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The Limiting Behavior of Non-cylindrical Elementary Cellular Automata

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Abstract. A classification of cellular automata (CAs) complementary to that of Wolfram was recently proposed by Binder. This classification is motivated by the limiting behavior seen in automata with fixed boundaries. In what follows, we show that for a number of elementary CA rules it is possible to obtain complete solutions for the periods and numbers of all limit cycles, plus, in some cases, information on the structure of the cycles themselves. In particular, we are able to confirm the existence of globally attractive limit cycles of fixed period for several CA rules.

1. Introduction

Cellular automata (CAs) [1–6] are mathematical models in which space, time, and state variables are discrete. A CA consists of a regular lattice of sites, with a discrete variable at each site. These variables change according to a single local rule, which may be either deterministic or probabilistic.

Elementary CAs are the simplest realizations of this concept. They consist of a chain of sites, where each site variable can take one of two values (0 or 1), and the local rule is a function of three variables, the value of a site and those of its nearest neighbors. Despite this apparent simplicity, elementary CAs display a wide range of behavior. The behavior of periodic (or cylindrical) elementary CAs has been classified by Wolfram [7]. The present paper was motivated by a recent numerical study and classification of elementary CAs (with both fixed and periodic boundary conditions) performed by Binder [8], which, in particular, considered all 88 elementary rules, all possible fixed boundary conditions, and all initial states. Rules were classified according to their limiting behavior. A deterministic system with a finite number of states must eventually enter a limit cycle, where it will remain. Techniques exist for both cylindrical [9] and non-cylindrical [10] CAs that give the number of such limit cycles of any given period for a particular rule. The aim of this paper is to demonstrate analytic techniques that give the periods of *all* limit cycles for all fixed boundaries for a selection of elementary rules; in fact, we present results for 46 of the 88 nonequivalent elementary rules. (The only other such studies of which we are aware are an analysis of rule 3 by Binder [11], a study of rule 232 [12], and work on rule 150 [13].) We note that all results given here have been checked against, and agree with, the extensive simulations performed by Binder.

1.1 Notation

We will consider a chain of L sites, each of which can take the value 0 or 1. At the left and right ends of the chain, in positions 0 and $L+1$, will be the fixed boundary sites, with values a_0 and a_{L+1}. a_i^t denotes the value of the ith site at time t. The value of this site is determined by the values of itself and its immediate neighbors at the previous time step, as follows:

$$a_i^t = f(a_{i-1}^{t-1}, a_i^{t-1}, a_{i+1}^{t-1}) \tag{1}$$

The action of the global function f on the eight possible triplets defines the rule. We will follow the conventional numbering system for the rules [3].

2. The shift Rules

Consider the set of rules of the form:

$$a_i^{t+1} = f(a_{i-1}^t, a_i^t, a_{i+1}^t) = a_{i+1}^t g(a_{i-1}^t, a_i^t) \tag{2}$$

or

$$a_i^{t+1} = a_{i-1}^t g(a_i^t, a_{i+1}^t) \tag{3}$$

There are fifteen nonequivalent rules of this form. Of these, numbers 128 and 170 are straightforward, not requiring any sophisticated analysis (for example, 170 acts as a pure translation), and numbers 15 and 34 will be considered in the next section. In this section, we will outline techniques that enable us to obtain solutions to the numbers and periods of all limit cycles for the eleven remaining rules (numbers 2, 8, 10, 32, 40, 42, 130, 138, 160, 162, and 168), plus three additional rules (numbers 24, 46, and 152) that are one bit away from being a rule in this class and can be analyzed using similar techniques.

We present the analysis of rule 24 as an example. The case $a_0 = a_{L+1} = 1$ has been considered previously [14]. We give here a full and more rigorous analysis which illustrates the techniques used to analyze all the above rules. Details of results for the other rules can be found in Appendix A.

2.1 Rule 24

The definition of rule 24 is

$$\begin{aligned} f(*10) = f(00*) = f(*01) = f(11*) &= 0 \\ f(011) = f(100) &= 1 \end{aligned} \tag{4}$$

where $*$ indicates an irrelevant bit.

