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Introduction to Terminal Dynamics

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Abstract. This paper introduces terminal dynamics as a set of ordinary differential equations which does not possess a unique solution, due to violation of the Lipschitz condition at equilibrium points. Each equilibrium point represents a terminal attractor that is approached in finite time or a terminal repeller for which the solution splits into two equally probable branches. This property introduces elements of stochasticity that are associated with the random walk paradigm. A relationship is established between the original dynamical model and the corresponding Fokker-Planck equation for probability density. A new type of attractor that represents a stochastic process is described. The relevance of the terminal model to irreversibility in Newtonian dynamics and to chaos theory is discussed.

1. Introduction

The governing equations of classical dynamics may be derived from Lagrangian functions, from variational principles, or directly from Newton's laws of motion, and they may be presented in various equivalent forms. However, there is one mathematical restriction on all such forms: the differential equations describing a dynamical system

$$\dot{x}_i = v_i(x_1, x_2, \ldots, x_n) \qquad i = 1, 2, \ldots, n \tag{1}$$

must satisfy the Lipschitz condition, which expresses that all the derivatives

$$\left| \frac{\partial v_i}{\partial x_j} \right| < \infty \tag{2}$$

must be bounded. This mathematical restriction guarantees the uniqueness of the solution to (1), subject to fixed initial conditions, and that uniqueness has proved to be very important for the application of dynamical systems to the modeling of energy transformations in mechanics, physics, and chemistry. However, attempts to exploit classical dynamics for the application of information processing to the modeling of biological and social behaviors have exposed certain limitations of the approach, due to determinism and reversibility of solutions. Mathematical and physical aspects of these limitations, as well as the consequences of their removal, are discussed in [7–17]. In this paper we present a general structure for dynamical systems that does not possess a unique solution, due to violation of condition (2) at equilibrium points.

2. Terminal limit sets

2.1 Terminal attractors and repellers

Terminal dynamics can be introduced as a set of nonlinear ordinary differential equations of the form

$$\dot{x}_i = v_i^k(x_1, x_2, \ldots, x_n) \qquad i = 1, 2, \ldots, n \tag{3}$$

in which

$$\left|\frac{\partial v_i}{\partial x_j}\right| < \infty \tag{4}$$

and $k < 1$. Therefore,

$$\left|\frac{\partial \dot{x}_i}{\partial x_j}\right| = k v^{(k-1)}(x_1, \ldots, x_n) \left|\frac{\partial v_i}{\partial x_i}\right| \to \infty \qquad \text{if } \dot{x}_i \to 0 \tag{5}$$

and the Lipschitz condition (2) is violated at all the equilibrium points

$$\dot{x}_i = 0$$

As in the classical case, the equilibrium points are attractors if the real parts of the eigenvalues of the matrix

$$m = \left\|\frac{\partial v_i}{\partial x_j}\right\| \tag{6}$$

are negative; that is, if

$$\mathrm{Re}\ \lambda_i < 0 \tag{7}$$

and they are repellers if some of the eigenvalues have positive real parts.

In order to emphasize the difference between classical and terminal equilibrium points, we will begin with the simplest terminal dynamical system, as follows.

$$\dot{x} = -x^{1/3} \tag{8}$$

This equation has an equilibrium point at $x = 0$, at which the Lipschitz condition (2) is violated:

$$\frac{d\dot{x}}{dx} = -\frac{1}{3}x^{-2/3} \to -\infty \qquad \text{at } x \to 0 \tag{9}$$

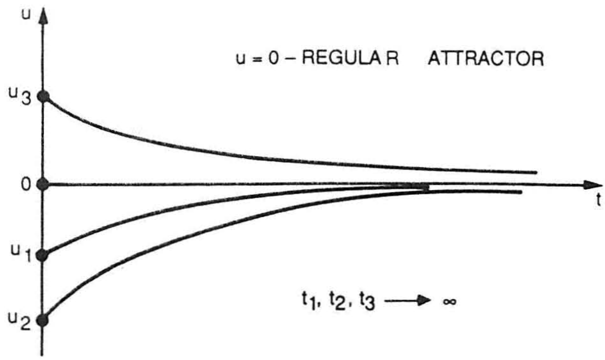

Figure 1: Convergence to regular attractor.

Because condition (7) is satisfied, that is,

$$\text{Re}\ \lambda \to -\infty < 0 \tag{10}$$

this point is an attractor of "infinite" stability.

The relaxation time for a solution with the initial condition $x = x_0 < 0$ to this attractor is finite:

$$t_0 = -\int_{x_0}^{x \to 0} \frac{dx}{x^{1/3}} = \frac{3}{2} x_0^{2/3} < \infty \tag{11}$$

Consequently, this attractor becomes terminal. It represents a singular solution which is intersected by all the attracted transients (see Figures 1 and 2).

For the equation

$$\dot{x} = x^{1/3} \tag{12}$$

the equilibrium point $x = 0$ becomes a terminal repeller, as follows.

$$\frac{d\dot{x}}{dx} \to \frac{1}{3} x^{-(2/3)} \to \infty \qquad \text{at } x \to 0 \tag{13}$$

that is,

$$\text{Re}\ \lambda \to \infty > 0$$

If the initial condition is infinitely close to this repeller, the transient solution will escape the repeller during a finite time period:

$$t_0 = \int_{\varepsilon \to 0}^{x_0} \frac{dx}{x^{1/3}} = \frac{3}{2} x_0^{2/3} < \infty \qquad \text{if } x < \infty \tag{14}$$

whereas, for a regular repeller, the time would be infinite.

As an alternative to (8) and (12), one can consider a more general case,

$$\dot{x} = \pm x^k \qquad k > 0 \tag{15}$$

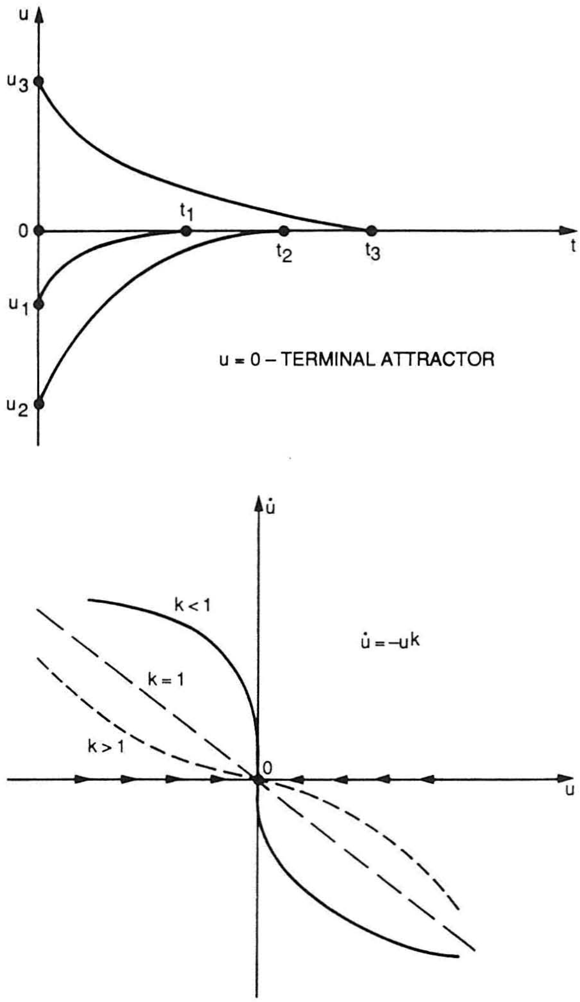

Figure 2: Convergence to terminal attractor.

for which the relaxation time for the attractor (or the escaping time for the repeller) is

$$t_0 \begin{cases} \to \infty & \text{if } k \geq 1 \\ = \dfrac{x_0^{1-k}}{1-k} & \text{if } k < 1 \end{cases} \tag{16}$$

As shown in the theory of differential equations, singular solutions in equations

$$F(x, y, y') = 0 \tag{17}$$

are found by eliminating y' from the system, as follows.

$$F(x, y, y') = 0 \qquad \frac{\partial F}{\partial y'} = 0 \tag{18}$$

Hence, static terminal attractors (if they exist in (17)) must be among the solutions to system (18).

