

## A Liquid-Crystal Model for Neural Networks

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**Abstract.** In this paper, the interaction between molecular free-point dipoles is proposed as an analog of the dynamics of randomly interconnected neurons. Typically, neural interaction has been described as being analogous to the stochastic aspects of the magnetic Ising spin model. For example, Hopfield's attractor neural network follows the zero-field spin-glass analogy and warrants the neural interconnections to have bilateral symmetric weights across the interacting neurons. But the actual neural interconnections may not pose such a symmetry, because the stochastic aspects of excitatory and inhibitory synaptic responses are not the same; and, in general, random asymmetry in synaptic couplings more closely approximates physiological reality. The interconnecting weights that decide the collective response across a neural arrangement are asymmetric both temporally as well as spatially. Lack of spatial symmetry effects in the specification of anisotropic proliferation of neural state-transitions has motivated the present work; the consistent requirement of symmetric weights in neural assembly modeling (analogous to the Ising spin-glass model) is thereby obviated. In the relevant considerations, neural interactions are depicted as being similar to those of molecular free-point dipoles—specifically, those of a liquid crystal in the nematic phase having a long-range orientational order. This partial anisotropy in spatial orientation incorporates an asymmetry in synaptic coupling activity, and is addressed via Langevin's theory of dipole orientation. A stochastically justifiable sigmoidal activation function is derived therefrom to represent the squashing action in the input-output relation of the complex dynamics pertinent to the cellular automata.

## 1. Introduction

The standard analogy of neurons with the aggregates of interacting magnetic spins contains certain inconsistencies. The analogy arises from the fact that the organization of neurons is a collective enterprise in which the activity of interactive cells constitutes a cooperative process similar to that of spin interactions in a magnetic system. The strengths of synaptic connections between the cells (representing the extent of interactive dynamics in the cellular automata) are considered to be analogous to the strengths of exchange interactions in magnetic spin systems. Synaptic activity, manifesting as the competition between the excitatory and inhibitory processes, is regarded as being equitable to the competition between the ferromagnetic and antiferromagnetic exchange interactions in spin-glass systems. Also, the threshold condition stipulated for the neural network is viewed as the analog of the condition of metastability against single spin flips in the Ising spin-glass model.

Despite similarities between neurons and magnetic spins, a major inconsistency persists regarding synaptic coupling versus spin interactions. Nearest-neighbor spin-spin interactions are symmetric; successive rows of the spin system can be added in any direction. On the contrary, the time-development of a neural network has a specific forward direction; in other words, the neural problem is inherently anisotropic. In addition, not only is the state of a neuron at any time-instant determined by the state of some or all of the neurons at the previous instant, but the state of a neuron at a specific location is dependent on the state of some or all the neurons at the previous location across the neurocellular arrangement. Thus, asymmetry in the synaptic coupling is governed by both the structured functional aspect pertinent to the processing of temporal sequences of synaptic events, and the physioanatomical considerations related to the cellular activity and structure. (The problem of synaptic asymmetry has even broader implications in neural network strategies, as discussed in [1].)

Within the framework of statistical mechanics considerations, the state-transition probabilities in magnetic spin systems between two states  $S_1$  and  $S_2$  (with probabilities specified with reference to the equilibrium state  $S_0$ ) are equal in both directions. That is, microscopic reversibility pertaining to magnetic spin interactions is inherent to the strength of coupling between symmetrical exchange interactions. However, in the neural system observed by Griffith [2], such symmetric weights of neural interaction are questionable. In an aggregate of  $M$  cells with a dichotomous state of activity (as in the spin system) having  $2^M$  possible states, identified as  $S = 1, 2, \dots, 2^M$ , there is only a superficial analogy with a quantum mechanics situation. (This analogy refers to a set of  $M$  subsystems, each having two possible quantum states; for example, a set of  $M$  atoms each having a spin  $\frac{1}{2}$ .) Each of the  $2^M$  states has a definite successor in time, such that the progress of the state-transition process corresponds to a sequence  $i_2 = z(i_1) \rightarrow i_3 = z(i_2) \dots$ . This sequence would eventually terminate in either of two dichotomous states,  $S_1 \Rightarrow +S_U$  or  $S_2 \Rightarrow -S_L$ , with the transitional probabilities  $\rho(S_{1,2})$ . The natural tendency

for this microscopic parameter is simply to tend toward  $\rho_0$ , the equilibrium value; there is no likelihood of the successor function  $z$  showing any bilateral symmetry. In other words, in the neural cycle of state transitions, the interconnecting weights between any two cells  $i$  and  $j$  are such that  $W_{i,j} \neq W_{j,i}$ . That is, there is no perceivable microscopic reversibility, implying that the synaptic connections in neural networks are seldom symmetric. Quite often, either  $W_{i,j}$  or  $W_{j,i}$  is nonzero. This asymmetry implies the existence of a disparity between the number of excitatory and inhibitory synapses on the one hand, and the corresponding synaptic activities on the other.