Theorem 2.1. *If $a_0 = 0$, then all limit cycles are fixed points.*

Lemma 2.1. *If $a_i^t = 0$ for all $t \geq t_0$, where $0 \leq i \leq L-2$, then $a_{i+1}^t = 0$ for all $t \geq t_0 + 2$.*

Proof. If $a_{i+1}^{t_0} = 0$, then $a_{i+1}^{t_0+1} = f(00*) = 0$. Hence $a_{i+1}^t = 0$ for all $t \geq t_0$. If $a_{i+1}^{t_0} = 1$ and $a_{i+2}^{t_0+1} = 0$, then $a_{i+1}^{t_0+1} = f(010) = 0$. Hence $a_{i+1}^t = 0$ for all $t \geq t_0 + 1$. If $a_{i+1}^{t_0} = a_{i+2}^{t_0} = 1$, then

$$\begin{aligned} a_{i+1}^{t_0+1} &= f(011) = 1 \\ a_{i+2}^{t_0+1} &= f(11*) = 0 \\ a_{i+1}^{t_0+2} &= f(010) = 0 \end{aligned} \tag{5}$$

Hence $a_{i+1}^t = f(00*) = 0$ for all $t \geq t_0 + 2$. ∎

Lemma 2.2. *If $a_{L-1}^t = 0$ for all $t \geq t_0$ and $a_{L+1} = 0$, then $a_L^t = 0$ for all $t \geq t_0 + 1$.*

Proof. If $a_L^{t_0} = 0$, then $a_L^{t_0+1} = f(000) = 0$. Hence $a_L^t = f(00*) = 0$ for all $t \geq t_0$. If $a_L^{t_0} = 1$, then $a_L^{t_0+1} = f(010) = 0$. Hence $a_L^t = f(00*) = 0$ for all $t \geq t_0$. ∎

Lemma 2.3. *If $a_{L-1}^t = 0$ for all $t \geq t_0$ and $a_{L+1} = 1$, then either $a_L^t = 0$ for all $t \geq t_0$, or $a_L^t = 1$ for all $t \geq t_0$.*

Proof. If $a_L^{t_0} = 0$, then $a_L^{t_0+1} = f(001) = 0$. Therefore, $a_L^t = 0$ for all $t \geq t_0$. If $a_L^{t_0} = 1$, then $a_L^{t_0+1} = f(011) = 1$. Therefore, $a_L^t = 1$ for all $t \geq t_0$. ∎

Consider now the boundary condition $a_0 = 0$. By repeated application of Lemma 2.1, we can see that all sites for $i = 1$ to $i = L-1$ eventually become fixed at zero. Using either Lemma 2.2 or 2.3, the last site also becomes fixed. If $a_0 = a_{L+1} = 0$, there is a global fixed point 0^L. If $a_0 = 0$ and $a_{L+1} = 1$, there are two fixed points, 0^L and $0^{L-1}1$. This completes the proof of Theorem 2.1.

We now consider the case $a_0 = 1$.

Theorem 2.2. *If $a_0 = 1$ then all limit cycles are of period three.*

Lemma 2.4. *If $a_i^t = 1$ where $2 \leq 1 \leq L-1$ and $t \geq 1$ then $a_{i-1}^t = a_{i+1}^t = 0$. Hence, if $a_i^t = 0$ where $1 \leq i \leq L-2$ and $t \geq 1$, then $a_{i+1}^{t+1} = 0$.*

Proof. If $a_i^t = 1$ then either

$$a_{i-1}^{t-1} = 0, \qquad a_i^{t-1} = a_{i+1}^{t-1} = 0 \tag{6}$$

hence $a_{i-1}^t = f(*01) = 0$ and $a_{i+1}^t = f(00*) = 0$, or

$$a_{i-1}^{t-1} = 0, \qquad a_i^{t-1} = a_{i+1}^{t-1} = 1 \tag{7}$$

hence $a_{i-1}^t = f(*01) = 0$ and $a_{i+1}^t = f(11*) = 0$. If $a_i^t = 0$, then (using the above result) $a_{i+1}^{t+1} = f(000)$, $f(001)$, or $f(010)$. Hence $a_{i+1}^{t+1} = 0$. This completes the proof. ∎

From this lemma, we can deduce that, for sites that are not adjacent to a boundary, updating proceeds as for rule 16 (which, by reflection, is equivalent to rule 2.)

The periodic behavior of a limit cycle depends on the periodic behavior of the limiting temporal sequences of which it is composed. A temporal sequence W_i is defined as follows:

$$W_i = [a_i^t : t \geq 0] \tag{8}$$

The sequences W_0 and W_{L+1} are thus the fixed boundaries. The sequences W_1 to W_L correspond to the evolving columns of the system. Since we are interested only in the limiting behavior of such sequences, we will label sequences according to this behavior. We define the following limiting sequences.

$$\begin{aligned} W_i^0 &= [a_i^{3m} = a_i^{3m+1} = 0,\ a_i^{3m+2} = 1;\ 3m \geq t_0] \\ W_i^1 &= [a_i^{3m+1} = a_i^{3m+2} = 0,\ a_i^{3m} = 1;\ 3m \geq t_0] \\ W_i^2 &= [a_i^{3m} = a_i^{3m+2} = 0,\ a_i^{3m+1} = 1;\ 3m \geq t_0] \end{aligned} \tag{9}$$

where $W_i = W_i^n$ if the appropriate condition holds for all $3m \geq t_0$, for some value of t_0.