2.2 Physical Interpretation of Terminal Attractors

As will be pointed out in the Conclusion to this paper, the mathematical formalism of terminal dynamics follows from a more general structure of the dissipation function which allows the existence of smooth transitions from static to kinetic friction. It should be emphasized that the behavior of the solutions around the equilibrium points in terminal dynamics is more "realistic" than in classical dynamics, because the actual time of convergence to equilibrium points is finite. However, in order to make it finite, the Lipschitz condition must be violated, because all the trajectories must intersect at the equilibrium point (see Figure 2). In classical dynamics, the Lipschitz condition is not violated, and the infinite time of convergence is accounted for by "small dissipative forces" that are always present. In fact, terminal dynamics incorporates these forces via the parameter k (see (3)), which can be found through measurement of the convergence time (see (16)).

It can be shown that the mathematical concept of the terminal attractor has other physical interpretations. One such interpretation is the energy-cumulation effect, in which case one deals with the finite time of convergence of a propagating wave rather than a motion of an individual particle. As an example, consider a propagation of an isolated pulse in an elastic continuum along the x axis. In general, the speed of propagation $\dot{x} = \lambda$ depends on x. Suppose there exists a point x^* such that $\lambda(x^*) = 0$. Then the time t^* during which the leading edge of the propagating pulse will approach x^* is expressed via the following integral.

$$t^* = \int_{x_0}^{x \to x^*} \frac{dx}{\lambda(x)} \tag{19}$$

If λ can be presented in the form

$$\lambda = (x^* - x)^k \qquad 0 < k < 1 \tag{20}$$

then this integral converges and, therefore, the time t^* is finite. It is easily verifiable that, in this case, the differential equation

$$\dot{x} = (x^* - x)^k \tag{21}$$

that describes the dynamics of the pulse propagation has a terminal attractor at $x = x^*$. But if the leading and the trailing edges of the propagating pulse approach the same point x^* during finite time, then the width of the pulse eventually will shrink to zero, and all the energy transported by the pulse will be distributed over a drastically diminishing length. Hence, the existence of a terminal attractor in such models leads to an unbounded concentration of energy in the neighborhood of the attractor.

Based upon this model, [2, 3] explain and describe the formation of a supersonic snap at a free end of a filament suspended in a gravity field, and the accumulation of shear strain energy at the soil surface in response to an underground explosion. In these models, the free end of the filament and the free surface of the soil serve as terminal attractors. Some terminal effects in fluid dynamics are introduced and discussed in [15].

2.3 Periodic terminal limit sets

Thus far, we have concentrated on static terminal attractors. We now demonstrate the existence of periodic terminal attractors. For that purpose, let us consider a dynamical system separable in polar coordinates r, θ, as follows.

$$\dot{r} = r(R - r)^{1/3} \qquad (r \leq R) \tag{22}$$

$$\dot{\theta} = \omega \tag{23}$$

In this case, $d\dot{r}/dr \to -\infty$ at $r \to R$ (compare with (9)) and, therefore, the solutions $r = R, \theta = \omega t + \theta(0)$ form a terminal limit cycle. Its basin is defined by the condition $r > 0$. For the solution with the initial condition $r_0 < R$ the relaxation time is finite, as follows.

$$t_0 = \int_{r_0}^{R} \frac{dr}{r(R-r)^{1/3}} < \int_{r_0}^{R} \frac{dr}{r_0(R-r)^{1/3}} = \frac{2}{3r_0}(R - r_0)^{2/3} < \infty \tag{24}$$

It is easily demonstrated that a periodic terminal repeller can be obtained by changing the sign in the right-hand side of (22).

The terminal analog of a chaotic attractor is introduced and discussed in [13, 15].

2.4 Unpredictability in terminal dynamics

The concept of unpredictability in classical dynamics was introduced in connection with the discovery of chaotic motions in nonlinear systems. Such motions are caused by the Lyapunov instability [4], which is characterized by a violation of the continuous dependence of solutions on the initial conditions during an unbounded time interval ($t \to \infty$). That is why the unpredictability in such systems develops gradually. Indeed, if two initially close trajectories diverge exponentially:

$$\epsilon = \epsilon_0 \exp \lambda t \qquad 0 < \lambda < \infty \tag{25}$$

then, for an infinitesimal initial distance $\epsilon_0 \to 0$, the current distance ϵ becomes finite only at $t \to \infty$. For this reason, the Lyapunov exponents (the mean exponential rate of divergence) are defined in an unbounded time interval, as follows.

$$\sigma = \lim \left(\frac{1}{t}\right) \ln \frac{\epsilon}{\epsilon_0} \qquad t \to \infty \tag{26}$$

In distributed dynamical systems, described by partial differential equations, there exists a stronger instability (discovered by Hadamard). In the course of this instability, a continuous dependence of a solution on the initial conditions is violated during an arbitrarily small time period. Such a "blowup" instability is caused by a failure of hyperbolicity and a transition to ellipticity [2]. In this section we show that a similar type of blowup instability that leads to "discrete pulses" of unpredictability can occur in dynamical systems which contain terminal repellers.

Let us analyze the transient escape from the terminal repeller in the equation

$$\dot{x} = x^{1/3} \qquad x_0 = x(0) \tag{27}$$

assuming that $|x_0| \to 0$. The solution to (27) reduces to the following.

$$x = \pm t^{3/2} \qquad x \neq 0 \tag{28}$$

Hence, two different solutions are possible for "almost the same" initial conditions. The fundamental property of this result is that the divergence of the solutions to (28) is characterized by an unbounded parameter, σ.

$$\sigma = \lim_{t \to t_0} \left(\frac{1}{t} \ln \frac{2t^{3/2}}{2|x_0|}\right) = \infty \qquad |x_0| \to 0 \tag{29}$$

where t_0 is an arbitrarily small (but finite) positive quantity. In contrast to (26), the rate of divergence in (29) can be defined in an arbitrarily small time interval, because the initial infinitesimal distance between the solutions becomes finite during this interval. Thus, a terminal repeller represents a drastically diminishing but infinitely powerful "pulse of unpredictability" which is "pumped" into the dynamical system.

To illustrate the unpredictability in such a non-Lipschitzian dynamics, we turn to the following equation.

$$\dot{x} - yx^{1/3} = 0 \tag{30}$$

where

$$y = \cos \omega t \tag{31}$$

Assuming that $x \to 0$ at $t \to 0$, we obtain the regular solutions

$$x = \pm \left(\frac{2}{3\omega} \sin \omega t \right)^{3/2} \qquad x \neq 0 \tag{32}$$

and a singular solution (an equilibrium point)

$$x = 0 \tag{33}$$

During the first time period

$$0 < t < \frac{\pi}{2\omega} \tag{34}$$

equilibrium point (33) is a terminal repeller (because $y > 0$). Therefore, within this period, solutions (32) have the same property as solutions (28): their divergence is characterized by an unbounded rate σ.

During the next time period

$$\frac{\pi}{2\omega} < t < \frac{3\pi}{2\omega}$$

equilibrium point (33) becomes a terminal attractor (because $y < 0$), and the system which approaches this attractor at $t = \pi\omega$ remains motionless until $t > 3\pi/2\omega$. After that point, the terminal attractor converts into the terminal repeller, and the system escapes again.