The inconsistency between neurons with inherent asymmetric synaptic couplings and symmetric spin-glass interactions led Griffith [2] to declare that the analogy between them had “no practical value.” Nevertheless, several compromises have been proposed [1, 2–4] that show the usefulness of the analogy. The assumption of symmetry, and the specific form of the synaptic coupling in a neural assembly, define what is generally known as the Hopfield model. This model demonstrates the basic concepts and functioning of a neural network, and serves as a starting point for a variety of models in which many underlying assumptions are relaxed to meet some of the requirements of real systems. For example, the question of  $W_{i,j}$  being unequal to  $W_{j,i}$  in a neural system was addressed in a proposal by Little [3], who defined a time-domain long-range order such that the corresponding anisotropy introduces bias terms in the Hamiltonian relation, making it asymmetric to match the neural Hamiltonian. That is, Little’s long-range order, with reference to neurons, corresponds to a time-domain-based long-time correlation of the states; these time-persistent states of a neural network are equated to the long-range (spatial) order in an Ising spin system. Little, therefore, attempted to incorporate the asymmetric considerations in the neural interactions via correlated time developments.

Little’s model, however, lacks the attributions of spatial correlation effects (the spatial anisotropy, depicting the state transition at one location being correlated to some or all of such transitions in the previous location). Because both temporal and spatial anisotropy coexist in the neural state-transition process, it is imperative to elucidate and incorporate a spatial persistency in the proliferation dynamics of state transitions. It is the motive of the present study to develop a stochastic model of the spatially anisotropic neurocellular (interconnected) arrangement, for adoption in conjunction with the temporal statistics of Little.

### Problem definition

The problem posed in this paper refers to the modeling of the spatial anisotropy in interconnected neural arrangements. The relevant method of attributing long-range order to neurons [3] is followed, except that it is referred to spatial (orientational) anisotropy. To facilitate this approach, free-point molecular dipole interactions are considered as an analogy, in place of magnetic spin interactions. Free-point molecular dipole interactions with partial

anisotropy in spatial arrangement describe the nematic phase in a liquid crystal. Hence, the present analysis equates neural statistics to those of a nematic phase system, in concurrence with the assertion that “the living cell is actually a liquid crystal” [6]. On the basis of the proposed spatial anisotropy, the squashing action upon the input-output relation pertaining to the dynamics of cellular automata is described appropriately by a stochastically justifiable sigmoidal function.

It should be noted that Little’s approach in facilitating asymmetry in the neural interconnection (via long-range time-correlation of state-transitional events) follows basically (pseudo) thermodynamic considerations; the desired temporal anisotropy is achieved by introducing a bias-term in the Hamiltonian relation depicting the temporal dynamics of the state-transitions. The present study is also based on the thermodynamics of the associated statistics. This approach stochastically justifies the input-output response and introduces a spatial (orientational) anisotropy in the neural state-transition process.

## 2. Free-point molecular dipole interactions

Suppose a set of polarizable molecules are anisotropic, with a spatial long-range orientational order corresponding to the nematic liquid crystal in the mesomorphic phase. This differs from an isotropic molecular arrangement, as in liquid, in that the molecules are spontaneously oriented with their long axes approximately parallel. The preferred direction or orientational order may vary from point to point in the medium, but in the long range, a specific orientational parallelism is retained.

In the nematic phase, the statistical aspects of dipole orientation in the presence of an externally applied field can be studied via Langevin’s theory with the following hypotheses.