Lemma 2.5. *If $a_0 = 1$, then $W_1 = W_1^n$ and $W_2 = W_2^{n+1}$, where $n = 0$, 1, or 2, and the addition is performed modulo 3.*

Proof. Consider the system at some time $t \geq 1$. Then, by Lemma 2.4, $a_1^t = a_2^t = 1$ is not possible. If $a_1^t = 1$ and $a_2^t = 0$, then $a_1^{t+1} = f(110) = 0$ and $a_2^{t+1} = f(10*) = 0$ or 1. If $a_1^t = 0$ and $a_2^t = 1$, then (by Lemma 2.4) $a_3^t = 0$, hence $a_1^{t+1} = f(101) = 0$ and $a_2^{t+1} = f(010) = 0$. If $a_1^t = 0$ and $a_2^t = 0$, then $a_1^{t+1} = f(100) = 1$ and $a_2^{t+1} = f(00*) = 0$. By Lemma 2.4, $a_3^{t+1} = 0$. Hence,

$$\begin{aligned} a_1^{t+2} &= f(110) = 0 \\ a_2^{t+2} &= f(100) = 1 \\ a_3^{t+2} &= f(00*) = 0 \\ a_1^{t+3} &= f(101) = 0 \\ a_2^{t+3} &= f(010) = 0 \end{aligned} \tag{10}$$

So a_1^0 and a_2^0, whatever their values, must both eventually become zero. Once this has occurred, they then cycle. This completes the proof of the lemma. ∎

Lemma 2.6. *If $W_i = W_i^n$, where $2 \leq i \leq L-4$, then $W_{i+1} = W_{i+1}^{n+1}$ and $W_{i+2} = W_{i+2}^{n+2}$, where $n = 0$, 1, or 2, and the addition is performed modulo 3.*

Proof. Consider the case $W_i = W_i^1$ at a time $t = 3m$. Then $a_i^{3m} = 1$, hence (using Lemma 2.4) $a_{i+1}^{3m} = 0$. Consider first the case $a_{i+2}^{3m} = 0$. Using the rule plus Lemma 2.4, we have the following.

$$\begin{aligned} a_{i+1}^{3m+1} &= f(100) = 1 \\ a_{i+2}^{3m+1} &= f(00*) = 0 \\ a_{i+3}^{3m+1} &= 0 \\ a_{i+1}^{3m+2} &= 0 \\ a_{i+2}^{3m+2} &= f(100) = 1 \end{aligned} \tag{11}$$

Hence, by Lemma 2.4, $a_{i+1}^{3m+3} = a_{i+2}^{3m+3} = 0$, and a cycle has been entered.

If $a_{i+2}^{3m} = 1$ then:

$$\begin{aligned} a_{i+1}^{3m+1} &= f(101) = 0 \\ a_{i+2}^{3m+1} &= 0 \end{aligned} \tag{12}$$

Thus, by Lemma 2.4, $a_{i+1}^{3m+2} = a_{i+2}^{3m+2} = 0$ and $a_{i+1}^{3m+3} = a_{i+2}^{3m+3} = 0$. The first half of the proof now applies. Hence W_i^1 is always followed by W_{i+1}^2 and W_{i+2}^0. W^0, W^1, and W^2 are equivalent under a shift in t, hence the proof is complete. ∎

Using Lemmas 2.5 and 2.6, we can now see that all temporal sequences for $i = 1$ to $i = L - 2$ have period-3 limiting behavior. It now remains to consider the last two columns.

Lemma 2.7. *If $W_{L-2} = W_{L-2}^n$ and $a_{L+1} = 0$, then $W_{L-1} = W_{L-1}^{n+1}$ and $W_L = W_L^{n+2}$.*

Proof. Let $W_{L-2} = W_{L-2}^1$, and consider a time $t = 3m$. By Lemma 2.4, $a_{L-1}^{3m} = 0$. If $a_L^{3m} = 0$, then

$$\begin{aligned} a_{L-1}^{3m+1} &= f(100) = 1 \\ a_L^{3m+1} &= f(00*) = 0 \\ a_{L-1}^{3m+2} &= f(010) = 0 \\ a_L^{3m+2} &= f(100) = 1 \\ a_{L-1}^{3m+3} &= f(00*) = 0 \\ a_L^{3m+3} &= f(010) = 0 \end{aligned} \tag{13}$$