It is important to notice that each time the system escapes the terminal repeller, the solution splits into two symmetrical branches; therefore, the total trajectory can be combined from 2^n pieces, where n is the number of cycles; that is, it is the integer part of the quantity $(t/2\pi\omega)$. The nature of this unpredictability is significantly different from the unpredictability in chaotic systems.

Motion (32) resembles chaotic oscillations known from classical dynamics: it combines random characteristics with the attraction to a center. However, in contrast to classical chaos, motion (32) is driven by a failure of uniqueness of the solution at the equilibrium point, and it has a well organized probabilistic structure. Because the time of approaching the equilibrium point $x = 0$ by solution (32) is finite, this type of chaos can be called terminal [13–15].

Equations (30) and (31) can be presented in autonomous form, as follows.

$$x = yx^{1/3} \tag{30a}$$
$$\dot{y} = -\omega z + y(1 - y^2 - z^2) \tag{31a}$$
$$\dot{z} = \omega y + z(1 - y^2 - z^2) \tag{31b}$$

If one takes into account that the last two equations have periodic attractors,

$$y = \cos \omega t \qquad z = -\sin \omega t$$

Although the structure of (30a), (31a), and (31b) in general resembles the structure of the Lorentz or Rossler attractors, the violation of the Lipschitz condition is important for the appearance of nondeterministic solutions. Indeed, if (30) is replaced by the following,

$$\dot{x} = yx \tag{30b}$$

then the solution to the system (30a), (31a), and (31b)

$$x = x_0 e^{(-1/\omega)\sin\omega t} \qquad x_0 = x(0) \text{ at } t \to \infty$$

becomes periodic.

2.5 Irreversibility of terminal dynamics

Classical dynamics describes processes in which time t plays the role of a parameter: it remains fully reversible, in the sense that the time-backward motion can be obtained from the governing equation by time inversion, $t \longrightarrow -t$. (This means that classical dynamics cannot explain the emergence of new dynamical patterns in nature.) However, there exists a class of phenomena for which past and future play different roles, and time is not invertible: by definition (the second low of thermodynamics), irreversibility is introduced into thermodynamics by postulating the increase of entropy. As stressed by Prigogine (1980), irreversible processes play a fundamental constructive role in the physical world; they are the basis of important coherent processes, which appear with particular clarity on the biological level.

In this connection, let us compare the dynamical behavior of solutions in small neighborhoods of classical and terminal repellers, respectively:

$$\dot{x} = x \tag{35}$$

and

$$\dot{x} = x^{1/3} \tag{36}$$

The solution to (35),

$$x_+ = x_0 e^t \tag{37}$$

which describes an escape from a classical repeller, is reversible because

$$u_- = x_0 e^{-t} \tag{38}$$

is a possible motion describing a convergence to a classical attractor $x = 0$. The solution to (36),

$$x_+ = \sqrt{\left(\frac{2}{3}t\right)^3} \tag{39}$$

is irreversible because the time-backward motion

$$x_- = \sqrt{-\left(\frac{2}{3}t\right)^3} \tag{40}$$

does not exist (x has imaginary value).

This mathematical formalism expresses deeper roots of the irreversibility of terminal dynamics, which can be understood if one turns to the solution of (30) and (31). This solution consists of regular and singular parts. When the regular solution (32) approaches the equilibrium point $x = 0$ (33) (in finite time), it switches to the singular solution $x \equiv 0$, and this switch is irreversible.

3. Probabilistic structure of terminal dynamics

As shown in [16], the terminal version of Newtonian dynamics is different from its classical version only within drastically diminishing neighborhoods of equilibrium states and, therefore, it contains classical mechanics as a special case. This means that terminal dynamics is not always unpredictable and irreversible: in some domains it is identical with classical dynamics. However, in this section our attention will be concentrated on effects specific to terminal dynamics and, in particular, on its probabilistic structure.

The fundamental difference between the probabilistic properties of terminal dynamics and those of stochastic or chaotic differential equations should be emphasized. The randomness of stochastic differential equations is caused by random initial conditions, random force, or random coefficients; in chaotic equations, small (but finite!) random changes of initial conditions are amplified by the mechanism of instability. However, in both cases the differential operator itself remains deterministic. In contrast, randomness in terminal dynamics results from the violation of the uniqueness of the solution at equilibrium points; therefore, the differential operator itself generates random solutions.

3.1 A terminal model of the random walk process

Random walk is a stochastic process in which changes occur only at fixed times. In this section we introduce a terminal dynamics that describes this process.

We begin with the following dynamical system.

$$\dot{x} = \gamma \sin^{1/3} \frac{\sqrt{\omega}}{\alpha} x \sin \omega t \qquad \gamma = \text{Const}, \, \omega = \text{Const}, \, \alpha = \text{Const} \tag{41}$$

At the equilibrium points

$$x_m = \frac{\pi m \alpha}{\sqrt{\omega}} \qquad m = \ldots, -2, -1, 0, 1, 2, \ldots \tag{42}$$

it can be verified that the Lipschitz condition is violated:

$$\partial \dot{x} / \partial x \to \infty \qquad \text{at } x \to x_m \tag{43}$$

If $x = 0$ at $t = 0$ then, during the first period

$$0 < t < \frac{\pi}{\omega} \tag{44}$$

the point $x_0 = 0$ is a terminal repeller because $\sin \omega t > 0$; the solution at this point splits into two branches (positive and negative) whose divergence is characterized by the unbounded parameter σ (see (29)). Consequently, x can move with equal probability in the positive or the negative direction. For the sake of concreteness, we assume that it moves in the positive direction. Then the solution approaches the second equilibrium point $x_1 = \pi\alpha/\sqrt{\omega}$ at

$$t^* = \frac{1}{\omega} \arccos \left[1 - \frac{B\left(\frac{1}{3}, \frac{1}{3}\right)}{2^{1/3}} \frac{\alpha\sqrt{\omega}}{\gamma} \right] \tag{45}$$

in which B is the Beta function.

It can be verified that the point x_1 will be a terminal attractor at $t = t_1$ if

$$t_1 \leq \pi/\omega$$

that is, if

$$\frac{\gamma}{\alpha} \geq \frac{B\left(\frac{1}{3}, \frac{1}{3}\right)}{2^{4/3}} \sqrt{\omega} \tag{46}$$

Therefore, x will remain at point x_1 until it becomes a terminal repeller; that is, until $t > t_1$. At that point, the solution splits again: one of the two possible branches approaches the next equilibrium point $x_2 = 2\pi\alpha/\sqrt{\omega}$, while the other returns to the point $x_0 = 0$, and so forth. The periods of transition from one equilibrium point to the next are the same length, and are given by (45).

It is important to notice that these transition periods t^* are bounded only because of the failure of the Lipschitz condition at the equilibrium points. Otherwise they would be unbounded, because the time of approaching a regular attractor is infinite (as is the time of escaping a regular repeller).

Thus, the evolution of x prescribed by (41) is totally unpredictable: it has 2^m different scenarios, where $m = E(t/t^*)$; whereas any prescribed value of x from (42) will appear eventually. This evolution is identical to random walk, and the probability $f(x, t)$ is governed by the following difference equation.

$$f\left(x, t + \frac{\pi}{\omega}\right) = \frac{1}{2} f\left(x - \frac{\pi\alpha}{\sqrt{\omega}}, t\right) + \frac{1}{2} f\left(x + \frac{\pi\alpha}{\sqrt{\omega}}, t\right) \tag{47}$$

For a better physical interpretation, we assume that

$$\frac{\pi\alpha}{\sqrt{\omega}} \ll L \qquad t^* \ll T \tag{48}$$

that is,

$$\omega \longrightarrow \infty$$

where L and T are the total length and the total time period of the random walk.

