\rangle$ of their velocity components $v_{\mu,i}$ (at a given place $\vec{x}$ and time t). Since it is formally equivalent and more easily comparable to fluid dynamics, we shall instead search equations for the *mass density*

$$\langle\varrho_\mu\rangle := m_\mu\langle\rho_\mu\rangle,$$

the mean *momentum density*

$$\langle\rho_\mu\rangle\langle m_\mu\vec{v}_\mu\rangle = \langle\varrho_\mu\rangle\langle\vec{v}_\mu\rangle,$$

and the mean *energy density* (in direction i)

$$\langle\epsilon_{\mu,i}\rangle := \langle\rho_\mu\rangle\left\langle \frac{m_\mu}{2} v^2_{\mu,i} \right\rangle = \langle\varrho_\mu\rangle\frac{\langle v_{\mu,i}\rangle^2}{2} + \langle\varrho_\mu\rangle\left\langle \frac{(\delta v_{\mu,i})^2}{2} \right\rangle.$$

By multiplication of (3) by $\psi_\mu(\vec{v}_\mu) = m_\mu$, $m_\mu\vec{v}_\mu$ or $m_\mu v^2_{\mu,i}/2$, and integration over $\vec{u}_\mu$, one can obtain the following equations (keeping in mind that the Gaussian surface integrals vanish; see chapter 2.10 of [18]):

$$\frac{\partial\langle\varrho_\mu\rangle}{\partial t} = -\frac{\partial}{\partial x_{\mu,\alpha}}\left(\langle\varrho_\mu\rangle\langle v_{\mu,\alpha}\rangle\right) + Q_\mu(1) \tag{17a}$$

$$+\sum_\nu \left[\frac{m_\mu}{m_\nu}\langle\varrho_\nu\rangle\chi^{\nu\mu}_\mu(1) - \langle\varrho_\mu\rangle\chi^{\mu\nu}_\mu(1)\right] \tag{17b}$$

for the mass density,

$$\frac{\partial\left(\langle\varrho_\mu\rangle\langle v_{\mu,\beta}\rangle\right)}{\partial t}$$

$$= -\frac{\partial}{\partial x_{\mu,\alpha}}\left(\langle\varrho_\mu\rangle\langle v_{\mu,\alpha}\rangle\langle v_{\mu,\beta}\rangle + p_{\mu,\alpha\beta}\right) + \langle\varrho_\mu\rangle\frac{f_{\mu,\beta}}{m_\mu} + Q_\mu(v_{\mu,\beta}) \tag{18a}$$

$$+ \langle\varrho_\mu\rangle\frac{1}{\tau_\mu}\left(\langle v^0_{\mu,\beta}\rangle - \langle v_{\mu,\beta}\rangle\right) \tag{18b}$$

$$+ \sum_\nu \langle\varrho_\mu\rangle\langle\varrho_\nu\rangle\frac{1}{m_\nu}\left[\chi_{\mu\nu}(v^*_{\mu,\beta}) - \chi_{\mu\nu}(v_{\mu,\beta})\right] \tag{18c}$$

$$+ \sum_\nu \left[\frac{m_\mu}{m_\nu}\langle\varrho_\nu\rangle\chi^{\nu\mu}_\mu(v_{\nu,\beta}) - \langle\varrho_\mu\rangle\chi^{\mu\nu}_\mu(v_{\mu,\beta})\right] \tag{18d}$$

for the momentum density, and

$$\frac{\partial\langle\epsilon_{\mu,i}\rangle}{\partial t}$$

$$= -\frac{\partial}{\partial x_{\mu,\alpha}}\left(\langle v_{\mu,\alpha}\rangle\langle\epsilon_{\mu,i}\rangle + p_{\mu,\alpha i}\langle v_{\mu,i}\rangle + j_{\mu,\alpha,i}\right) + \langle\varrho_\mu\rangle\langle v_{\mu,i}\rangle\frac{f_{\mu,i}}{m_\mu} + Q_\mu\left(\frac{v^2_{\mu,i}}{2}\right) \tag{19a}$$

$$+ \langle\varrho_\mu\rangle\frac{1}{\tau_\mu}\left(\langle v^0_{\mu,i}\rangle^2 - \langle v_{\mu,i}\rangle^2\right) \tag{19b}$$

$$+ \langle\varrho_\mu\rangle\frac{1}{\tau_\mu}\left(\langle(\delta v^0_{\mu,i})^2\rangle - \langle(\delta v_{\mu,i})^2\rangle\right) \tag{19c}$$

$$+ \sum_\nu \langle\varrho_\mu\rangle\langle\varrho_\nu\rangle\frac{1}{m_\nu}\left[\chi_{\mu\nu}\left(\frac{v^{*\,2}_{\mu,i}}{2}\right) - \chi_{\mu\nu}\left(\frac{v^2_{\mu,i}}{2}\right)\right] \tag{19d}$$

$$+ \sum_\nu \left[\frac{m_\mu}{m_\nu}\langle\varrho_\nu\rangle\chi^{\nu\mu}_\mu\left(\frac{v^2_{\nu,i}}{2}\right) - \langle\varrho_\mu\rangle\chi^{\mu\nu}_\mu\left(\frac{v^2_{\mu,i}}{2}\right)\right] \tag{19e}$$

for the energy density. We have used the Einsteinian summation convention to sum over terms in which the Greek indices α, β, or γ occur twice.

3.1 Interpretation

Equations (17a), (18a), and (19a) are the well-known hydrodynamic equations (see chapters 2.4 and 2.10 of [18]). Equations (18c) and (19d) describe the effects of interactions between two individuals of type μ and ν (for details see section 6). These terms do not vanish, as they would if conservation of momentum and energy were fulfilled in a *strict* sense (see chapter 2.10 of [18]). However, since the individuals try to approach the intended velocity $\vec{v}^{\,0}_\mu$, there is a *tendency* to restore momentum and energy that is described by (18b), (19b), and (19c).

Equations (17b), (18d), and (19e) are additional terms due to individuals who change their type of motion. In the following we will assume the special case that these terms (as well as the terms $Q_\mu(\psi_\mu(\vec{v}_\mu)/m_\mu)$, due to individuals entering or leaving the system $\mathcal{M}$) vanish, by compensation of inflow into and outflow from μ. For concrete situations the quantities $\chi_\mu^{;}$ and $Q_\mu(.)$ must be obtained empirically.

In thermodynamics, $p_{\mu,\alpha\beta}$ is termed the tensor of *pressure*, and $\vec{j}_{\mu,i}$ is called the *heat flow*. For pedestrians $\vec{j}_{\mu,i}$ describes the tendency of the velocity variance $\langle(\delta v_{\mu,i})^2\rangle$ to equalize over time (see (31)). The variance

$$\theta_{\mu,i} := \langle(\delta v_{\mu,i})^2\rangle \equiv k_{\scriptscriptstyle B} T_{\mu,i}/m_\mu$$

is the thermodynamic equivalent to the *absolute temperature* $T_{\mu,i}$ in direction i. $p_{\mu,\alpha\beta}n_\beta$ represents the force used by individuals of type μ to change their movement when crossing a line of unit length l (or, more precisely, the component of this force in the direction $\vec{n}$ perpendicular to the line). Approximate expressions for $p_{\mu,\alpha\beta}$ and $\vec{j}_{\mu,i}$ are derived in sections 4 and 5.