1. The molecules are point-dipoles with a prescribed extent of anisotropy.
2. The ensemble average taken at an instant is the same as the time average taken on any element (ergodicity property).
3. The characteristic quantum numbers of the problem are sufficiently high that the system obeys the classical statistics of Maxwell-Boltzmann (the limit of quantum statistics for systems with high quantum numbers). The present characterization of paraelectricity, therefore, differs from spin paramagnetism, in which the quantum levels are restricted to two values only.
4. The dipole molecules, in general, when subjected to an external electric field  $\mathbf{E}$ , experience a moment  $\boldsymbol{\mu}_E = \alpha_E \mathbf{E}$ , where  $\alpha_E$  by definition refers to the polarizability of the molecule. The dipole orientation contributing to the polarization of the material is quantified as  $\mathbf{P} = N \langle \boldsymbol{\mu}_E \rangle$ , where  $N$  is the dipole concentration.

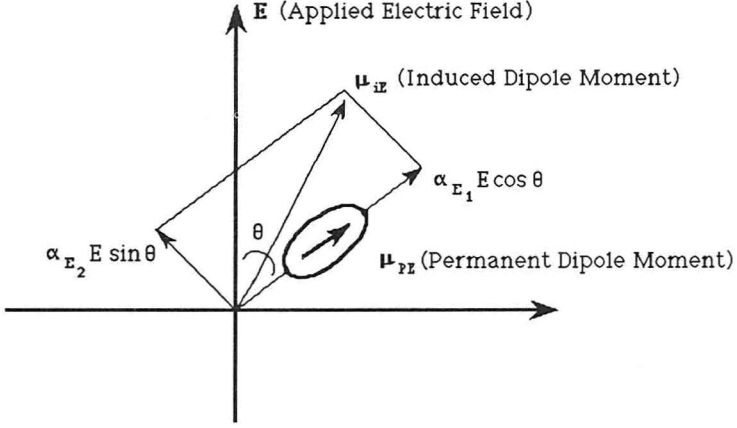


Figure 1: Free-point dipole and its moments.

5. In an anisotropic system such as the liquid crystal, there is a permanent dipole moment  $\mu_{PE}$ , the direction of which is assumed along the long-axis of a nonspherical dipole configuration. Consequently, two orthogonal polarizability components exist, namely  $\alpha_{E1}$  along the long axis and  $\alpha_{E2}$  perpendicular to this long axis.

### 3. The order parameter

The dipole moments in an anisotropic molecule are depicted in Figure 1. Projecting along the applied electric field  $\mathbf{E}$ , the net induced electric polarization moment is

$$\begin{aligned}\mu_E &= \mu_{PE} \cos \theta + (\alpha_{E1} \cos^2 \theta + \alpha_{E2} \sin^2 \theta) E \\ &= \mu_{PE} \cos \theta + (\Delta \alpha_E \cos^2 \theta + \alpha_{E2}) E\end{aligned}\quad (3.1)$$

where  $\Delta \alpha_E$  is a measure of anisotropy.

The corresponding energy of the polarized molecule in the presence of an applied field  $\mathbf{E}$  is constituted by (i) the potential energy  $W_{PE}$  due to the permanent dipole, given by

$$W_{PE} = -\mu_{PE} \cdot \mathbf{E} = -\mu_{PE} E \cos \theta \quad (3.2)$$

and (ii) the potential energy due to the induced dipole given by

$$W_{iE} = -\frac{1}{2}(\alpha_{e1} \cos^2 \theta + \alpha_{E2} \sin^2 \theta) E^2 \quad (3.3)$$

Hence, the total energy is equal to  $W_T = W_{PE} + W_{iE}$ . Furthermore, the statistical average of  $\mu_E$  can be specified by

$$\langle \mu_E \rangle = \frac{\int \mu_E \exp[-W_{PE}/k_B T] d\Omega}{\int \exp[-W_i/k_B T] d\Omega} \quad (3.4)$$

where  $d\Omega$  is the elemental solid angle around the direction of  $\mathbf{E}$ . That is,  $d\omega = 2\pi \sin(\theta) d\theta$ . By performing the integration of (3.4) using (3.1),

$$\langle \mu_E \rangle = \mu_{PE} \langle \cos \theta \rangle + (\Delta \alpha_E \langle \cos^2 \theta \rangle + \alpha_{E_2}) E \quad (3.5)$$

where the quantity  $\langle \cos^2 \theta \rangle$  varies from  $\frac{1}{3}$  (for randomly oriented molecules) to 1 (for the case where all the molecules are parallel, or antiparallel, to the field  $\mathbf{E}$ ). On the basis of these limits, the following parameter is defined.

$$\begin{aligned} S^0 &= \frac{3}{2} \langle \cos^2 \theta \rangle - \frac{1}{2} \\ &= 0 \quad (\text{for } \langle \cos^2 \theta \rangle = \frac{1}{3}) \\ &= 1 \quad (\text{for } \langle \cos^2 \theta \rangle = 1) \end{aligned} \quad (3.6)$$

**Definition.** The parameter  $S^0$ , which is bounded between 0 and 1 under the conditions of equation (3.6), represents the “order parameter” of the system [6].