Setting

$$\frac{\pi\alpha}{\sqrt{\omega}} \to 0 \qquad t^* \to 0 \tag{49}$$

we arrive at the Fokker-Planck equation:

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2} D^2 \frac{\partial^2 f(x,t)}{\partial x^2} \qquad D^2 = \pi\alpha^2 \tag{50}$$

The unrestricted solution of (50), given the initial condition that random walk starts from the origin $x = 0$ at $t = 0$, is

$$f(x,t) = \frac{1}{\sqrt{(2\pi D^2 t)}} \exp\left(-\frac{x^2}{2D^2 t}\right) \tag{51}$$

This solution qualitatively describes the evolution of the probability distribution for dynamical equation (41). It is worth noticing that one should turn to difference equation (47) for the exact solution, because $\omega < \infty$.

Equation (47) can be presented in operator form, as follows.

$$\left[E_t - \frac{1}{2}(E_x + E_x^{-1})\right] f = 0 \tag{52}$$

where E_t and E_x are the shift operators

$$E_t f(x,t) = f(x, t+\tau) \qquad E_x f(x,t) = f(x+h, t) \qquad h = \frac{\pi\alpha}{\sqrt{\omega}} \tag{53}$$

Utilizing the relationships between the shift operators and the differential operator D,

$$E_t^r = e^{r\tau D_t} \qquad E_x^r = e^{rhD_x} \qquad D_t = \frac{\partial}{\partial t} \qquad D_x = \frac{\partial}{\partial x} \tag{54}$$

we can transfer from (47) to (50) if $\omega \to \infty$ (that is, if $\tau, h \to 0$).

For further analysis it will be more convenient to modify (41) as follows.

$$\dot{x} = \gamma \sin^k\left(\frac{\sqrt{\omega}}{\alpha} x\right) \sin \omega t \tag{55}$$

assuming that

$$k = \frac{1}{2n+1} \qquad n \to \infty \tag{56}$$

where n is an integer. This replacement does not change the qualitative behavior of dynamical system (55): it changes only its quantitative behavior between the critical points, such that we have explicit control over the period of transition from one critical point to another. Indeed, given that

$$\lim_{n\to\infty} \sin^{1/2n+1} X = \operatorname{sgn} \sin X$$

we obtain the solution for x which is valid between critical points $x^{(m)}$ and $x^{(m+1)}$:

$$x = \frac{\gamma}{\omega}(1 - \cos \omega t) \tag{57}$$

It is evident that the distances between the equilibrium points will not depend upon the step m:

$$h_m = x_m - x_{m-1} = \frac{\pi \alpha m}{\sqrt{\omega}} - \frac{\pi \alpha (m-1)}{\sqrt{\omega}} = \frac{\pi \alpha}{\sqrt{\omega}} \tag{58}$$

The period of transition from the $(m-1)$st to the mth critical point follows from (57) and (58):

$$t^* = \frac{1}{\omega} \arccos \left(1 - \frac{h_m}{\gamma}\right) \leq \frac{\pi}{\omega} \tag{59}$$

which means that

$$\delta \geq \omega h_m \tag{60}$$

because it should not exceed the period between the conversions of terminal attractors into terminal repellers (and vice versa).

3.2 Multidimensional systems

The results presented in the previous sections can be generalized to multidimensional dynamics. For that purpose, consider the following terminal dynamical system.

$$\dot{x}_i = \gamma_i \sin^k \left(\frac{\sqrt{\omega}}{\alpha_i} \sum T_{ij} x_j\right) \sin \omega t \qquad T_{ij} = \text{Const} \tag{61}$$

assuming that

$$T_{ij} = T_{ji}, T_{11} > 0, \begin{vmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{vmatrix} > 0, \ \ldots \tag{62}$$

that is, that $|T_{ij}|$ is a symmetric positive-definite matrix; and where k is defined as in (56). Properties (62) provide stability (if $\sin \omega t < 0$) or instability (if $\sin \omega t > 0$) of system (61) at the terminal equilibrium points $\overset{*}{x}_i$:

$$\overset{*}{x}_i = \lambda_i \sum_{j=1}^{n} m_j \frac{\partial \Delta}{\partial T_{ij}} \tag{63}$$

where

$$\lambda_i = \frac{\pi \alpha_i}{\Delta \sqrt{\omega}} \qquad \Delta = \det |T_{ij}| \tag{64}$$

m_i is the number of steps made by the variable x_i, and $\partial \Delta / \partial T_{ij}$ is a cofactor of the element T_{ij}.

After one step of a variable x_i, the corresponding value of m_i will change to $m_i + 1$ or $m_i - 1$ with the same probability. Thus, the length of a step h_i made by the variable x_i will have 2^n equally probable values:

$$h_i = \lambda_i \sum_{j=1}^{n} (\pm\beta_{ij}) \frac{\partial \Delta}{\partial T_{ij}} \qquad \beta_{ij} = 1 \tag{65}$$

dependent on 2^n combinations of the signs of β_{ij} in (65).

Denoting each of these combinations by q ($q = 1, 2, \ldots, 2^n$), and introducing a shift operator E_i for each variable x_i:

$$E_i f(t, x_1, \ldots, x_i, \ldots, x_n) = f(t, x_1, \ldots, x_i + 1, \ldots, x_n) \tag{66}$$

we arrive at the following governing equation for the joint probability density of the solution to (61).

$$\left(E_t - 2^{-n} \sum_{q=1}^{2^n} \prod_{i=1}^{n} E_i^{h_{iq}} \right) f = 0 \tag{67}$$

where h_{iq} is a particular value of h_i taken from (62) at a particular q.

It follows from (67) that with increase of n the dynamics of (61) becomes less and less predictable. For $n = 2$, (67) reduces to

$$\left[E_t - \frac{1}{4} (E_1^{h_{11}} E_2^{h_{21}} + E^{h_{12}} E^{h_{22}} + E^{h_{13}} E^{h_{23}} + E^{h_{14}} E^{h_{24}}) \right] f = 0 \tag{68}$$

where

$$\begin{aligned}
h_{11} &= -h_{14} = \frac{\pi}{\Delta\sqrt{\omega}} (\alpha_1 T_{22} - \alpha_2 T_{12}) \\
h_{12} &= -h_{13} = \frac{\pi}{\Delta\sqrt{\omega}} (\alpha_1 T_{22} + \alpha_2 T_{12}) \\
h_{21} &= -h_{24} = \frac{\pi}{\Delta\sqrt{\omega}} (\alpha_2 T_{11} - \alpha_1 T_{12}) \\
h_{22} &= -h_{23} = \frac{\pi}{\Delta\sqrt{\omega}} (\alpha_2 T_{11} + \alpha_1 T_{12})
\end{aligned} \tag{69}$$

If $\omega \to \infty$ (that is, if $h_{ij}, t^* \to 0$), (68) transforms into a two-dimensional Fokker-Planck equation, as follows.

$$\frac{\partial f}{\partial t} = \frac{1}{2} \left(D_{11} \frac{\partial^2 f}{\partial x_1^2} + D_{12} \frac{\partial^2 f}{\partial x_1 \partial x_2} + D_{22} \frac{\partial^2 f}{\partial x_2^2} \right) \tag{70}$$

where

$$\begin{aligned}
D_{11} &= \frac{\pi}{\Delta^2} (\alpha_1^2 T_{22}^2 + \alpha_2^2 T_{12}^2) \\
D_{12} &= \frac{2\pi T_{12}}{\Delta^2} (\alpha_1^2 T_{22} + \alpha_2^2 T_{11}) \\
D_{22} &= \frac{\pi}{\Delta^2} (\alpha_2^2 T_{11}^2 + \alpha_1^2 T_{12}^2)
\end{aligned} \tag{71}$$

We should point out that all the coefficients D_{ij} in (67)—which governs the evolution of the probability density f—are uniquely defined by the fully deterministic parameters T_{ij} of the original dynamical system (61).