3.2 Problems of small densities

The densities $\hat{\rho}_\mu$ of pedestrian crowds are usually very small. As a consequence, equation (3) will not be fulfilled very well, and a discrete formulation would be more appealing (see [11]). However, we can begin with the continuity equation (20), which holds better since the densities ρ_μ are only moderately small. The macroscopic equations will still be better fulfilled, because they are equations for the mean values $\langle\varrho_\mu\rangle$, $\langle\vec{v}_\mu\rangle$, and $\langle\epsilon_{\mu,i}\rangle$; they could also be set up by plausibility considerations.

In order to have small fluctuations of the variables $\langle\varrho_\mu\rangle$, $\langle\vec{v}_\mu\rangle$, and $\langle\epsilon_{\mu,i}\rangle$ over time, $\hat{\rho}_\mu$ in equation (1) must be averaged over a finite area A and a finite volume V, which should be sufficiently large. If T denotes the time scale (apart from fluctuations) for the temporal change of $\langle\varrho_\mu\rangle$, $\langle\vec{v}_\mu\rangle$, and $\langle\epsilon_{\mu,i}\rangle$, these variables can also be averaged over time intervals $\Delta t \ll T$, as follows:

$$\overline{\langle\varrho_\mu(\vec{x},t)\rangle\langle\psi_\mu(\vec{x},t)\rangle} := \frac{1}{\Delta t}\int\limits_{t-\Delta t/2}^{t+\Delta t/2} \langle\varrho_\mu(\vec{x},t')\rangle\langle\psi_\mu(\vec{x},t')\rangle\, dt'.$$

Then, equations (17) through (19) will be proper approximations for the movement of pedestrians.

Another complication associated with low densities is that *Knudsen corrections* [12] must be taken into account. According to these corrections the "temperature" $\theta_{\mu,i}$ and the tangential velocity $\langle v_{\mu,\|}\rangle$ change discontinuously at a boundary $\partial\mathcal{M}$, which seems, therefore, to be shifted by a small distance ξ that is comparable to the mean interaction-free path (see §14 of [19]).

4. Pedestrians in equilibrium

In order to calculate $p_{\mu,\alpha\beta}$, $\vec{j}_{\mu,i}$, and $\chi_{\mu\nu}$, we need the explicit form of ρ_μ (see and (12) through (15)). ρ_μ is the density of individuals of type μ at place $\vec{x}$ and time t having the actual velocity $\vec{v}_\mu$ but arbitrary intended velocity

$\vec{v}^0_\mu$ (see (10)). It is directly measurable in pedestrian crowds. By integration of (3) over $\vec{v}^0_\mu$ we obtain the following theoretical dependence (compare to [27, 26]):

$$\frac{d\rho_\mu}{dt} \equiv \frac{\partial \rho_\mu}{\partial t} + \nabla_{\vec{x}} \left(\rho_\mu \vec{v}_\mu\right) + \nabla_{\vec{v}_\mu} \left(\rho_\mu \frac{\vec{f}_\mu}{m_\mu}\right)$$
$$= \frac{\rho^0_\mu - \rho_\mu}{\tau_\mu} + \sum_\nu S_{\mu\nu} + \sum_\nu C_{\mu\nu} + q_\mu \tag{20}$$

The temporal development of the density ρ_μ is given by a tendency to walk with the intended velocity $\vec{v}^0_\mu$ (see (4)), by the effects $S_{\mu\nu}$ of pair interactions (see (7)), by the effects $C_{\mu\nu}$ of pedestrians changing their type of motion (see (9)), and by the effect q_μ of pedestrians entering or leaving the system $\mathcal{M}$ (see (11) and (16)). The last two effects shall be be neglected in the following discussion (see the comment in section 3.1).

Equation (20) can be solved in a suitable approximation by the recursive method of Chapman and Enskog [4, 5]. The lowest order approximation presupposes the condition $d\rho^e_\mu/dt = 0$ of *local equilibrium*, which is approximately fulfilled by the Gaussian distribution

$$\rho^e_\mu(\vec{x}, \vec{v}_\mu, t) = \langle\rho_\mu\rangle \cdot \frac{1}{2\pi b_\mu \theta_{\mu,\|}} \mathrm{e}^{-[(v_{\mu,\|}-\langle v_{\mu,\|}\rangle)^2/(2\theta_{\mu,\|})+(v_{\mu,\perp}-\langle v_{\mu,\perp}\rangle)^2/(2\theta_{\mu,\perp})]} \tag{21}$$

according to empirical data [13, 14, 23]. The quantity

$$(b_\mu)^2 := \frac{\theta_{\mu,\perp}}{\theta_{\mu,\|}} \le 1$$

describes the fact that the velocity variance $\theta_{\mu,i}$ perpendicular ($\perp$) to the mean intended direction of movement $\langle\vec{v}^0_\mu\rangle$ is normally less than that parallel ($\|$) to it [23].

For each type μ of motion let us perform a particular transformation, as follows:

$$\vec{x} \longrightarrow \vec{X}_\mu := \begin{pmatrix} x_{\mu,\|} \\ x_{\mu,\perp}/b_\mu \end{pmatrix}$$

$$\vec{v}_\mu \longrightarrow \vec{V}_\mu := \begin{pmatrix} v_{\mu,\|} \\ v_{\mu,\perp}/b_\mu \end{pmatrix}$$

$$\vec{f}_\mu \longrightarrow \vec{F}_\mu := \begin{pmatrix} f_{\mu,\|} \\ f_{\mu,\perp}/b_\mu \end{pmatrix}$$

$$\langle\varrho_{\mu,\perp}(\vec{x})\rangle := \frac{1}{\Delta x_\|} \int\limits_{x_\|-\Delta x_\|/2}^{x_\|+\Delta x_\|/2} \langle\varrho_\mu\rangle dx_{\mu,\|} \longrightarrow b_\mu \langle\varrho_{\mu,\perp}(\vec{X}_\mu)\rangle$$

$$p_{\mu,\alpha\beta} =: \epsilon_{\mu,\alpha\gamma} P_{\mu,\gamma\beta} \longrightarrow P_{\mu,\alpha\beta}$$

$$j_{\mu,\alpha,i} =: \epsilon_{\mu,\alpha\beta} J_{\mu,\beta,i} \longrightarrow J_{\mu,\alpha,i}$$

with

$$\underline{\epsilon}_\mu \equiv (\epsilon_{\mu,\alpha\beta}) := \begin{pmatrix} 1 & 0 \\ 0 & b_\mu^2 \end{pmatrix}.$$