Appropriate to the nematic phase,  $S^0$  specifies the long-range orientational parameter pertaining to a liquid crystal of rod-like molecules, as follows. Assuming that the distribution function of the molecules is cylindrically symmetric about the axis of preferred orientation,  $S^0$  defines the degree of alignment: for perfectly parallel (or antiparallel) alignment  $S^0 = 1$ , while for random orientations  $S^0 = 0$ . In the nematic phase,  $S^0$  has an intermediate value that is strongly temperature-dependent.

$S^0 = 0$  refers to an isotropic statistical arrangement of random orientations, such that for each dipole pointing in one direction, there is (statistically) a corresponding molecule pointing in the opposite direction (see Figure 2). In the presence of an external electric field  $\mathbf{E}$ , the dipoles experience a torque and tend to polarize along  $\mathbf{E}$ , so that the system becomes slightly anisotropic; eventually, under a strong field ( $\mathbf{E}$ ), the system becomes totally anisotropic with  $S^0 = 1$ .

#### 4. Collective stochastic response of neurons under activation

By means of the analogy to a random, statistically isotropic dipole system, the graded response of neurons under activation was modeled previously by the authors [8], applying the concepts of Langevin’s theory of dipole polarization. The continuous graded response of neuron activity (corresponding to the stochastic interaction between incoming excitations that produce true, collective, nonlinear effects) was elucidated in terms of a sigmoidal function, specified by the Langevin function  $L(\lambda \sigma_i)$  where  $\lambda = \beta/k_B T$ , with  $\beta$  being the scaling factor and  $\sigma_i$  depicting the neural state-vector. Explicitly,  $L(x) = \coth(x) - 1/x$ . Further,  $k_B T$  depicts the Boltzmann energy corresponding to the (pseudo) temperature,  $T$ .

In the present considerations, neurons are specifically depicted as similar to the nematic phase of liquid crystals, and are assumed to possess an inherent, long-range spatial order. In other words, it is suggested that  $0 < S^0 < 1$

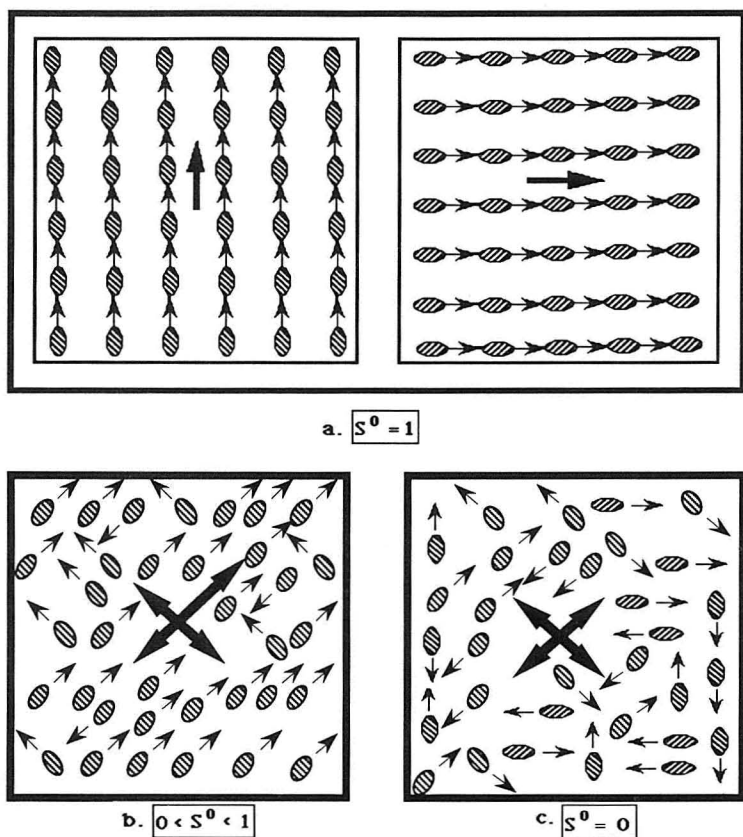


Figure 2: Types of disorders in spatial free-point molecular arrangements: (a) Completely ordered (total anisotropy); Parallel and antiparallel arrangements. (b) Partial long-range order (partial anisotropy): Nematic phase arrangement. (c) Total absence of long-range order (total isotropy): Random arrangement.