Hence a_{L-1}^t and a_L^t have entered a cycle. If $a_L^{3m} = 1$, then

$$\begin{aligned} a_{L-1}^{3m+1} &= f(101) = 0 \\ a_L^{3m+1} &= f(010) = 0 \\ a_{L-1}^{3m+2} &= f(00*) = 0 \\ a_L^{3m+2} &= f(00*) = 0 \\ a_{L-1}^{3m+3} &= f(00*) = 0 \\ a_L^{3m+3} &= f(00*) = 0 \end{aligned} \tag{14}$$

Hence the first section of the proof now applies. We can thus conclude that if $W_{L-2} = W^1_{L-2}$, then $W_{L-1} = W^2_{L-1}$ and $W_L = W^0_L$. A shift in t then completes the proof. ∎

Using Lemmas 2.5 and 2.6, we can see that in the case $a_0 = 1$ and $a_{L+1} = 0$ there is a global period-3 attractor. It now remains to consider the case $a_0 = 1$ and $a_{L+1} = 1$.

Lemma 2.8. *If $a_{L+1} = 1$ and $W_{L-2} = W^n_{L-2}$ then either $W_{L-1} = W^{n+1}_{L-1}$ and $a^t_L = 0$ for all $t \geq t_0$ for some t_0, or $a^t_{L-1} = 0$ and $a^t_L = 1$ for all $t \geq t_0$ for some t_0.*

Proof. Let $W_{L-2} = W^1_{L-2}$. Then, as before, $a^{3m}_{L-1} = 0$. If $a^{3m}_L = 1$, then

$$\begin{aligned} a^t_{L-1} &= f(*01) = 0 \\ a^t_L &= f(011) = 1 \end{aligned} \tag{15}$$

for all $t \geq 3m$. If $a^{3m}_L = 0$, then

$$\begin{aligned} a^{3m+1}_{L-1} &= f(100) = 1 \\ a^{3m+1}_L &= f(00*) = 0 \\ a^{3m+2}_{L-1} &= f(010) = 0 \\ a^{3m+2}_L &= f(101) = 0 \\ a^{3m+3}_{L-1} &= f(00*) = 0 \\ a^{3m+3}_L &= f(00*) = 0 \end{aligned} \tag{16}$$

hence $a^t_L = 0$ for all $t \geq 3m$ and $W_{L-1} = W^2_{L-1}$. As before, a shift in t completes the proof. ∎

We can now see that for the case $a_{L+1} = 1$, either a^t_L or a^t_{L-1} eventually becomes fixed at zero. Hence the system exhibits two period-3 cycles. This now completes the proof of theorem 2.2.

3. The two-neighbor rules

Two-neighbor rules are defined such that either

$$a^{t+1}_i = f(a^t_i, a^t_{i-1}) \tag{17}$$

or

$$a^{t+1}_i = f(a^t_i, a^t_{i+1}) \tag{18}$$

Of the sixteen such elementary rules, there are nine nonequivalent rules (0, 3, 12, 15, 34, 51, 60, 136, and 204). Of these, rules 0, 15, 51, and 204 are trivial, being members of the subclass of rules that are a function of only one of the variables a^t_i, a^t_{i-1}, or a^t_{i+1}, or, in the case of rule 0, of none. Of the five remaining, the limiting behavior of rule 3 has been studied previously by Binder [11]. We will present an analysis of rule 136 as an example; all but one of the other rules can be analyzed in a similar manner. Rule 60, which requires a more sophisticated technique, which will be detailed in Appendix B.

Rule no.	(a_0, a_{L+1})	$p = 1$	$p = 2$	Definitions
12	$(0,0)$	y_L	0	$y_n = y_{n-1} + y_{n-2}$
	$(0,1)$	y_L	0	$y_1 = 2$ and $y_2 = 3$.
	$(1,0)$	y_{L-1}	0	
	$(1,1)$	y_{L-1}	0	
34	$(0,0)$	1	0	
	$(0,1)$	0	1	
	$(1,0)$	1	0	
	$(1,1)$	0	1	

Table 1: Numbers and periods of all limit cycles for rules 12 and 34, as a function of system size and boundary conditions.

3.1 Rule 136

The definition of rule 136 is as follows:

$$\begin{aligned} f(*11) &= 1 \\ f(*00, *01, *10) &= 0 \end{aligned} \tag{19}$$

where $*$ indicates an irrelevant bit.