4. Stochastic attractors in terminal dynamics

All the dynamical systems considered thus far exhibit an unrestricted random walk. As a result, the joint probability density of their solutions vanishes at $t \to \infty$. In this section we will describe a new phenomenon—an attraction of the solution to a stationary stochastic process whose joint density function is uniquely defined by the parameters of the original dynamical system.

4.1 One-dimensional restricted random walk

We begin with the following one-dimensional dynamical system,

$$\dot{x} = \gamma \sin^k \left(\frac{\sqrt{\omega}}{\alpha} \sin x \right) \sin \omega t \tag{72}$$

which has the following equilibrium points:

$$\overset{*}{x}_m = \arcsin \left(\frac{\pi \alpha}{\sqrt{\omega}} m \right) \qquad m = \ldots, -1, -1, 0, 1, 2, \ldots \tag{73}$$

It is clear that the distances between these points depend upon the number of steps m:

$$h_m = \overset{*}{x}_m - \overset{*}{x}_{m-1} = \arcsin \left(\frac{\pi \alpha}{\sqrt{\omega}} m \right) - \arcsin \left[\frac{\pi \alpha}{\sqrt{\omega}} (m-1) \right] \tag{74}$$

We introduce a new variable, y.

$$y = \sin x \tag{75}$$

Thus,

$$\overset{*}{y}_m = \frac{\pi \alpha}{\sqrt{\omega}} m \qquad \overset{*}{y}_m - \overset{*}{y}_{m-1} = \frac{\pi \alpha}{\sqrt{\omega}} \tag{76}$$

and (76) becomes identical to (58). This means that the probability as a function of y satisfies the following equation.

$$\left[E_t - \frac{1}{2}(E_y + E_y^{-1}) \right] f(t, y) = 0 \tag{77}$$

However, in contrast to x in (52), y is bounded:

$$|y| = |\sin x| \leq 1 \tag{78}$$

The solution of (77), subject to the boundary condition (78), is

$$f = \overset{*}{f}(t, y) \tag{79}$$

therefore, the solution to the original problem (i.e., to equation (72)) is

$$f = \overset{*}{f}[t, y(\sin x)]|\cos x| \tag{80}$$

For a better physical interpretation of (80), we consider a limit case where

$$\sqrt{\omega} \to \infty \tag{81}$$

that is,

$$\tau, h_m \to 0$$

Thus, (77) transfers to the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} D^2 \frac{\partial^2 f}{\partial y^2} \tag{82}$$

with the boundary conditions

$$\left.\frac{\partial f}{\partial y}\right|_{y=1} = \left.\frac{\partial f}{\partial y}\right|_{y=1-} = 0 \tag{83}$$

Subject to initial conditions

$$f(0, y) = \varphi(y) \qquad \varphi(y) \geq 0 \qquad \text{and} \qquad \int_{-1}^{1} \varphi(y)\, dy = 1 \tag{84}$$

the solution to (82) is

$$f(t, y) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{1}{2}\pi^2 D^2 n^2 t} \cos \frac{n\pi}{2}(y+1) \qquad |y| \leq 1 \tag{85}$$

$$a_n = 2 \int_{-1}^{1} \varphi(z) \cos \frac{n\pi}{2}(z+1)\, dz \qquad n = 1, 2, \ldots \tag{86}$$

therefore,

$$f(t, y) \to \frac{1}{2} \qquad \text{at } t \to \infty,\ |y| \leq 1 \tag{87}$$

Returning to the original variable x, we obtain (in place of (87))

$$\begin{aligned} f(x) = 0.5|y'| &= 0.5 \cos x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ (x) &= 0 \qquad \text{otherwise} \end{aligned} \tag{88}$$

Hence, any solution that originates within the interval

$$-\frac{\pi}{2} < x < \frac{\pi}{2} \tag{89}$$

always approaches stationary stochastic process (88), which plays the role of a stochastic attractor.

We emphasize that this is a phenomenon that does not exist in the classical version of nonlinear dynamics. Unlike the situation for chaotic attractors, the probability density can be uniquely controlled by the parameters of the original dynamical system, and the limit stochastic process does not depend upon the initial conditions if they are within the basin of attraction.

The preceding results were obtained under the assumption of (80), which allowed us to replace the original difference equation (77) with differential equation (82). But if $\sqrt{\omega}$ is finite and, therefore, steps (74) also are finite, the solution to (72) in some cases can overcome the barrier of (88) and, after a slow diffusion, eventually approach the "universal" attractor

$$f = 0 \tag{90}$$

We investigate such a possibility in detail. Turning to condition (60), which synchronizes the conversions of terminal attractors into terminal repellers (and vice versa), we assume that the following condition is violated.

$$\gamma = \omega h_m - \varepsilon^2, \ \varepsilon \ll 1 \qquad \text{at } h_m < \overset{*}{h}_m \tag{91}$$

Invoking (73), we conclude that if

$$\left|\frac{\pi}{2} - \overset{*}{h}_m\right| < \overset{*}{h}_m \tag{92}$$

then the solution to (72) can surpass the barrier $|x| = \pi/2$, and escape region (89). Conversely, if

$$\left|\frac{\pi}{2} - \overset{*}{h}_m\right| > \overset{*}{h}_m \tag{93}$$

then this solution will be trapped within the region

$$|x| \leq \left|\frac{\pi}{2} - \overset{*}{h}_m\right| \tag{94}$$

Qualitatively, the solution to (72) under condition (93) behaves as solution (88), representing a stochastic attractor. Clearly, (72) has an infinite number of such attractors, with basins

$$\frac{\pi n}{2} < x < \frac{\pi(n+2)}{2} \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots \tag{95}$$

Under condition (92), this solution will penetrate the barriers and diffuse through all the basins (95), approaching the attractor (90).

Now we may generalize (72) by requiring that its solution have a stochastic attractor with a prescribed density function $f(x)$, with the only restrictions being that

$$f(x) = 0 \ \text{for} |x| > N \qquad N < \infty \qquad \text{and} \qquad \int_{-N}^{N} f(x)\, dx = 1 \tag{96}$$

Based upon (88), we arrive at the following equation in place of (72).

$$\dot{x} = \gamma \sin^k \left[\frac{\sqrt{\omega}}{\alpha} p(x)\right] \sin \omega t \qquad p(x) = 2\int_{-N}^{x} f(\xi)\, d\xi - 1 \tag{97}$$

In fact, introducing a new variable y (compare with (75)):

$$y = p(x) \qquad y(-N) = -1 \qquad y(N) = 1$$

we obtain, in place of (88),

$$f(x) = \frac{1}{2}|y'| = \frac{1}{2}\frac{dp}{dx}$$

We have not yet discussed the fact that the solution to (82) must satisfy the constraint

$$\int_{-1}^{1} f(y)\, dy = 1$$

in addition to boundary conditions (83). To illustrate that this constraint does not overdetermine the solution, we integrate (82) over y, as follows.

$$\int_{-1}^{1} \frac{\partial f}{\partial t}\, dy = \frac{\partial}{\partial t}\int_{-1}^{1} f\, dy = \frac{D^2}{2}\bigg|_{-1}^{1} \frac{\partial f}{\partial y}\, dy = 0$$

that is,

$$\int_{-1}^{1} f\, dy = \text{Const}$$

This means that if the initial conditions satisfy this constraint, then the solution will satisfy it automatically.