is a more valid order function for the neural assembly than  $S^0 = 0$ . In terms of  $S^0 = \frac{3}{2}\langle \cos^2 \theta \rangle - \frac{1}{2}$ , the term  $\langle \cos^2 \theta \rangle$  should correspond to a value between  $\frac{1}{3}$  and 1 (justifying the spatial anisotropy), rather than the lower limit, namely  $\frac{1}{3}$ .

To determine an appropriate squashing function for this intermediate range of  $\langle \cos^2 \theta \rangle$  (between  $\frac{1}{3}$  and 1, or, for  $0 < S^0 < 1$ ), the quantity  $\langle \cos^2 \theta \rangle$  is replaced by  $(\frac{1}{3} + \frac{1}{3q})$  in defining the order parameter  $S^0$ . Hence,

$$S^0 = \frac{3}{2} \left( \frac{1}{3} + \frac{1}{3q} \right) - \frac{1}{2} \quad (4.1)$$

where  $q \rightarrow \infty$  and  $q = \frac{1}{2}$  set the corresponding limits of  $S^0 = 0$  and  $S^0 = 1$ , respectively.

Resorting again to statistical mechanics,  $q = \frac{1}{2}$  refers to dichotomous states if the number of states are specified by  $(2q + 1)$ . For the dipoles or neural alignments, it corresponds to the two totally discrete anisotropic orientations (parallel or antiparallel). In a statistically isotropic, randomly oriented system, however, the number of possible discrete alignments would approach infinity, as dictated by  $q \rightarrow \infty$ .

For the intermediate  $(2q + 1)$  number of discrete orientations, the extent of dipole alignment to an external field (or, correspondingly, the output response of a neuron to excitation) would be decided by the probability of a discrete orientation being realized. It can be specified by [9]

$$\begin{aligned} L_q(x) &= \frac{\sum_{m=-q}^{+q} \frac{m}{q} \exp(\frac{m}{q}x)}{\sum_{m=-1}^{+q} \exp(\frac{m}{q}x)} \\ &= \left(1 + \frac{1}{2q}\right) \coth \left[ \left(1 + \frac{1}{2q}\right) x \right] - \frac{1}{2q} \coth \left( \frac{1}{2q} x \right) \end{aligned} \quad (4.2)$$

Equation 4.2 is a modified Langevin function and is also known as the Bernoulli function. The traditional Langevin function  $L(x)$  is the limit of  $L_q(x)$  for  $q \rightarrow \infty$ . The other limiting case, namely  $q = \frac{1}{2}$  (which exists for dichotomous states), corresponds to  $L_{1/2}(x) = \tanh(x)$ .

Thus, the sigmoidal function  $F(x)$  that decides the neural output response to an excitation has two bounds. With  $F(x) = \tanh(x)$ , it corresponds to the assumption that there exists a total orientational long-range order in the neural arrangement. As previously discussed by the authors [8],  $F(x) = \tanh(x)$  has been conventionally regarded as the squashing function for neural nets purely on empirical considerations: the input-output nonlinear relation being S-shaped (which remains bounded between two logistic limits, and follows a continuous monotonic functional form between those limits). In terms of the input variable  $\sigma_i$  and the gain/scaling parameter  $\lambda$  of an  $i$ th neuron, the sigmoidal function specified as the hyperbolic tangent function is  $\tanh(\lambda\sigma_i)$ . The logistic operation that compresses the range of the input so



that the output remains bounded between the logical limits is also specified alternatively by an exponential form  $F(x) = 1/[1 + \exp(-x)]$ , with  $x = \lambda\sigma_i$ .

The adoption of the hyperbolic tangent or the exponential form in neural network analyses has been purely empirical, with no justifiable reasoning attributed to their choice (except for their being sigmoidal). Hence, the authors developed statistical-mechanics-based considerations (as applied to the state vector of a neural unit) to elucidate a justifiable sigmoidal function [8]. As a result, by analogy with particle dynamics (wherein the collective response is attributed to nonlinear dependence of forces on positions of particles), the corresponding statistics due to Maxwell-Boltzmann were extended to neural response to describe the stochastic aspects of the neural state vector  $\sigma_i$ . Hence, as elaborated by the authors in [8], the Langevin function  $L(\lambda\sigma_i)$  was derived as the sigmoid representing the neural input-output response with  $\lambda = \beta/k_B T$ , where  $\beta$  is a scale factor and  $k_B T$  represents the pseudo-Boltzmann energy, as before.