Theorem 3.1. *All limit cycles are fixed points.*

Lemma 3.1. *If $a_i^t = 0$ for all $t \geq t_0$, where $i \geq 2$, then $a_{i-1}^t = 0$ for all $t \geq t_0 + 1$.*

Proof. If $a_{i-1}^{t_0} = 1$, then $a_{i-1}^{t_0+1} = f(*10) = 0$. If $a_{i-1}^{t_1} = 0$ for some $t_1 \geq t_0$, then, since $f(*00) = 0$, $a_{i-1}^t = 0$ for all $t \geq t_1$. Since either $a_{i-1}^{t_0} = 0$ or $a_{i-1}^{t_0+1} = 0$, the lemma is proved. ∎

Lemma 3.2. *If $a_i^t = 1$ for all $t \geq t_0$ then either $a_{i-1}^t = 0$ for all $t \geq t_0$ or $a_{i-1}^t = 1$ for all $t \geq t_0$.*

Proof. $f(*01) = 0$ and $f(*11) = 1$. Hence, whatever the value of $a_{i-1}^{t_0}$, it remains fixed for all subsequent time. ∎

From these lemmas, we can see that if a single row is fixed, all rows to the left of it must also become fixed. Since we always have such a row (namely, the right boundary), the theorem is proved. Moreover, we also can deduce the structure of the fixed points. If $a_{L+1} = 0$, there is a single fixed point 0^L. If $a_{L+1} = 1$, there are $L + 1$ fixed points of the form $0^n 1^{L-n}$, where n can take any value between 0 and L. The results for rules 12 and 34 are given in table 1.

4. Rules related to the two-neighbor rules

We consider now those rules that differ from a two-neighbor rule by one or two bits. For example, rule 7 differs from rule 3 and rule 170 by one bit, and is defined by

$$\begin{aligned} f(00*) &= f(010) = 0 \\ f(1**) &= f(011) = 1 \end{aligned} \tag{20}$$

where $*$ represents an irrelevant bit. We have found that for many such rules an analysis of the limit cycles is possible, such analysis proceeding in terms of columns or sets of columns. Of those rules that are one bit away and that do not fall into either of the previously considered classes, we have obtained proofs of the numbers and periods of all limits cycles with all fixed boundary conditions for rules 1, 4, 7, 11, 13, 14, 19, 28, 38, 44, 50, 62, 140, and 200. We have obtained similar results for some rules that are two bits away (numbers 5, 23, 29, 30, 33, 36, 72, 76, 78, 108, 132, 156, and 178) and also for rule 104, which is three bits away. The results for the numbers of fixed points and period-2 cycles for rules 7, 23, 29, and 30 have been published [10]. However, the methods detailed below can prove that these are the only limit cycles occurring for these particular rules, and also yield the structure of these cycles. We will now demonstrate these points in the analysis of rule 11.

4.1 Rule 11

The definition of rule 11 is

$$\begin{aligned} f(1**) &= f(010) = 0 \\ f(00*) &= f(011) = 1 \end{aligned} \tag{21}$$

where $*$ represents an irrelevant bit.

Theorem 4.1. *With fixed boundaries, all limit cycles are of period 3, provided that the system is sufficiently large.*

As in section 2.1, we will consider temporal sequences, and label them by their limiting behavior. For example, the definition of W^A is

$$W_i = W_i^A \tag{22}$$

if $a_i^t = 0$ for all $t \geq t_0$, for some t_0. It will be convenient to define the following limiting temporal sequences:

$$\begin{aligned} W_i^B &= [a_i^t = 1 : t \geq t_0] \\ W_i^{C_0} &= [a_i^{3m} = 0, a_i^{3m+1} = a_i^{3m+2} = 1 : 3m \geq t_0] \\ W_i^{C_1} &= [a_i^{3m+1} = 0, a_i^{3m} = a_i^{3m+2} = 1 : 3m \geq t_0] \\ W_i^{C_2} &= [a_i^{3m+2} = 0, a_i^{3m} = a_i^{3m+1} = 1 : 3m \geq t_0] \\ W_i^{D_0} &= [a_i^{3m+1} = 1, a_i^{3m} = a_i^{3m+2} = 0 : 3m \geq t_0] \\ W_i^{D_1} &= [a_i^{3m+2} = 1, a_i^{3m} = a_i^{3m+1} = 0 : 3m \geq t_0] \\ W_i^{D_2} &= [a_i^{3m} = 1, a_i^{3m+1} = a_i^{3m+2} = 0 : 3m \geq t_0] \end{aligned} \tag{23}$$

Obviously, $W_i^{C_0}$, $W_i^{C_1}$, and $W_i^{C_2}$ describe the same periodic structure, but shifted with respect to phase; similarly for $W_i^{D_0}$, $W_i^{D_1}$, and $W_i^{D_2}$. In the lemmas that follow, unless specifically stated, $i \leq L-2$.

Lemma 4.1. *If* $W_i = W_i^B$, *then* $W_{i+1} = W_i^A$.