4.2 Multidimensional restricted random walk

In order to illustrate the existence of stochastic attractors in multidimensional systems, we consider the following two-dimensional case.

$$\dot{x}_1 = \gamma_1 \sin^k \left[\sqrt{\omega}\sin(x_1 + x_2)\right] \sin \omega t \tag{98}$$

$$\dot{x}_2 = \gamma_2 \sin^k \left[\sqrt{\omega}\sin(x_1 - x_2)\right] \sin \omega t \tag{99}$$

Denoting

$$x_1 + x_2 = u_1$$
$$x_1 - x_2 = u_2$$

we can introduce a dynamical system

$$\dot{u}_1 = \gamma_1^* \sin^k \sqrt{\omega} \sin u_1 \sin \omega t \tag{100}$$

$$\dot{u}_2 = \gamma_2^* \sin^k \sqrt{\omega} \sin u_2 \sin \omega t \tag{101}$$

that has the same critical points

$$\sin u_1, \sin u_2 = \frac{\pi m}{\sqrt{\omega}} \qquad m = \ldots, -2, -1, 0, 2, \ldots \tag{102}$$

and, therefore, the same probability distribution of the solution as the original dynamical system.

Equations (100) and (101) have the form of (72) and, therefore, their formal solutions follow from (88):

$$\begin{aligned} &f(u_1) = 0.5|\mathrm{cos} u_1| \quad \frac{\pi m_1}{2} < u_1 < \frac{\pi(m_1+2)}{2}, \; m_1 = \ldots, -1, 0, 1, 2, \ldots \\ &f(u_1) = 0 \qquad \text{otherwise} \end{aligned} \tag{103}$$

$$\begin{aligned} &f(u_2) = 0.5|\mathrm{cos} u_2| \quad \frac{\pi m_2}{2} < u_2 < \frac{\pi(m_2+2)}{2}, \; m_2 = \ldots, -1, 0, 1, 2, \ldots \\ &f(u_2) = 0 \qquad \text{otherwise} \end{aligned} \tag{104}$$

However, not all of these solutions are stable. Applying stability conditions (see [17]) to linearized versions of (98) and (99) yields

$$\cos x_1 - \cos x_2 < 0 \qquad \cos x_1 \cos x_2 < 0 \tag{105}$$

that is,

$$\cos x_1 < 0 \qquad \cos x_2 > 0$$

therefore, the solutions are stable if

$$m_1 = \ldots, -7, -3, 1, 5, 9, \ldots \qquad m_2 = \ldots, -5, -1, 3, 5, 7, \ldots \tag{106}$$

in (103) and (104). Returning to the original variables, we obtain

$$f(x_1, x_2) = 0.5|\cos(x_1 + x_2)\cos(x_1 - x_2)| \tag{107}$$

$$\frac{\pi m_1}{2} < x_1 + x_2 < \frac{\pi(m_1+2)}{2} \qquad \frac{\pi m_2}{2} < x_1 - x_2 < \frac{\pi(m_2+2)}{2} \tag{108}$$

Solution (107) represents a stationary stochastic process which attracts all solutions with initial conditions within area (108). Each pair m_1 and m_2 from sequences (106) defines a corresponding stochastic attractor with joint density (107). Clearly, those solutions for which m_1 and m_2 do not belong to (106) are unstable and, eventually, will be attracted to one of the stochastic attractors (108).

Turning to an n-dimensional dynamical system, we confine ourselves by the use of the special form

$$\dot{x}_i = \gamma_i \sin^k \left[\frac{\sqrt{\omega}}{\alpha} p_i(y_i) \right] \sin \omega t \tag{109}$$

where

$$y_i = \sum_{j=1}^{n} T_{ij} x_j \qquad T_{ij} = \text{Const} \tag{110}$$

We assume that

$$\frac{dp_i}{dy_i} \begin{cases} > 0 & \text{for } |y_1| < N_i \\ = 0 & \text{for } |y_i| > N_i \end{cases} \quad N_i < \infty \tag{111}$$

and that the T_{ij} form a symmetric positive-definite matrix, that is, that conditions (62) are satisfied.

Based upon conditions (62) and (111), we conclude that system (109) is locally stable (or locally unstable, depending upon the sign of $\sin\omega t$), and that it synchronizes the conversions of terminal attractors into terminal repellers (and vice versa).

Exploiting (97), we find that the solution to (109) has the following density functions, in terms of the variables y_i.

$$f(y_1, \ldots, y_n) = \prod_{i=1}^{n} p_i'(y_i) \qquad p' = \frac{dp}{dy} \tag{112}$$

In terms of the variables x_i, the joint density of the solution is

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_i'(y_i). \det |T_{ij}| \tag{113}$$

where y_i is expressed via x_i by (110).

4.3 Examples

1. We begin with the following problem: Find a dynamical system whose solution is attracted to a stochastic process with the normal density

$$f(x) = z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{(x-\mu)^2/(2\sigma^2)} \tag{114}$$

where μ and σ are the mean and the standard deviation, respectively, and $z(y)$ is the standard normal density function.

To apply (97), we must first modify (114), because it does not satisfy restriction (96). We introduce a truncated standard normal density function

$$\tilde{z}(y) = \begin{cases} z(y) & \text{if } |y| < N \\ 0 & \text{if } |y| > N, \end{cases} \quad N < \infty \tag{115}$$

Then, with reference to (97), we obtain

$$\dot{x} = \gamma \sin^k \left[\frac{\sqrt{\omega}}{\alpha} \widetilde{\text{erf}} \left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right] \sin\omega t \qquad \widetilde{\text{erf}}(y) = \frac{2}{\sqrt{\pi}} \int_0^y \tilde{z}(u) du \tag{116}$$

Thus, (116) represents a dynamical system whose solution is attracted to a stochastic process with density function (115). For sufficiently large N, it will approximate a Gaussian process, with μ and σ as the mean and the standard deviation, respectively.

2. Let us assume that the density $f(x)$ of a desired stochastic process is characterized by $\mu = \mu_0, \sigma = \mu_1$, and higher central moments μ_r. Utilizing the Gram-Charlier series expansion [20]

$$f(x) = \frac{1}{\sigma}\sum_{r=0}^{\infty} c_r \tilde{z}^{(r)}\left(\frac{x-\mu}{\sigma}\right) \tag{117}$$

where

$$c_0 = 1 \qquad c_1 = c_2 = 0 \qquad c_3 = -\frac{1}{3!}\mu_3 \qquad c_4 = \frac{1}{4!}(\mu_4 - 3) \tag{118}$$
$$c_5 = -\frac{1}{5!}(\mu_5 - 10\mu_6) \qquad c_6 = \frac{1}{6!}(\mu_6 - 15\mu_4 + 30) \qquad \text{and so forth}$$

and

$$\tilde{z}^{(r)} = \frac{d^r \tilde{z}(y)}{dy^r} \tag{119}$$

and applying (97), we obtain

$$\dot{x} = \gamma \sin^k \left\{\frac{N\sqrt{\omega}}{\alpha}\left[\widetilde{\text{erf}}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right) + \sum_{r=3}^{\infty} c_r \tilde{z}^{r-1}\left(\frac{x-\mu}{\sigma}\right)\right]\right\} \sin \omega t \tag{120}$$

Hence, the solution to dynamical system (120) is attracted to a stochastic process whose density function is characterized by the moments μ_r.

3. In this example we pose the following problem: Find a dynamical system whose solutions $x_i(t)$ are attracted to a stochastic process characterized by the column of means and the matrix of moments

$$Mx_i = \mu_i \qquad \sigma_{ij} = M(x_i - \mu_i)(x_j - \mu_j) \qquad i, j = 1, 2, \ldots, n \tag{121}$$

We can find an orthogonal transformation

$$y_i = \eta_i + \sum_{j=1}^{n} T_{ij}(x_j - \mu_j) \tag{122}$$

such that

$$My_i = \eta_i = 0 \qquad \sigma'_{ik} = \sum_{j=1}^{n}\sum_{\ell=1}^{n} \sigma_{j\ell} T_{ij} T_{kl} = \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{of } i \neq k \end{cases} \tag{123}$$

where y_i are non-correlated standard normally distributed variables.