This transformation stretches the direction perpendicular to $\langle \vec{v}_\mu^0 \rangle$ by the factor $1/b_\mu$, and simplifies (17) through (19) to *isotropic* equations (equations with local rotational symmetry). With

$$\theta_\mu := \langle (dV_{\mu,\alpha})^2 \rangle = \theta_{\mu,\|}$$

we get

$$P^e_{\mu,\alpha\beta} = \langle \varrho_\mu \rangle \theta_\mu \delta_{\alpha\beta} =: P^e_\mu \delta_{\alpha\beta} \tag{22}$$

for the pressure, and

$$J^e_{\mu,\alpha,i} = 0 \tag{23}$$

for the "heat flow" (see chapter 2.10 of [18]). In addition, for $\vec{F}_\mu = 0$ the equations

$$\frac{\partial \langle \varrho_\mu \rangle}{\partial t} + \langle V_{\mu,\alpha} \rangle \frac{\partial \langle \varrho_\mu \rangle}{\partial X_{\mu,\alpha}} = -\langle \varrho_\mu \rangle \frac{\partial \langle V_{\mu,\alpha} \rangle}{\partial X_{\mu,\alpha}}$$

$$\frac{\partial \langle V_{\mu,\beta} \rangle}{\partial t} + \langle V_{\mu,\alpha} \rangle \frac{\partial \langle V_{\mu,\beta} \rangle}{\partial X_{\mu,\alpha}} = -\frac{1}{\langle \varrho_\mu \rangle} \frac{\partial P^e_\mu}{\partial X_{\mu,\beta}} + \frac{k^1_\mu}{\tau_\mu} \Big(\langle V^e_{\mu,\beta} \rangle - \langle V_{\mu,\beta} \rangle \Big)$$

$$\frac{\partial \theta_\mu}{\partial t} + \langle V_{\mu,\alpha} \rangle \frac{\partial \theta_\mu}{\partial X_{\mu,\alpha}} = -\frac{P^e_\mu}{\langle \varrho_\mu \rangle} \frac{\partial \langle V_{\mu,\alpha} \rangle}{\partial X_{\mu,\alpha}} + \frac{k^2_\mu}{\tau_\mu} (\theta^e_\mu - \theta_\mu)$$

can be derived approximately from (17) through (19), (22), and (23) (see chapter 16.2 of [31]). Obviously, $\langle V_{\mu,\beta} \rangle$ will vary around $\langle V^e_{\mu,\beta} \rangle$ after some time. Therefore, we transform the above equations to *moving coordinates* $\vec{X}'_\mu(t) := \vec{X}_\mu - \langle \vec{V}^e_\mu \rangle t$. After some steps one obtains:

$$\frac{d\langle \varrho_\mu \rangle}{dt} := \frac{\partial \langle \varrho_\mu \rangle}{\partial t} + \langle V_{\mu,\alpha} \rangle \frac{\partial \langle \varrho_\mu \rangle}{\partial X_{\mu,\alpha}} = -\langle \varrho_\mu \rangle \frac{\partial \langle V_{\mu,\alpha} \rangle}{\partial X_{\mu,\alpha}} \tag{24}$$

$$\frac{d\langle V_{\mu,\beta} \rangle}{dt} := \frac{\partial \langle V_{\mu,\beta} \rangle}{\partial t} + \langle V_{\mu,\alpha} \rangle \frac{\partial \langle V_{\mu,\beta} \rangle}{\partial X_{\mu,\alpha}} = -\frac{1}{\langle \varrho_\mu \rangle} \frac{\partial P^e_\mu}{\partial X_{\mu,\beta}} - \frac{k^1_\mu}{\tau_\mu} \langle V_{\mu,\beta} \rangle \tag{25}$$

$$\frac{d\theta_\mu}{dt} := \frac{\partial \theta_\mu}{\partial t} + \langle V_{\mu,\alpha} \rangle \frac{\partial \theta_\mu}{\partial X_{\mu,\alpha}} = -\frac{P^e_\mu}{\langle \varrho_\mu \rangle} \frac{\partial \langle V_{\mu,\alpha} \rangle}{\partial X_{\mu,\alpha}} + \frac{k^2_\mu}{\tau_\mu} (\theta^e_\mu - \theta_\mu) \tag{26}$$

Equations (24) through (26) agree with the *Euler equations* if the last terms of (25) and (26) are negligible, which shall be assumed in the following. The quantities $\langle V^e_{\mu,\alpha} \rangle \equiv \langle V^e_{\mu,\alpha} \rangle(\langle \varrho_\mu \rangle)$ and $\theta^e_\mu \equiv \theta^e_\mu(\langle \varrho_\mu \rangle)$ are the stationary and homogeneous solutions of (18) and (19), for which the temporal and spatial derivatives vanish.

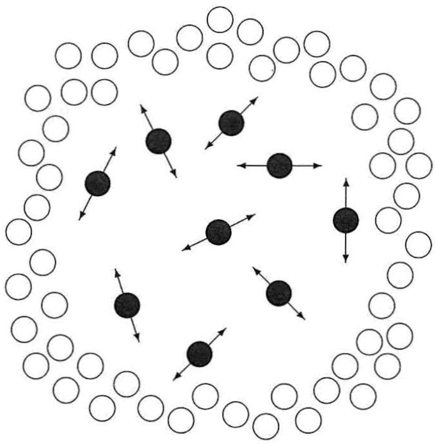

Figure 1: Behavior on a dance floor. Dancing individuals (filled circles) show a lower density than individuals standing around (empty circles), since they intend to move with a greater velocity variance.

4.1 Behavior on a dance floor

On a dance floor like that of a discotheque, two types of motion can be observed: type 1 describing dancing individuals, and type 2 describing individuals standing around and looking on. We can assume an isotropic case, which implies $b_\mu = 1$ and $\theta_{\mu,\|} = \theta_{\mu,\perp}$. According to (19c), the variance $\langle(\delta v^0_{\mu,i})^2\rangle$ of the intended velocities $v^0_{\mu,i}$ is *causal* for the temperature $\theta_{\mu,i}$, in other words, for the variance $\langle(\delta v_{\mu,i})^2\rangle$ of the actual velocities $v_{\mu,i}$. Thus, for the temperatures θ_1 and θ_2 of individuals dancing and individuals standing around,

$$\theta_1 > \theta_2,$$

since the dancing individuals intend to move with higher variance $\langle(\delta v^0_{1,i})^2\rangle > \langle(\delta v^0_{2,i})^2\rangle$. (This is so even in the case of equilibrium, since we must take into account the effect of the Knudsen corrections; see section 3.2.) The equilibrium condition of equal pressure

$$P^e_1 = P^e_2$$

now leads to

$$\langle\varrho_1\rangle = \frac{\theta_2}{\theta_1}\langle\varrho_2\rangle < \langle\varrho_2\rangle$$

(see (22)). Therefore, the dancing individuals will exhibit a lower density than the individuals standing around (see figure 1). This effect can readily be observed.

4.2 Propagation of density waves

In the case of very large τ_μ we have $\langle\vec{V}_\mu\rangle \approx \langle\vec{V}^e_\mu\rangle(\langle\varrho_\mu\rangle)$, and equation (24) can be put into the form

$$\frac{\partial\langle\varrho_\mu\rangle}{\partial t} + C_\mu(\langle\varrho_\mu\rangle)\nabla_{\vec{X}_\mu}\langle\varrho_\mu\rangle = 0.$$

This equation describes nonlinear density waves propagating with velocity

$$C_\mu(\langle\varrho_\mu\rangle) := \left(\langle\vec{V}^e_\mu\rangle + \langle\varrho_\mu\rangle\frac{\partial\langle\vec{V}^e_\mu\rangle}{\partial\langle\varrho_\mu\rangle}\right),$$

and has been discussed by Whitham in detail (see [20] and chapter 2.1 of [35]).