Proof. By definition, $a_i^t = 1$ for all $t \geq t_0$, for some t_0. From the definition of the rule, $f(1 * *) = 0$, therefore $a_{i+1}^t = 0$ for all $t \geq t_0 + 1$. Hence $W_{i+1} = W_{i+1}^A$. ∎

Lemma 4.2. *If* $W_i = W_i^A$, *then either* $W_{i+1} = W_{i+1}^{C_n}$ *or* $W_{i+1} = W_{i+1}^{D_n}$, *where* $n = 0$, *1, or 2.*

Lemma 4.3. *If* $W_i = W_i^{C_n}$, *then either*

$$i = L-1,\ W_L = W_L^{D_n},\ \text{and } W_{L+1} = W_{L+1}^A$$

or

$$i = L-1,\ W_L = W_L^{D_n},\ \text{and } W_{L+1} = W_{L+1}^B$$

or

$$W_{i+1} = W_{i+1}^{D_n} \text{ and } W_{i+2} = W_{i+2}^{C_{n-1}}$$

or

$$W_{i+1} = W_{i+1}^{D_n} \text{ and } W_{i+2} = W_{i+2}^{D_{n-1}}$$

where $n = 0$, *1, or 2, and all subtraction is performed modulo 3.*

Lemma 4.4. *If* $W_i = W_i^{D_n}$ *then either*

$$W_{i+1} = W_{i+1}^{C_{n-1}} \text{ and } W_{i+2} = W_{i+2}^{D_{n-1}}$$

or

$$i = L-1,\ W_L = W_L^{D_{n-1}},\ \text{and } W_{L+1} = W_{L+1}^{D_{n-1}}$$

or

$$W_{i+1} = W_{i+1}^{D_{n-1}} \text{ and } W_{i+2} = W_{i+2}^{D_{n-2}}$$

where $n = 0, 1$ *or 2 and the subtraction is performed modulo 3.*

In all cases, the proofs follow by considering W_i as a boundary, and considering all possible initial values for the next three or four bits.

From these four lemmas, it can be seen that whatever the values of the fixed boundaries (that is, $W_0 = W_0^A$ or W_0^B, and $W_{L+1} = W_{L+1}^A$ or W_{L+1}^B), all the columns W_1 to W_L must exhibit the limiting behavior of $W_i^{C_n}$ or $W_i^{D_n}$, where $n = 0$, 1, or 2. All these have period three, hence the theorem is proved. Furthermore, these lemmas also yield the allowed sequences. This enables us to derive expressions for the numbers of limit cycles, for all fixed boundary conditions, and for L sufficiently large. These results are given in table 2. Similar techniques can be applied to all other rules given above. The results are presented in Appendix A.

(a_0, a_{L+1})	$p = 3$	Definitions
$(0,0)$	x_{L+1}	$x_n = \frac{1}{2}[n - 1 - f_n]$
$(0,1)$	1	$f_n = \frac{1}{2}[1 + (-1)^n]$
$(1,0)$	x_L	
$(1,1)$	1	

Table 2: Numbers of limit cycles for rule 11 as a function of system size and boundary conditions.

5. Discussion

Thus far we have considered only constant boundary conditions. But there are other possibilities: a_0 and a_{L+1} could vary with time. Consider first the case where the boundaries are initially set to one of the four constant boundary conditions, but then noise is gradually introduced. In this case, will there be definite limit cycles? For the case of rule 140, the answer is yes. This rule possesses multiple fixed points for each of the constant boundary conditions. Moreover, there is a set of configurations that are fixed points for any of the four. It is possible to show that, with the introduction of noise, this set of configurations becomes the limiting set. Hence, the general behavior (that is, multiple fixed points) is preserved.

However, for other rules, we can see that when noise is introduced there will not be a stable set of limiting configurations. Consider the case of rule 38 with noise. At some point in an infinite run, each of the four fixed boundary conditions will occur for a length of time sufficient for the zero-noise cycles to appear. So, for example, a period-8 cycle may appear for some length of time, while later a period-4 will arise. Thus, there will be times when the automata passes through all of the cycles seen in the case of constant fixed boundaries. However, since there is no cycle that is common to all constant boundary conditions, the system will never enter a stable cycle. Instead, we will see portions of each of the possible cycles.

We will not consider here the other case, that of temporally periodic fixed boundaries. It would be interesting to see the effect such a boundary had on the periods of limit cycles, and whether the general type of limiting behavior was sensitive to such a change.

6. Conclusions

In this paper, we have demonstrated techniques that yield the periods of all limit cycles on arbitrarily large lattices for 46 of the 88 elementary rules. We have been able to confirm the behavior seen by Binder [11] for these rules, and verify that it does indeed hold for all system sizes. We note here that the general approach used, that of an analysis in terms of the limiting behavior of temporal sequences, is possible because we already know two such sequences (that is, the boundaries).