Combining (109), (110), and (116) we obtain

$$\dot{x}_i = \gamma_i \sin^k \left[\frac{\sqrt{\omega}}{\alpha}\,\widetilde{\text{erf}}\left(\frac{y_i}{\sqrt{2}}\right)\right] \sin \omega t \qquad y_i = \sum_{j=1}^{n} T_{ij}(x_j - \mu_j) \tag{124}$$

Some comments concerning the stability of (124) must be made. Since T_{ij} is an orthogonal matrix, it does not satisfy conditions (62). However, the real parts of the eigenvalues of T_{ij} are

$$R_e\lambda_i = 1 \qquad \text{or} \qquad R_e\lambda_i = \cos\varphi_i > 0 \qquad \text{for } 0 \leq \varphi < \frac{\pi}{2} \tag{125}$$

where φ_i are the angles of rotation of the coordinate axes. Because

$$\frac{d}{dy_i}\,\widetilde{\text{erf}}\,(y_i) > 0 \qquad \text{for } |y| < N_i \tag{126}$$

(that is, condition (111) is satisfied), display (124) (when linearized with respect to its equilibrium points) has eigenvalues whose real parts are all positive (if $\sin\omega t > 0$) or negative (if $\sin\omega t < 0$). This synchronizes conversions from terminal attractors to terminal repellers (and vice versa).

Thus, the solution to the dynamical system is attracted to a stochastic process with the probabilistic structure prescribed in (121) if the initial conditions are within the basin of attraction $|y_i| < N_i$.

5. Self-organization in terminal dynamics

A dynamical system is considered self-organizing if it acquires a coherent structure without specific interference from the outside. In this section we show that terminal dynamics possesses a powerful tool for self-organization, based on the possibility of coupling between the original dynamical system and its own associated probability density dynamics.

We begin with dynamical system (116), represented in the form of (41).

$$\dot{x} = \gamma \sin^k\left(\frac{\sqrt{\omega}}{\alpha}y\right)\sin\omega t \qquad y = \widetilde{\text{erf}}\left(\frac{x}{\sqrt{2}\sigma}\right) \tag{127}$$

The probability density function $f(y,t)$ satisfies (50), as follows.

$$\frac{\partial f}{\partial t} = \frac{\pi\alpha^2}{2}\frac{\partial^2 f}{\partial y^2} \qquad -N \leq y \leq N \tag{128}$$

Its solution (subject to boundary and initial conditions (83) and (84), respectively), is given by (85). In terms of x, this solution is

$$f_*(x,t) = \left\{\frac{1}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{1}{2}\pi^3\alpha^2 n^2 t}\cos\frac{n\pi}{2}\left[\widetilde{\text{erf}}\left(\frac{x}{\sqrt{2}\sigma}\right)\right]\right\}\tilde{z}\left(\frac{x}{\sigma}\right) \tag{129}$$

where $\tilde{z}$ is defined by (114) and (115).

In all the preceding cases, the parameter σ was prescribed in advance, denoting the variance of the stationary density

$$f_*(x) = \tilde{z}\left(\frac{x}{\sigma}\right) \qquad \text{at } t \to \infty \tag{130}$$

Let us assume that the parameter σ depends upon moments of the current density (129)—for instance,

$$\sigma^2 = \sigma_0^2 - \mathrm{Var}(x) \qquad \sigma_0 = \mathrm{Const} \tag{131}$$

where

$$\mathrm{Var}(x) = \int_{-N}^{N} x^2 f_*(x,t)\, dx \tag{132}$$

We point out immediately that the time scale t' of changing σ is defined by (129), and has the order

$$t' \sim \frac{1}{\alpha^2} \tag{133}$$

Because the time scale t'' of changing x in (127) has the order $t'' \sim 1/\sqrt{\omega} \to 0$ (see (46)) and, therefore,

$$t' \gg t'' \tag{134}$$

the variable σ can be considered as a slowly changing parameter in (127) (but not in (129)!).

Thus, dynamical system (127) is guided by the probability density via the parameter σ. This parameter is obtained from (131), after the substitution of (131) in the integrand. For the final stationary state ($t \to \infty$), we obtain (from (131) and (132))

$$\mathrm{Var}(x) = \sigma^2 = \sigma_0^2 - \mathrm{Var}(x) \tag{135}$$

hence,

$$\sigma^2 = \frac{1}{2}\sigma_0^2 \qquad \text{at } t \to \infty \tag{136}$$

Therefore, the solution to dynamical system (127) approaches a stochastic attractor with the probability density $\tilde{z}(x/(\sigma_0/\sqrt{2}))$. We stress that this attractor has not been "stored" in the prescribed coefficients of (127): the dynamical system "found" it as a result of coupling with its "own" probability equations.

In the general case, parameters of dynamical system (109) can be coupled with moments of probability density (113) to lead to new self-organizing architectures.

6. Discussion and conclusion

6.1 Relevance of terminal dynamics to the determinism of Newtonian dynamics

Classical dynamics describes processes in which the future can be derived from the past, and in which the past can be traced from the future by time inversion, $t \to -t$. Because of such determinism and reversibility, classical dynamics becomes fully predictable and, therefore, cannot explain the emergence of new dynamical patterns in nature (as nonequilibrium thermodynamics can). This major flaw in classical dynamics has attracted the attention of many outstanding scientists (Gibbs, Planck, and Prigogine, among others; see [19]).

Considering the governing equations of classical dynamics,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} - \frac{\partial R}{\partial \dot{q}_i} \qquad i = 1, 2, \ldots, n \tag{137}$$

(where L is the Lagrangian, $q_i, \dot{q}_i$ are the generalized coordinates and velocity, and R is the dissipation function), we should recall that the structure of $R(\dot{q}_i, \ldots, \dot{q}_n)$ is not prescribed by Newton's laws: some additional assumptions must be made in order to define it. The "natural" assumption (which has been never challenged) is that these functions can be expanded in Taylor series with respect to equilibrium states $\dot{q}_i = 0$. Clearly, this requires the existence of the derivative

$$\left|\frac{\partial^2 R}{\partial \dot{q}_i\, \partial \dot{q}_i}\right| < \infty \qquad \text{at } \dot{q}_i \to 0$$

A departure from that condition is proposed in [15], where the following dissipation function is introduced.

$$R = \frac{1}{k+1}\sum_i \alpha_i \left|\sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j\right|^{k+1} \tag{138}$$

in which

$$k = \frac{p}{p+2} < 1 \qquad p \gg 1 \tag{139}$$

where p is a large odd number. By selecting large p, we can make k close to 1, so that (138) is almost identical to the classical assumption (when $k = 1$) everywhere excluding a small neighborhood of the equilibrium point $\dot{q}_j = 0$; whereas, at that point,

$$\left|\frac{\partial^2 R}{\partial \dot{q}_i\, \partial \dot{q}_j}\right| \to \infty \qquad \text{at } \dot{q}_j \to 0 \tag{140}$$

Thus, the Lipschitz condition is violated; the friction force $F_i = -(\partial R/\partial \dot{q}_i)$ grows sharply at the equilibrium point, and then it gradually approaches its "classical" value. This effect can be interpreted as a mathematical representation of a jump from static to kinetic friction, where the dissipation force does not vanish with the velocity.