We shall instead study the case of small τ_μ, were the Euler equations are applicable. In a nearly homogenuous pedestrian crowd with small density variations one can assume

$$\langle\vec{V}_\mu\rangle\cdot\nabla_{\vec{X}_\mu}\langle\varrho_\mu\rangle \approx 0, \quad \langle\vec{V}_\mu\rangle\nabla_{\vec{X}_\mu}\langle\vec{V}_\mu\rangle \approx \vec{0}, \quad \langle\vec{V}_\mu\rangle\nabla_{\vec{X}_\mu}\theta_\mu \approx 0.$$

From the Euler equations (24) through (26) the equation

$$\frac{\partial^2\langle\varrho_\mu\rangle}{\partial t^2} - \Delta_{\vec{X}_\mu}P^e_\mu = 0$$

can then be derived (see chapter 16.2 of [31]). Subtracting $\langle\varrho_\mu\rangle/\theta_\mu\times(26)$ from (24), and making use of (22), the *adiabatic law*

$$\frac{P^e_\mu}{\langle\varrho_\mu\rangle^2} = \text{constant}$$

can be shown, which leads to the *linear wave equation*

$$\langle\varrho_\mu\rangle\kappa^S_\mu\frac{\partial^2}{\partial t^2}\langle\varrho_\mu\rangle - \Delta_{\vec{X}_\mu}\langle\varrho_\mu\rangle = 0 \tag{27}$$

with the *adiabatic compressibility*

$$\kappa^S_\mu := \frac{1}{\langle\varrho_\mu\rangle}\left(\frac{\partial P^e_\mu}{\partial\langle\varrho_\mu\rangle}\right)_S = \frac{1}{2}\frac{\langle\varrho_\mu\rangle}{P^e_\mu} = \frac{1}{2\langle\varrho_\mu\rangle\theta_\mu}$$

(see chapter 16.2 of [31]). Equation (27) describes the propagation of density waves with velocity

$$c_\mu = \frac{1}{\sqrt{\langle\varrho_\mu\rangle\kappa^S_\mu}} \tag{28}$$

relative to $\vec{X}'_\mu(t) = \vec{X}_\mu - \langle\vec{V}^e_\mu\rangle t$. On the other hand, the velocity of propagation is given by the mean distance $d_\mu = 1/\sqrt{\langle\rho_\mu\rangle}$ of a succeeding individual, divided by its mean reaction time ζ_μ, as follows:

$$c_\mu = \frac{1}{\sqrt{\langle\rho_\mu\rangle}\zeta_\mu}.$$

Inserting this result into (28), it follows that for small densities $\langle\rho_\mu\rangle$ the adiabatic compressibility grows with the mean reaction time ζ_μ of individuals according to

$$\kappa_\mu^S = \frac{(\zeta_\mu)^2}{m_\mu}.$$

5. Nonequilibrium equations

In cases of deviations

$$\delta\rho_\mu(\vec{X}_\mu, \vec{V}_\mu, t) := \rho_\mu(\vec{X}_\mu, \vec{V}_\mu, t) - \rho_\mu^e(\vec{X}_\mu, \vec{V}_\mu, t) \tag{29}$$

from local equilibrium $\rho_\mu^e(\vec{X}_\mu, \vec{V}_\mu, t) := \rho_\mu^e(\langle\rho_\mu\rangle, \langle\vec{V}_\mu\rangle, \theta_\mu)$, we must find a higher order approximation of equation (20) than in section 4. If the deviations $\delta\rho_\mu$ remain small compared to ρ_μ, we can linearize equation (20) around ρ_μ^e and get

$$\frac{d\rho_\mu^e}{dt} \approx \frac{d\rho_\mu^e}{dt} + \frac{d\delta\rho_\mu}{dt} = \frac{d\rho_\mu}{dt} \approx -\frac{\delta\rho_\mu}{\tau_\mu} + \sum_\nu \frac{\delta\rho_\nu}{\tau_{\mu\nu}}$$

(see chapter 15 of [17]). Here, $\tau_{\mu\nu}$ is the mean interaction-free time between an individual of type μ and individuals of type ν (see (35), and chapter 16.2 of [31]). From (21), (22), and (24) through (26), one finds the following (see [31]):

$$\begin{aligned}\frac{d\rho_\mu^e}{dt} &= \frac{\partial\rho_\mu^e}{\partial\langle\rho_\mu\rangle}\frac{d\langle\rho_\mu\rangle}{dt} + \left(\nabla_{\langle\vec{V}_\mu\rangle}\rho_\mu^e\right)\cdot\frac{d\langle\vec{V}_\mu\rangle}{dt} + \frac{\partial\rho_\mu^e}{\partial\theta_\mu}\frac{d\theta_\mu}{dt} \\ &= \rho_\mu^e\left[\frac{\delta\vec{V}_\mu}{\theta_\mu}\cdot\nabla_{\vec{X}_\mu}\theta_\mu\left(\frac{(\delta\vec{V}_\mu)^2}{2\theta_\mu} - 2\right)\right] \\ &\quad + \rho_\mu^e\left[\frac{1}{\theta_\mu}\left(\delta V_{\mu,\alpha}\frac{\partial\langle V_{\mu,\beta}\rangle}{\partial X_{\mu,\alpha}}\delta V_{\mu,\beta} - \frac{(\delta\vec{V}_\mu)^2}{2}\nabla_{\vec{X}_\mu}\langle\vec{V}_\mu\rangle\right)\right].\end{aligned}$$

If

$$(\tau_\mu\delta_{\mu\nu} + \overline{\tau}_{\mu\nu})$$

denotes the inverse matrix of

$$\left(\frac{1}{\tau_\mu}\delta_{\mu\nu} - \frac{1}{\tau_{\mu\nu}}\right),$$

the relation

$$\delta\rho_\mu = -\tau_\mu\frac{d\rho_\mu^e}{dt} - \sum_\nu \overline{\tau}_{\mu\nu}\frac{d\rho_\nu^e}{dt} \tag{30}$$

leads (because of $\rho_\mu = \rho_\mu^e + \delta\rho_\mu$) to the corrected tensor of pressure

$$P_{\mu,\alpha\beta} = P_\mu^e \delta_{\alpha\beta} - \eta_\mu \Lambda_{\mu,\alpha\beta} - \sum_\nu \eta_{\mu\nu} \Lambda_{\nu,\alpha\beta}$$

(see (13)), and the corrected heat flow

$$J_{\mu,\alpha} = -\kappa_\mu \frac{\partial \theta_\mu}{\partial X_{\mu,\alpha}} - \sum_\nu \kappa_{\mu\nu} \frac{\partial \theta_\nu}{\partial X_{\nu,\alpha}} \tag{31}$$

(see (14)). Here,

$$\Lambda_{\mu,\alpha\beta} := \left(\frac{\partial \langle V_{\mu,\alpha} \rangle}{\partial X_{\mu,\beta}} + \frac{\partial \langle V_{\mu,\beta} \rangle}{\partial X_{\mu,\alpha}} - \frac{\partial \langle V_{\mu,\alpha} \rangle}{\partial X_{\mu,\alpha}} \delta_{\alpha\beta} \right)$$

is the *tensor of strain*,

$$\eta_\mu = \tau_\mu \theta_\mu \langle \varrho_\mu \rangle \qquad \text{and} \qquad \eta_{\mu\nu} = \overline{\tau}_{\mu\nu} \theta_\nu \langle \varrho_\nu \rangle$$

are coefficients of the *shear viscosity*, and

$$\kappa_\mu = 2\tau_\mu \theta_\mu \langle \varrho_\mu \rangle \qquad \text{and} \qquad \kappa_{\mu\nu} = 2\overline{\tau}_{\mu\nu} \theta_\nu \langle \varrho_\nu \rangle$$

are coefficients of the *thermal conductivity*.