It is clear that subsets of rules that possess some simplifying factor should admit an analysis. The techniques developed here were originally applied solely to the two-neighbor rules. The fact that their use could be extended was fortuitous. The problems of extending the method can be seen in the analysis of rule 11, where, in the proofs of Lemmas 4.2–4.4, we were required to consider the next three or four columns. If we attempt to apply these techniques to arbitrary rules, we will certainly encounter cases in which we need to consider more than four columns. In fact, there will be cases in which we will be forced to consider the whole system. In such cases, the application of these techniques is inappropriate, as is the case for rules that possess variable or large cycle lengths. In short, these techniques are useful for a large number of rules, but the selection of appropriate rules is largely nonsystematic.

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Appendix A: Tables of results

(a_0, a_{L+1})			
28	$p=1$	$p=2$	Definitions
$(0,0)$	$\frac{L+3-f_L}{2}$	$1+\frac{1}{2}\left[\left(\sum_{n=1}^{L-1} g_{n+2}\right)-\frac{L+3-f_L}{2}\right]$	$f_n=\frac{1}{2}[1+(-1)^n]$
$(0,1)$	$L+1$	$\frac{1}{2}\left[\left(\sum_{n=1}^{L}(g_{L-n}+f_{L+1-n})\right)-L\right]$	$g_n=g_{n-2}+2g_{n-3}$
$(1,0)$	f_L	$\frac{1}{2}[g_{L+2}-f_L]$	$g_0=1,\ g_1=0,\ g_2=1.$
$(1,1)$	1	$\frac{1}{2}[g_L+g_{L+1}-1]$	
30	$p=1$	$p=2$	$p=4$
$(0,0)$	$2-f_L$	f_L	0
$(0,1)$	1	$1-f_L$	f_L
$(1,0)$	f_L	$1-f_L$	0
$(1,1)$	1	f_L	$1-f_L$
33	$p=2$	$p=4$	Definitions
$(0,0)$	b_L	0	$b_n=b_{n-1}+b_{n-2}$
$(0,1)$	0	b_{L-2}	$b_2=1,\ b_3=2.$
$(1,0)$	0	b_{L-2}	
$(1,1)$	1	$2b_{L-4}$	
38	$p=1$	$p=4$	$p=8$
$(0,0)$	2	0	0
$(0,1)$	0	$1-f_L$	f_L
$(1,0)$	1	0	0
$(1,1)$	0	1	0
62	$p=1$	$p=6$	Definitions
$(0,0)$	1	$c_{L-5}+c_{L-6}$	$c_n=c_{n-1}+c_{n-3}+c_{n-4}$
$(0,1)$	0	c_{L-4}	$c_1=1,\ c_2=1,$
$(1,0)$	0	$c_{l-4}+c_{L-5}$	$c_3=2,\ c_4=4.$
$(1,1)$	0	c_{L-3}	
104	$p=1$	$p=2$	Definitions
$(0,0)$	d_L	0	$d_n=d_{n-1}+d_{n-4}$
$(0,1)$	e_L	0	$e_n=e_{n-1}+e_{n-4}$
$(1,0)$	e_L	0	$d_0=1,\ d_1=1,$
$(1,1)$	$e_{L-2}+e_{L-3}$	$1-f_L$	$d_2=2,\ d_3=3.$
			$e_1=2,\ e_2=2,$
			$e_3=2,\ e_4=3.$

The table above gives the complete results for number and periods of all limit cycles for a selection of rules. Note that the results do not necessarily apply to the smallest lattices. The numbers of fixed points and period-2 cycles were checked using matrix methods [10]. For the remainder of the rules, we will give only the allowed periods of limit cycles for each choice of fixed boundary conditions. The exact numbers of such cycles can be obtained by using either the methods in the text, or matrix techniques [10].

Rule no.	Allowed periods				Rule no.	Allowed periods			
	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
1	2	2	2	2	46	1	3	1	3
2	1	3	1	3	50	1, 2	2	2	2
4	1	1	1	1	72	1	1	1	1
5	1, 2	1, 2	1, 2	1, 2	76	1	1	1	1
7	1, 2	1, 2	1, 2	1, 2	78	1	1	1	1
8	1	1	1	1	108	1, 2	1, 2	1, 2	1, 2
10	1	4	1	4	130	1	3	1	1,3
13	1	1	1	1	132	1	1	1	1
14	1	1	1	1	138	1	1	1	1
19	2	2	2	2	140	1	1	1	1
23	1, 2	1, 2	1, 2	1, 2	152	1	1	3	1, 2, 3
29	1, 2	1, 2	1, 2	1, 2	156	1, 2	1, 2	1, 2	1, 2
32	1	1	1	1, 2	160	1	1	1	1, 2
36	1	1	1	1	162	1	2	1	1, 2
40	1	1	1	1	168	1	1	1	1
42	1	3	1	3	178	1, 2	2	2	1, 2
44	1	1, 2	1	1, 2	200	1	1	1	1