Pursuant to the discussion of Equation (4.2),  $L(x) = L_{q \rightarrow \infty}(x)$  specifies the system in which the randomness is totally isotropic; that is, it is implicit that the anisotropy is zero. This refers to an extensive situation, assuming that the neural configuration poses no spatial anisotropy or long-range order whatsoever. Likewise, the intuitive modeling of  $F(x) = \tanh(x)$  (as commonly adapted) depicts a totally anisotropic system wherein the long-range order  $S^0$  attains 1. In other words,  $\tanh(x) = L_{q \rightarrow 1/2}(x)$ , with the dichotomous discrete orientations (parallel or antiparallel) being specified by  $(2q + 1) \rightarrow 2$ .

In the nematic phase, neither of the functions just discussed (namely,  $\tanh(x)$  and  $L(x)$ ) is commensurable, since a partial long-range order (depicting a partial anisotropy) is imminent in such systems. Thus, with  $\frac{1}{2} < q < \infty$ , the true sigmoid of a neural arrangement (with an inherent nematic, spatial long-range order) should be  $L_q(x)$ .

**Remark.** It can be concluded that the conventional sigmoid (namely, the hyperbolic tangent or its variations) and the Langevin function [8] constitute the upper and lower bounds, respectively, of the state-vector squashing characteristics of a neural unit. The relevant considerations of the foregoing discussions are summarized in Table 1.

## 5. Hamiltonian of the neural arrangement with spatial long-range order

In general, the anisotropy of a disorder leads to a Hamiltonian that can be specified in two ways as follows.

(a) Suppose the exchange Hamiltonian is given by

$$H = - \sum W_{xx} S_i^x S_j^x + W_{yy} S_i^y S_j^y + W_{zz} S_i^z S_j^z \quad (5.1)$$

where  $W_{xx}$ ,  $W_{yy}$ , and  $W_{zz}$  are diagonal elements of the exchange matrix  $\mathbf{W}$

Spatial Configuration of Neurons	Order Parameter $S^0$	Sigmoidal Function $F(x)$	Slope at the Origin $\left. \frac{dF(x)}{dx} \right _{x=0}$	Remark	Number of Orientational States (Analogous to Quantum No.)	Sigmoids $F(x)$
Totally Isotropic (Zero Long-Range Order)	0	Langevin Function $L_{q \rightarrow \infty}(x) = L(x)$ $= \coth(x) - \frac{1}{x}$	$\frac{1}{3}$	Lower Bound Sigmoid	$\infty$ ( $q \rightarrow \infty$ )	
Totally Anisotropic (Long-Range Order to Total Extent)	1	Hyperbolic Tangent $L_{q \rightarrow 1/2}(x) = \tanh(x)$	1	Upper Bound Sigmoid	2 ( $q \rightarrow 1/2$ )	
Partially Anisotropic (Partial Long-Range Order as in Nematic Phase)	$\frac{3}{2} \left[ \frac{1}{3} + \frac{1}{3q} \right] - \frac{1}{2}$ ( $0 < S^0 < 1$ )	Bernoulli Function $L_q(x) = (1+y) \coth[(1+y)x]$ $- y \coth(yx)$ $y = 1/2q$	$\left[ \frac{1}{3} + \frac{1}{3q} \right]$	Nematic Phase Neuronal Configuration with Partial Long-Range Order	$(2q+1)$	
Zero Interaction	—	Signum Function $L_{q \rightarrow 0}(x) = \text{sgn}(x)$	0	McCulloch-Pitts Model	—	

Table 1: Types of spatial disorder in neural configuration, and sigmoidal functions of the nonlinear input-output squashing characteristics of a neural assembly.

(with the off-diagonal elements being zero). If  $W_{xx} = W_{yy} = 0$  and  $W_{zz} \neq 0$ , it is a symmetric anisotropy (with dichotomous states as in the Ising model). Note that the anisotropy arises if the strength of at least one of the exchange constants is different from the other two. If  $W_{xx} = W_{yy} \neq 0$  and  $W_{zz} = 0$ , it corresponds to an isotropic xy model; if  $W_{xx} = W_{yy} = W_{zz}$ , it is known as the isotropic Heisenberg model.