Note that the main effect of restoring the local equilibrium distribution ρ_μ^e results from the tendency to approach the intended velocity distribution $\rho_\mu^0(\vec{X}_\mu, \vec{V}_\mu, t)$ with a time constant τ_μ, but not, as usual, from interaction processes (see chapter 13.3 of [30]). Therefore, in contrast to ordinary fluids, the viscosity η_μ is dependent on the density $\langle \rho_\mu \rangle$ (see pages 323 and 327 in [31]). For vanishing densities $\langle \rho_\nu \rangle \longrightarrow 0$ the interaction rates $1/\tau_{\mu\nu}$ become negligible (see (35)), and $\overline{\tau}_{\mu\nu}$, $\eta_{\mu\nu}$, and $\kappa_{\mu\nu}$ vanish in comparison with τ_μ, η_μ, and κ_μ, respectively. According to (30), the deviation $\delta\rho_\mu$ from the local equilibrium distribution ρ_μ^e, and, therefore, the viscosity and the thermal conductivity, are all consequences of finite relaxation times τ_μ and $\tau_{\mu\nu}$.

5.1 Effect of viscosity

Where pedestrians are concerned, the effect of viscosity is not compensated for by a gradient of pressure as in ordinary fluids, but instead by the tendency of pedestrians to reach their intended velocity described by (18b). In the case of a stationary flow in one direction (that is, of one type of motion) parallel to the boundaries $\partial\mathcal{M}$, we have essentially the equation

$$0 = \frac{\partial \langle \varrho_\mu \rangle \langle V_{\mu,\|} \rangle}{\partial t} = \eta_\mu \frac{\partial^2}{\partial X_{\mu,\perp}^2} \langle V_{\mu,\|} \rangle + \langle \varrho_\mu \rangle \frac{1}{\tau_\mu} \left(\langle V_{\mu,\|}^0 \rangle - \langle V_{\mu,\|} \rangle \right), \tag{32}$$

if $\eta_\mu \gg \eta_{\mu\mu}$ (see (18a) and (18b)). For a lane of effective width $2W$ (with the origin $X_{\mu,\perp} = 0$ in the middle), equation (32) has the *hyperbolic* solution

$$\langle V_{\mu,\|} \rangle = \langle V_{\mu,\|}^0 \rangle \left[1 - \frac{\cosh(X_{\mu,\perp}/D_\mu)}{\cosh(W/D_\mu)} \right] \tag{33}$$

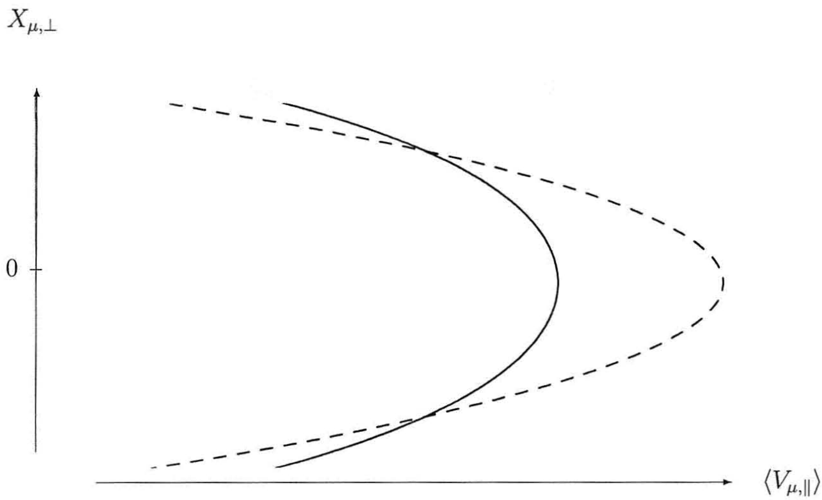

Figure 2: Effect of viscosity (internal friction). Ordinary fluids show a parabolic velocity profile (broken line). In contrast to this, a hyperbolic velocity profile is expected for pedestrian crowds (solid line). Whereas in ordinary fluids the internal friction is compensated for by a pressure gradient, in pedestrian crowds this role is played by the accelerating effect of the intended velocity. Due to the Knudsen corrections the fluid slips at the boundary (in other words, the *effective* width is greater than the *actual* width).

with a *boundary layer* of width

$$D_\mu = \sqrt{\frac{\eta_\mu \tau_\mu}{\langle \varrho_\mu \rangle}} = \tau_\mu \sqrt{\theta_\mu}.$$

In comparison to this, a pressure gradient

$$\frac{\partial P^e_{\mu,\|\|}}{\partial X_{\mu,\|}} := \frac{\Delta P^e_\mu}{L}$$

generating the driving force, would lead instead to the *parabolic* solution

$$\langle V_{\mu,\|} \rangle = \frac{\Delta P^e_\mu}{\eta_\mu L}(W^2 - X^2_{\mu,\perp}), \tag{34}$$

and the mean tangential velocity $\langle V_{\mu,\|} \rangle$ would depend on the length L of the lane (see chapter 3.3 of [21]). Both the hyperbolic solution (33) and the parabolic solution (34) are depicted in figure 2.

6. Effects of interactions

The *scattering rates* $\sigma^*_{\mu\nu}$ of interactions (see (7) and (8)) are proportional to the relative velocity $\|\vec{v}_\mu - \vec{v}_\nu\|$ of the interacting pedestrians and to the

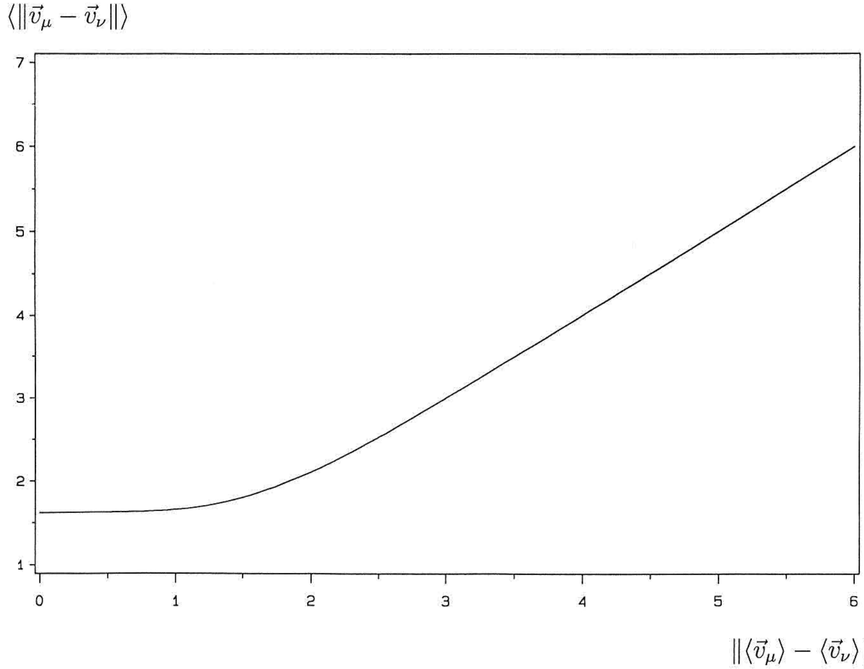

Figure 3: The mean relative velocity $\langle \|\vec{v}_\mu - \vec{v}_\nu\| \rangle$, dependent upon $\|\langle \vec{v}_\mu \rangle - \langle \vec{v}_\nu \rangle\|$, for the special case $\theta_{\mu,i} = 1 = \theta_{\nu,i}$.

scattering cross section $l_{\mu\nu}$ (which is a length of the order of a pedestrian's stride) [18]:

$$\sigma^*_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu) = l_{\mu\nu} \|\vec{v}_\mu - \vec{v}_\nu\| P_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu).$$