Appendix B: Rule 60

Rule 60 is defined as

$$f(00*) = f(11*) = 0$$
$$f(01*) = f(10*) = 1$$

This can also be expressed in the additive form

$$a_i^{t+1} = a_i^t + a_{i-1}^t$$

where the addition is performed modulo 2. This rule can be generalized to the case where the addition is performed modulo k, corresponding to a radius-1 k-state additive rule. We will perform the analysis for $k = 2$, but the same methods can be applied to arbitrary k. Note that in the periodic case the evolution of states is linear, but is not always so in the case of fixed boundaries.

Following Martin et al. [15]. we define the generating function for a state as

$$A^t(x) = \sum_{i=1}^{L} a_i^t x^i$$

Two polynomials are equivalent if they describe the same state. Therefore

$$A(x) \approx B(x)$$

if

$$A(x) = B(x) + O(x^{L+1})$$

The action of the rule on a state can be expressed in terms of operations on polynomials, as

$$A^{t+1}(x) \approx (1 + x)A^t(x) + a_0 x$$

where the polynomial coefficients are defined modulo 2, and a_0 is the value of the left boundary. Applying this repeatedly gives the result

$$A^t(x) = (1 + x)^t A^0(x) + a_0[(1 + x)^t - 1]$$

where $A^0(x)$ represents the initial state. If $A(x)$ lies in a limit cycle, then the following must hold for some value of $t \geq 1$:

$$A(x) \approx (1 + x)^t A(x) + a_0[(1 + x)^t - 1]$$

Hence, we have the limit cycle equation

$$[(1 + x)^t - 1][A(x) + a_0] \approx 0$$

The smallest value of t for which this equation holds is the period of the limit cycle. The enumeration of limit cycles then corresponds to solving the above relation for t and the polynomial coefficients of $A(x)$.

We note that the above equation always possesses a solution, valid for all a_0 and $A(x)$, given by

$$[(1 + x)^t - 1] \approx 0$$

Hence, we can define the maximum cycle length $t = p_{\max} = 2^\beta$, where $2^{\beta-1} \leq L < 2^\beta$. Since this is a solution that is independent of $A(x)$, we can deduce that all states lie in limit cycles, the periods of which must be divisors of the maximum cycle length $p_{\max}$. This is in contrast to the periodic case, where certain states (for example, all states composed of a single one) do not lie in any limit cycle.

The non-linear case: $a_0 = 1$

In this case, the limit cycle equation reduces to

$$[(1 + x)^t - 1][A(x) + 1] \approx 0$$

The only solutions to this are solutions of the form

$$[(1 + x)^t - 1] \approx 0$$

Hence all states lie in cycles of period $p_{\max}$ as defined above.

The linear case: $a_0 = 0$

In this case, we have to solve

$$A(x)[(1+x)^t - 1] \approx 0$$

Let m be an odd positive integer, and α be a positive integer. Then, either $t = m$, or $t = m2^\alpha$. In the first case, the equation reduces to the m-independent form

$$[x + \cdots]A(x) \approx 0$$

which has only two solutions. These are the fixed points $A(x) \approx 0$ and $A(x) \approx x^L$ $(t = 1)$, which correspond to the states 0^L and $0^{L-1}1$.

In the second case, the equation reduces to

$$[x^{2^\alpha} + \cdots]A(x) \approx 0$$

and is again independent of m. There are two classes of solutions to this equation. If $2^\alpha \geq L$, then all $A(x)$ satisfy this equation. If $2^\alpha < L$, there are some solutions $A(x)$, namely those corresponding to all states with leading zeros, the number of such being at least $(L - 2^\alpha)$, and less than $(L - 2^{\alpha-1})$.

To summarize, for a system of size L, where $2^\alpha < L \leq 2^{\alpha+1}$, phase space is partitioned as follows.

There are two fixed points $A(x) \approx 0$ and $A(x) \approx x^L$.

For $m = 1$ to α, there exist $N(m)$ cycles of period $p = 2^m$ where

$$N(m) = \frac{1}{p}[2^p - 2^{\frac{p}{2}}]$$

The remaining states lie in cycles of period $p = 2^{\alpha+1}$, which is either $p_{\max}$ or $p_{\max}/2$, depending on L.

It is obvious that all these periods are divisors of $p_{\max}$, as required.

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