(b) Given that the system has an anisotropy due to partial long-range order, as in the nematic phase representation of the neural arrangement, the corresponding Hamiltonian is

$$H = -\mathbf{W} \sum S_i S_j + H_a \quad (5.2)$$

where  $H_a$  refers to the anisotropic contribution, which can be specified by an inherent constant  $h_i^0$  related to the order parameter  $S^0$ , so that

$$H = - \sum_i \sum_j W_{ij} S_i S_j - \sum_i h_i^0 S_i \quad (5.3)$$

While the interactions  $W_{ij}$  are local,  $H$  refers to an extensive quantity corresponding to the long-range orientational (spatial) interconnections in the neural arrangement.

## 6. Spatial persistence of synaptic transmission in the nematic phase representation of neural arrangement

The nematic phase modeling of the neural arrangement specifies a long-range spatial anisotropy, which may pose a persistence (or, preferably, a directional routing) of the synaptic transmission. Pertinent analysis would be similar to the time-domain persistence in neural firing patterns demonstrated by Little [3].

Considering  $(2q+1)$  possible spatial orientations (or states) pertaining to  $M$  interacting neurons as represented by  $\Psi(\alpha)$ , the probability of obtaining the state  $\Psi(\alpha')$ , having started with a preceding  $\Psi(\alpha)$   $m$  spatial intervals ( $x$ ), can be written in terms of a transfer matrix [3] as

$$\Psi(\alpha') \mathbf{T}_M^m \Psi(\alpha) \quad (6.1)$$

where  $\Psi(\alpha)$  can be expressed in terms of  $(2q+1)^M$  orthonormal eigenvectors  $\vartheta_r$  (with eigenvalues  $\lambda_r$ ) of the operator  $\mathbf{T}_M$ . Each  $\vartheta_r$  has  $(2q+1)M$  components, one for each configuration  $\alpha$ :

$$\Psi(\alpha) = \sum_{r=1}^{(2q+1)^M} \vartheta_r(\alpha) \quad (6.2)$$

Hence,

$$\langle \Psi(\alpha') | \mathbf{T}_M | \Psi(\alpha) \rangle = \sum_r \lambda_r \vartheta_r(\alpha') \vartheta_r(\alpha) \quad (6.3)$$

Analogous to Little's time-domain persistent order analysis [3], it is of interest to find a particular state  $\alpha_1$  after  $m$  spatial steps, having commenced at an arbitrary spatial location in the neural topology. The probability of obtaining the state  $\alpha_2$  after  $l$  spatial steps, given  $\alpha_1$  after  $m$  spatial steps from the commencement location, can be written as [3]:

$$\Gamma(\alpha_1, \alpha_2) = \Gamma(\alpha_1) \Gamma(\alpha_2) \quad (6.4)$$

which explicitly specifies no spatial correlation between the states  $\alpha_1$  and  $\alpha_2$ . However, if the maximum eigenvalue  $\lambda_{\max}$  is degenerate, the above factorization of  $\Gamma(\alpha_1, \alpha_2)$  is not possible and there will be a spatial correlation in the synaptic transmission behavior. Such a degeneracy in spatial order can be attributed to any possible transition from isotropic to anisotropic nematic phase in the neural configuration. That is, should there be a persistent or orientational linkage/interaction of neurons in the path of synaptic transmission, degeneracy may set in. In the spin system, a similar degeneracy refers to the transition from a paramagnetic to a ferromagnetic phase. In a neural system, considering the persistence in the time-domain, Little [3] observes that long-range time-ordering is related to short-term memory considerations as dictated by intracellular biochemical process(es).

## 7. Discussions

In essence, the possibility has been proposed of a long-range spatial order prevailing in neural arrangement, so that a persistence in random synaptic signal proliferation (partially oriented in a specific direction) could exist. The following inferences result.