The mean rate of interactions of an individual of type μ with individuals of type ν is

$$\begin{aligned} \frac{1}{\tau_{\mu\nu}} &:= \frac{1}{\langle \rho_\mu \rangle} \iint \rho_\mu(\vec{x}, \vec{v}_\mu, t) \rho_\nu(\vec{x}, \vec{v}_\nu, t) l_{\mu\nu} \|\vec{v}_\mu - \vec{v}_\nu\| d^2 \vec{v}_\mu \, d^2 \vec{v}_\nu \\ &= \langle \rho_\nu \rangle l_{\mu\nu} \langle \|\vec{v}_\mu - \vec{v}_\nu\| \rangle \end{aligned} \tag{35}$$

where $\tau_{\mu\nu}$ is the corresponding mean interaction-free time [18]. For the mean relative velocity $\langle \|\vec{v}_\mu - \vec{v}_\nu\| \rangle$ (see figure 3), the following limits can be calculated by making use of (21) and (29), and neglecting terms of order $O(\delta\rho_\mu)$:

$$\langle \|\vec{v}_\mu - \vec{v}_\nu\| \rangle \approx \begin{cases} \sqrt{\pi \theta_\mu} & \text{if } \langle \vec{v}_\mu \rangle \approx \langle \vec{v}_\nu \rangle, \, \theta_\mu \approx \theta_\nu \\ \|\langle \vec{v}_\mu \rangle - \langle \vec{v}_\nu \rangle\| & \text{if } \|\langle \vec{v}_\mu \rangle - \langle \vec{v}_\nu \rangle\| \gg \theta_\mu, \theta_\nu. \end{cases} \tag{36}$$

Let us introduce

$$\tau^*_{\mu\nu} = \tau^*_{\mu\nu}(\langle\vec{v}_\mu\rangle, \langle\vec{v}_\nu\rangle, \theta_\mu, \theta_\nu) := \tau_{\mu\nu}\langle\varrho_\nu\rangle = \tau_{\mu\nu} m_\nu \langle\rho_\nu\rangle, \tag{37}$$

and the *total rate of interactions*

$$\frac{1}{\hat{\tau}_\mu} := \sum_\nu \frac{1}{\tau_{\mu\nu}}. \tag{38}$$

Then,

$$r_\mu = \mathrm{e}^{-\Delta t_\mu/\hat{\tau}_\mu} \tag{39}$$

is the probability of having the opportunity to pass an individual on the right or left, if this requires an interaction-free time of at least Δt_μ (see chapter 12.1 of [30]).

$$P_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu) = \sum_k P^k_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu)$$

is the probability that two individuals of types μ and ν have velocities $\vec{v}_\mu$ and $\vec{v}_\nu$ before their interaction, and the individual of type μ has the velocity $\vec{v}^*_\mu$ thereafter. We shall distinguish three types k of interaction, as follows.

If an individual of type μ is hindered by another individual of type ν, he tries to pass the other to the right with probability $p_{\mu\nu}$, or to the left with probability $1 - p_{\mu\nu}$:

$$P^1_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu) = r_\mu \left[p_{\mu\nu}\delta(\vec{v}^*_\mu - \underline{S}_{\beta_{\mu\nu}}\vec{v}_\mu) + (1 - p_{\mu\nu})\delta(\vec{v}^*_\mu - \underline{S}^{-1}_{\beta_{\mu\nu}}\vec{v}_\mu)\right].$$

$\vec{v}^*_\mu = \underline{S}_{\beta_{\mu\nu}}\vec{v}_\mu$ describes a rotation of velocity $\vec{v}_\mu$ to the right side by an angle $\beta_{\mu\nu}$, in order to avoid the hindering pedestrian; $\vec{v}^*_\mu = \underline{S}^{-1}_{\beta_{\mu\nu}}\vec{v}_\mu$ describes the inverse rotation to the left side.

In cases where it is impossible to avoid an individual of type ν having a velocity $\vec{v}_\nu$, the individual of type μ tries to walk with velocity $\vec{v}^*_\mu = \vec{v}_\nu$, if $\vec{v}_\nu$ has a positive component $\vec{v}_\nu \cdot \vec{e}_\mu > 0$ in the intended direction of motion $\vec{e}_\mu := \vec{v}^0_\mu / \|\vec{v}^0_\mu\|$:

$$P^2_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu) = (1 - r_\mu)\delta(\vec{v}^*_\mu - \vec{v}_\nu)\Theta(\vec{v}_\nu \cdot \vec{e}_\mu > 0).$$

This corresponds to situations in which one moves for a short time within a gap behind a pedestrian who is in the way (or sometimes, for different directions $\vec{e}_\mu \neq \vec{e}_\nu$, in front of him). The *decision function* Θ is defined by

$$\Theta(x) := \begin{cases} 1 & \text{if } x \text{ is fulfilled} \\ 0 & \text{otherwise.} \end{cases}$$

If $\vec{v}_\nu \cdot \vec{e}_\mu < 0$ (the case of a negative component of the hindering pedestrian's velocity $\vec{v}_\nu$ with respect to the intended direction $\vec{e}_\mu$ of movement), it is better for an individual to stop ($\vec{v}^*_\mu = \vec{0}$):

$$P^3_{\mu\nu}(\vec{v}_\mu, \vec{v}_\nu; \vec{v}^*_\mu) = (1 - r_\mu)\delta(\vec{v}^*_\mu - \vec{0})\Theta(\vec{v}_\nu \cdot \vec{e}_\mu < 0).$$

This results in

$$\frac{1}{m_\nu}\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle\left[\chi^k_{\mu\nu}(\vec{v}^*_\mu) - \chi^k_{\mu\nu}(\vec{v}_\mu)\right]$$

$$\approx \frac{\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle}{\tau^*_{\mu\nu}} \cdot \begin{cases} r_\mu[p_{\mu\nu}\langle \underline{S}_{\beta_{\mu\nu}}\vec{v}_\mu\rangle + (1-p_{\mu\nu})\langle \underline{S}^{-1}_{\beta_{\mu\nu}}\vec{v}_\mu\rangle - \langle\vec{v}_\mu\rangle] & k=1 \quad (40a) \\ (1-r_\mu)\langle\Theta_{\mu\nu}\rangle[\langle\vec{v}_\nu\rangle - \langle\vec{v}_\mu\rangle] & k=2 \quad (40b) \\ -(1-r_\mu)(1-\langle\Theta_{\mu\nu}\rangle)\langle\vec{v}_\mu\rangle & k=3 \quad (40c) \end{cases}$$

and

$$\frac{1}{m_\nu}\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle\left[\chi^k_{\mu\nu}\left(\frac{(v^*_{\mu,i})^2}{2}\right) - \chi^k_{\mu\nu}\left(\frac{(v_{\mu,i})^2}{2}\right)\right]$$

$$\approx \frac{\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle}{\tau^*_{\mu\nu}} \cdot \begin{cases} r_\mu[p_{\mu\nu}\langle(\underline{S}_{\beta_{\mu\nu}}\vec{v}_\mu)^2_i\rangle + (1-p_{\mu\nu})\langle(\underline{S}^{-1}_{\beta_{\mu\nu}}\vec{v}_\mu)^2_i\rangle - \langle v^2_{\mu,i}\rangle] \quad (41a) \\ \qquad k=1 \\ (1-r_\mu)\langle\Theta_{\mu\nu}\rangle[\langle v^2_{\nu,i}\rangle - \langle v^2_{\mu,i}\rangle] & k=2 \quad (41b) \\ -(1-r_\mu)(1-\langle\Theta_{\mu\nu}\rangle)\langle v^2_{\mu,i}\rangle & k=3 \quad (41c) \end{cases}$$

which explicitly allows the calculation of (18c) and (19d). We have used the abbreviation

$$\langle\Theta_{\mu\nu}\rangle = \langle\Theta_{\mu\nu}\rangle(\vec{e}_\mu, \langle\vec{v}_\nu\rangle, \theta_\nu) := \langle\Theta(\vec{v}_\nu \cdot \vec{e}_\mu > 0)\rangle$$

$$\approx \begin{cases} 1 - \frac{1}{2}e^{-y^2_{\mu\nu}/(2\theta_\nu)} + \frac{y_{\mu\nu}}{\sqrt{2\pi\theta_\nu}}\left[1 - \Phi\left(\frac{y_{\mu\nu}}{\sqrt{2\theta_\nu}}\right)\right] & \text{if } y_{\mu\nu} \ge 0 \\ \frac{1}{2}e^{-y^2_{\mu\nu}/(2\theta_\nu)} - \frac{|y_{\mu\nu}|}{\sqrt{2\pi\theta_\nu}}\left[1 - \Phi\left(\frac{|y_{\mu\nu}|}{\sqrt{2\theta_\nu}}\right)\right] & \text{if } y_{\mu\nu} < 0 \end{cases}$$

(see figure 4), with

$$y_{\mu\nu} := \langle\vec{v}_\nu\rangle \cdot \vec{e}_\mu$$

and the Gaussian *error function*

$$\Phi(z) := \int_0^z \frac{2}{\sqrt{\pi}} \mathrm{e}^{-x^2}\, dx.$$

$(1 - r_\mu)(1 - \langle\Theta_{\mu\nu}\rangle)$ is the relative frequency of stopping processes.

6.1 Interpretation

(a) Development of lanes

According to (40a), an asymmetrical avoidance probability $p_{\mu\nu} \neq 1 - p_{\mu\nu}$ (see [9]) leads to a momentum density that tends to the right (for $p_{\mu\nu} > 1/2$) or to the left (for $p_{\mu\nu} < 1/2$). This momentum density vanishes when the products $\langle\varrho_\mu\rangle\langle\varrho_\nu\rangle$ of the densities $\langle\varrho_\mu\rangle$ and $\langle\varrho_\nu\rangle$ have become zero, causing a separation of different types $\mu \neq \nu$ of motion into several lanes (see figure 5). This effect

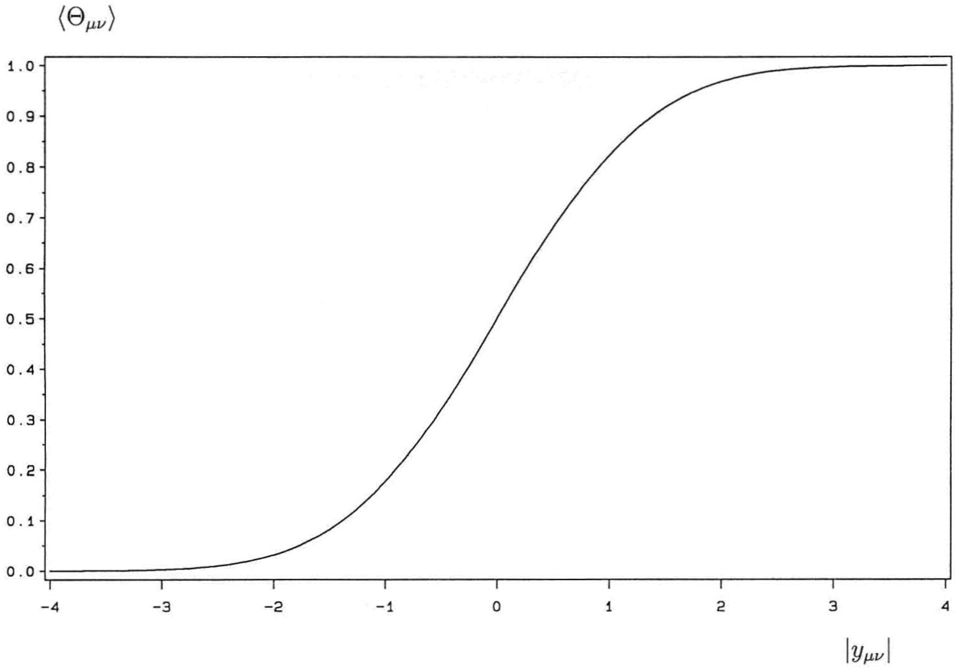

Figure 4: The function $\langle\Theta_{\mu\nu}\rangle$, dependent upon $|y_{\mu\nu}| = |\langle\vec{v}_\nu\rangle \cdot \vec{e}_\mu|$, for the special case $\theta_{\nu,i} = 1$.

can be observed, at least for high densities $\langle\varrho_\mu\rangle$ and $\langle\varrho_\nu\rangle$ [9, 23, 24, 25], and has the advantage of reducing the total rate $1/\hat{\tau}_\mu$ of interactions.

The width of the lanes of two opposing directions 1 and 2 can be calculated from the equilibrium condition of equal pressure:

$$P_1^e = P_2^e \qquad \Longrightarrow \qquad \langle\varrho_1\rangle\theta_1 = \langle\varrho_2\rangle\theta_2.$$

For a lane of width W_μ and length L consisting of N_μ individuals the relation

$$\langle\varrho_\mu\rangle = m_\mu\langle\rho_\mu\rangle = m_\mu\frac{N_\mu}{W_\mu L}$$

holds for the mass density $\langle\varrho_\mu\rangle$; therefore, we get

$$\frac{N_1}{N_2} \approx \frac{B_1}{B_2},$$

if $m_1\theta_1 \approx m_2\theta_2$. Consequently, the lane width B_μ will be proportional to the number N_μ of individuals (see [23], taking into account the Knudsen corrections described in section 3.2).

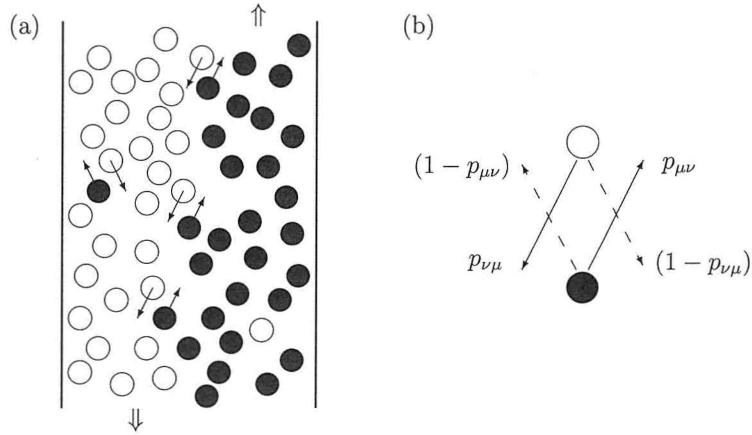

Figure 5: (a) Opposite directions of motion normally use separate lanes. Avoidance maneuvers are indicated by arrows. (b) For pedestrians with opposite directions of motion it is advantageous if both prefer either the right-hand side or the left-hand side when attempting to pass each other. Otherwise, they will have to stop in order to avoid a collision. The probability $p_{\mu\nu}$ for choosing the right-hand side is usually different from the probability $(1-p_{\mu\nu})$ for choosing the left-hand side.