1. Because random asymmetry in synaptic coupling is a physiological reality, it refers implicitly to the existence of spatial anisotropy in neural interactions with a persistent long-range order; this spatial long-range order in neural arrangement mimics the nematic phase orientations of liquid crystals. Neural interactions can therefore be regarded as analogous to an electric dipole interaction model pertaining to the anisotropic (random) collection of molecular free-point dipoles.
2. With a prescribed anisotropicity, the disordered system of neurons can be specified by an order parameter  $S^0$ , such that  $0 < S^0 < 1$ , where  $S^0$  defines the degree of alignment in the preferred direction. For a perfectly parallel (or antiparallel) polarization,  $S^0 = 1$ ; for isotropically random orientations,  $S^0 = 0$ .
3. With the limits of  $S^0$  being 0 and 1, the corresponding anisotropic order function is defined by  $S^0 = \frac{3}{2}(\frac{1}{3} + \frac{1}{3q}) - \frac{1}{2}$ , where  $q \rightarrow \infty$  and  $q = \frac{1}{2}$  are the factors corresponding to the lower and upper limits of  $S^0$ , respectively.

4. The nonlinear neural response with a squashed output for a given input is normally specified by a sigmoidal function. For  $q = \frac{1}{2}$  (or  $S^0 = 1$ ), the hyperbolic tangent function and its variations are conventionally adapted as the sigmoid, despite the fact that the relevant neural configuration refers only to an extensive upper limit, being totally anisotropic with parallel (or antiparallel) arrangement of neural cells. For  $q \rightarrow \infty$  (or  $S^0 = 0$ ), the Langevin function  $L(x) = \coth(x) - 1/x$  has been demonstrated elsewhere by the authors [8] as a stochastically justifiable sigmoid to represent the neural disorder system (in place of the hyperbolic tangent function). However,  $L(x)$  represents only the other extreme of the order parameter, namely  $S^0 = 0$  (corresponding to a totally isotropic random configuration of the neural cells).

Hence, the present work suggests that in a true representation of the neural arrangement, wherein there is a nematic phase attribution of the neuron with a long-range spatial order, it may be necessary to consider the factor  $q$  being such that  $\frac{1}{2} < q < \infty$ , which prescribes a partial anisotropy to the neural system. As a consequence, the synaptic transmission across the partially disordered, interconnected neural cells corresponds to a persistent, directional proliferation. That is, a spatial persistence of neural transmission consistent with a nematic phase modeling holds true.

5. The relevant sigmoid for the partial (anisotropic) disorder situation has been indicated in the present work as  $L_q(x)$ , known as the modified Langevin function or the Bernoulli function. The limiting cases of  $q \rightarrow \infty$  and  $q = \frac{1}{2}$  correspond to the lower and upper bounds, respectively, of the squashing characteristic of the neural response.
6. By modeling the neural system as a partially anisotropic spatial order, the corresponding transfer matrix depicting the neural interactions automatically becomes asymmetric; therefore, the question of artificially imposing the restriction of bidirectional symmetry (that is,  $W_{ij} = W_{ji}$ ) on the weighting factor (of the interconnections) does not arise. The relevant criticisms posed thereof (such as Griffith's [2]) are therefore avoided.
7. With anisotropy in the spatial arrangement, the corresponding Hamiltonian has an anisotropic contribution in portraying the extensive behavior of the neural interactions.
8. Further, by considering a partial (spatial) anisotropy, the state transitions at any given neural sites  $i$  and  $j$  (both  $i$  and  $j$  falling in the preferred direction/orientation) would exhibit a correlation. It refers to a degeneracy in the spatial order and the existence of the nematic phase.

This correlation, if it exists, stipulates a spatial persistence in synaptic transmission, analogous to a long-range time-ordering that may exist in the neural system as observed by Little [3], or in the Ising model of ferromagnetic phase subjected to an external field.

## 8. Conclusion

It has been proposed that, in addition to the time-domain persistent long-range order (as conceived by Little [3]), there is a spatial long-range order coexisting in the neural system, with the neural activity assuming a nematic phase. The physiological aspect of asymmetry in synaptic coupling and the corresponding bidirectional asymmetry in the neural interactions is viewed therefore as due to considerations of both temporal and spatial anisotropy. The temporal anisotropy refers to persistent time correlations in the firing characteristics associated with the cells, and the spatial anisotropy provides a preferential routing of neural activity across the assembly of neurons. Hence, the complex dynamic activities in cellular automata correspond to the joint spatiotemporal long-range persistent behavior of interconnected neurons. Furthermore, the weighted sum of the neural inputs directed anisotropically at a cell (both spatially and temporally) constitutes the argument of a nonlinear activation function. This function is shown as the Bernoulli function consistent with basic thermodynamic considerations.

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