(b) Crossings

If the direction μ of motion is crossed by the direction ν of motion, it suffers a change in momentum density of magnitude (40b), which causes the individuals of type μ to be "pushed" partly in direction $\langle \vec{v}_\nu \rangle$ of type ν. (For the *delay effect* of crossings see [22].)

(c) Pedestrian jams

In order to investigate the consequences of (40c) (stopping processes), we can consider the equation

$$\frac{\partial \langle \varrho_\mu \rangle \langle \vec{v}_\mu \rangle}{\partial t} = \frac{1}{\tau_\mu} \langle \varrho_\mu \rangle (\langle \vec{v}_\mu^0 \rangle - \langle \vec{v}_\mu \rangle) - \sum_\nu \langle \varrho_\mu \rangle \langle \varrho_\nu \rangle s_{\mu\nu}^3 \langle \vec{v}_\mu \rangle, \tag{42}$$

which describes the tendency to walk with the intended velocity $\vec{v}_\mu^0$, as well as stopping processes (see (18b) and (18c)). Here, the quantity

$$s_{\mu\nu}^3 := \frac{1 - r_\mu}{\tau_{\mu\nu}^*} (1 - \langle \Theta_{\mu\nu} \rangle)$$

has been introduced for brevity. The stationary solution of equation (42) is given by

$$\langle \vec{v}_\mu \rangle = \frac{1/\tau_\mu}{1/\tau_\mu + \sum_\nu \langle \varrho_\nu \rangle s_{\mu\nu}^3} \langle \vec{v}_\mu^0 \rangle =: k_\mu \langle \vec{v}_\mu^0 \rangle$$

(compare with [25]). According to (35) through (39), we find

$$\frac{\partial s^3_{\mu\nu}}{\partial \langle \rho_\nu \rangle} > 0$$

and, using the abbreviation $\langle \delta v_{\mu\nu} \rangle := \langle \|\vec{v}_\mu - \vec{v}_\nu\| \rangle$,

$$\frac{\partial s^3_{\mu\nu}}{\partial \langle \delta v_{\mu\nu} \rangle} > 0, \qquad s^3_{\mu\nu}(\langle \delta v_{\mu\nu} \rangle = 0) = 0. \tag{43}$$

According to (43), a development of pedestrian jams ($k_\mu < 1$) is *caused by the variation* $\delta\vec{v}_{\mu\nu} := \vec{v}_\mu - \vec{v}_\nu$ *of the velocities.* This is the case even for a lane consisting of individuals of one type μ only (where $\langle \varrho_\nu \rangle = 0$ for $\nu \neq \mu$, see (a)), since $s^3_{\mu\mu}$ is growing with the velocity variance θ_μ:

$$\frac{\partial s^3_{\mu\mu}}{\partial \theta_\mu} > 0, \qquad s^3_{\mu\mu}(\theta_\mu = 0) = 0.$$

(d) Decrease of variance

Equation (41c), describes a loss of variance (a loss of "temperature") by stopping processes.

7. Applications

7.1 Optimal motion

From sections 5.1 and 6.1 we can conclude that the motion of pedestrians can be optimized by

- avoiding crossings of different directions μ of motion—for example, by bridges or roundabout traffic;
- separation of opposite directions of movement—for example, by different lanes for each direction (the right lane being preferred [23, 24, 25]), or walking through a narrow passage *by turns*;
- avoiding great velocity variances θ_μ—for example, by walking in formation (as done by the military) [23];
- avoidance of obstacles, narrow passages, and great densities.

These rules are applicable to town- and traffic-planning.

7.2 Maximal diversity of perceptions

In a museum or supermarket, for instance, individuals will perceive more details (and probably buy more goods) if they walk slowly. So, the opposite of the rules in 7.1 could be applied to planning museums, markets, and so forth.

7.3 Critical situations

In critical situations pedestrians may panic. If the mean total interaction-free time $\hat{\tau}_\mu$ (see (38)) is less than the mean reaction time ζ_μ, the danger of falling and getting injured is great.

$$\hat{\tau}_\mu > \zeta_\mu$$

gives a condition for the critical density $\langle\rho_\mu\rangle^{\text{crit}}$ of pedestrians that should not be exceeded (see (35) through (38)).

8. Conclusions

Starting from the microscopic view of explicit gas-kinetic equations, we have derived some fluid-dynamic equations for the movement of pedestrians. These equations are anisotropic (that is, without local rotational symmetry). They resemble equations for ordinary fluids, but they are coupled equations for *several* fluids (that is, for several types of motion μ), each consisting of individuals having approximately the same intended velocity $\vec{v}^{\,0}_\mu$. They also contain additional terms that are characteristic for *pedestrian* fluids. These terms arise from the tendency of pedestrians to walk with an intended velocity and to change their type (direction) μ of motion, and from interactions between pedestrians (avoidance maneuvers).

For high densities $\langle\rho_\mu\rangle$ the interactions between pedestrians are very important. As a consequence, the development of pedestrian jams and of separate lanes for different directions of motion can be expected. Pedestrian jams can be understood as a deceleration effect due to avoidance maneuvers, which worsen as velocity variance increases. Separation into several lanes is caused by asymmetrical probabilities for avoiding a pedestrian to the right or to the left. This asymmetry creates the advantageous effect of reducing the number of situations in which hindering avoidance behavior is necessary.

For pedestrian crowds the mechanism of approaching equilibrium is essentially given by the tendency to walk with the intended velocity, not by interaction processes (as in ordinary fluids). As a consequence, the viscosity η_μ (the coefficient of internal friction) grows with the pedestrian density. In addition, we have seen that variations within pedestrian density will show wave-like propagation, with a velocity c_μ that depends on the mean reaction time.

Finally, quantities such as "temperature" and "pressure" play a different role than in ordinary fluids. It can be shown that the "temperature" (the velocity variance) θ_μ is produced by the variance of the intended velocities. As a consequence, two contacting groups of individuals belonging to different types of motion can show different "temperatures." This is the case, for example, on a dance floor. On the other hand, whereas a pressure gradient compensates for the effect of internal friction in ordinary fluids, for pedestrian crowds this role is played by the accelerating effect of the intended velocity. Therefore, a hyperbolic stationary velocity profile is found, rather than a parabolic one.

9. Outlook

Current investigations of pedestrian movement are concerned with the problem of specifying the forces $\vec{f}_\mu$ and the rates $\chi_{\mu}^{\cdot\cdot}$ of pedestrians changing their type of motion. This problem calls for a detailed model of the intentions of pedestrians.

Pedestrian intentions can be modeled according to stochastic laws. They are functions of

- a pedestrian's demand for certain commodities;
- the city center entry points (parking lots, metro stations, bus stops, and so forth);
- the location of stores offering the required commodities;
- expenditures (for example, prices and time); and
- unexpected attractions (shop windows, entertainment, and so forth).

Models of this kind have been developed and empirically tested by Borgers and Timmermans [2, 3]. A model that takes into account pedestrian intentions as well as gas-kinetic aspects is presented in [11]. This model can be formulated in such a way that it is also suitable for Monte Carlo simulations of pedestrian dynamics by computer. Computer simulations of this kind are an ideal tool for town- and traffic-planning. Their results can be directly compared with films of pedestrian crowds.

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