It appears that this "small" difference between the friction forces at $k = 1$ and $k < 1$ leads to fundamental changes in Newtonian dynamics. In order to demonstrate this, we consider the relationship betwen the total energy E and the dissipation function R.

$$\frac{dE}{dt} = -\sum_i \dot{q}_i \frac{\partial R}{\partial \dot{q}_i} = -(k+1)R \tag{141}$$

Within a small neighborhood of an equilibrium state (where the potential energy can be set to zero), the energy E and the dissipation function R have the respective orders

$$E \sim \dot{q}_i^2, \; R \sim \dot{q}_i^{k+1} \qquad \text{at } E \to 0 \tag{142}$$

Hence, the asymptotic form of (141) can be presented as

$$\frac{dE}{dt} = AE^{k+1/2} \qquad \text{at } E \to 0, \; A = \text{Const} \tag{143}$$

If $A > 0$ and $k < 1$, the equilibrium state $E = 0$ is an attractor where the Lipschitz condition ($|d\dot{E}/dE| \to \infty$ at $E \to 0$) is violated. Such a terminal attractor is approached by the solution originated at $E = \Delta E_0 > 0$, in finite time, as follows.

$$t_0 = \int_{\Delta E_0}^{o} \frac{dE}{AE^{(k+1)/2}} = \frac{2\Delta E_0^{(1-k)/2}}{(1-k)|A|} < \infty \tag{144}$$

Clearly, this integral diverges in the classical case $k \geq 1$, where $t_0 \to \infty$. The motion described by (143) has a singular solution $E \equiv 0$, and a regular solution

$$E = \left[\Delta E_0^{(1-k)/2} + \frac{1}{2}A(1-k)t\right]^{2/(1-k)} \tag{145}$$

In a finite time, the motion can reach the equilibirum and switch to the singular solution $E \equiv 0$, and this switch is irreversible.

The coefficient k can be found from experimental observations of the time t_0. In order to illustrate this, we consider a plane-incompressible flow, with a stream function ψ and the constitutive law

$$\sigma_{xy} = \mu_1 \left(\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right)^{k+1} \qquad v_x = \frac{\partial \psi}{\partial y}, \; v_y = -\frac{\partial \psi}{\partial x}, \; k < 1 \tag{146}$$

where σ_{xy}, v_x, and v_y are viscous stress and Cartesian projections of velocity. Based upon the relationship between the rate of change of the kinetic energy and the dissipation function, we obtain

$$\frac{\rho}{2}\frac{\partial}{\partial t}\int_V \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right] dx\, dy = -\mu_1 \int_V \left(\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right)^{k+1} dx\, dy \tag{147}$$

where ρ is density, μ_1 is viscosity, and V is the volume occupied by the fluid.

Suppose that $\psi(t,x,y)$ can be represented as a product $\psi = \tilde{\psi}(t)\bar{\psi}(x,y)$. Then (147) reduces to the ordinary differential equation with respect to $\varphi(t) = \tilde{\psi}^2(t)$, as follows.

$$\dot{\phi} = -\gamma\nu_1\varphi^k \tag{148}$$

and

$$\gamma = \frac{\int_V \left(\frac{\partial^2\bar{\psi}}{\partial y^2} - \frac{\partial^2\bar{\psi}}{\partial x^2}\right)^{k+1} dx\,dy}{\int_V \left[\left(\frac{\partial\bar{\psi}}{\partial x}\right)^2 + \left(\frac{\partial\bar{\psi}}{\partial y}\right)^2\right] dx\,dy} = \text{Const} \qquad \nu_1 = \frac{\mu_1}{\rho}$$

Equation (148) describes the damping of the fluid motion due to viscous stress (146). The equilibrium state represents a terminal attractor which is approached in a finite time:

$$t_0 = \frac{\varphi_0^{1-k}}{\gamma\nu_1(1-k)} \qquad \varphi_0 = \varphi(0) \tag{149}$$

Equation (149) allows one to evaluate k and ν_1 from experimental measurements of t_0.

In conclusion, we stress again that all the new effects of terminal dynamics emerge within drastically diminishing neighborhoods of equilibrium states, which are the only domains where the governing equations are different from the classical models.

6.2 Relevance to chaos

One of the central problems of Newtonian dynamics is the explanation of the fact that a motion that is described by fully deterministic governing equations can be random. To discuss this, let us consider the exponential growth of a variable α,

$$\alpha = \alpha_0 e^{\lambda t} \qquad 0 < \lambda < \infty \tag{150}$$

Clearly, the solution with infinitely close initial condition

$$\tilde{\alpha} = \alpha + \varepsilon \qquad \varepsilon \to 0 \tag{151}$$

will remain infinitely close to the original solution,

$$|\tilde{\alpha} - \alpha_0| = \varepsilon e^{\lambda t} \to 0 \qquad \text{if } \varepsilon \to 0,\ t < \infty \tag{152}$$

during all bounded time intervals. This means that random solutions can result only from random initial conditions when ε in (151) is small but finite, rather than infinitesimal. In other words, classical dynamics can explain amplifications of random motions, but cannot explain their origin. According to the terminal modification of Newtonian dynamics, random motions are generated by unstable equilibrium states at which dissipation forces do not vanish with velocities, that is, at which the Lipschitz condition is violated. We recall that the evolution of these random motions amplified by the mechanism of instability can be predicted by the use of the stabilization principle discussed in [4–6].

Because of the finite precision with which initial conditions are known, terminal equilibrium points can be incorporated into classical dynamics in the following way. Let us consider a first-order dynamical equation,

$$\dot{v} + \alpha v = 0 \tag{153}$$

and assume that the variable v can be observed with a finite error

$$|v_*| \ll v_0 \tag{154}$$

where v_0 is a representative value of v characterizing the scale of motion. The actual time of approaching the attractor

$$|v| \leq |v_*| \tag{155}$$

is finite:

$$t_1 = \frac{1}{\alpha} \ln \left| \frac{v_0}{v_*} \right| < \infty \tag{156}$$

A terminal version of (156) that describes the same process,

$$\dot{v} + \alpha v \left[1 + (1-k) \left(\frac{v}{v_0} \right)^{k-1} \right] = 0 \tag{157}$$

has a solution which, at $k \to 1$, is infinitely close to the solution of (153) everywhere, except in a small neighborhood of the attractor $v = 0$. The time of approaching this attractor is

$$t_2 = \frac{1}{\alpha(1-k)^2} \tag{158}$$

Equating t_1 from (20) and t_2 from (22), we find the order of an "equivalent" value of k,

$$k \sim 1 - \frac{1}{\sqrt{\ln \left| \dfrac{v_0}{v_*} \right|}} \tag{159}$$

Thus, the fact that dynamical parameters cannot be observed or measured with infinite precision is mathematically formalized by introducing terminal equilibrium points; the parameter $k < 1$ is defined by the relative error v_*/v_0. The terminal version of dynamical system (1),

$$\dot{x}_i = v_i \left[1 + (1-k) \left(\frac{v_i}{v_0} \right)^{k-1} \right] \qquad v_0 \gg v_i \tag{160}$$

allows us to explain the appearance of random solutions without random inputs within the framework of differentiable dynamics. However, additional terms in (160)(such as dissipation forces (138)) cannot be interpreted as physical quantities, because they are not invariant with respect to coordinate transformations. This fact emphasizes a computational origin of "classical" chaos, in contrast to a physical origin of terminal chaos.

6.3 Conclusion

We have discussed a new mathematical model for nonlinear dynamics—terminal dynamics. In this model, the dissipation function is reshaped, such that the time of approaching equilibrium points becomes theoretically finite due to violation of the Lipschitz condition. As a side effect of this property, terminal dynamics becomes irreversible and probabilistic.

We have given special attention to well organized terminal dynamical systems that are driven by a global rhythm, generated by a periodic attractor. Such systems have a relatively simple probabilistic structure based upon a random-walk paradigm; they are more appropriate for describing evolutions in biological and social systems, in which the coupling between variables is more flexible and, therefore, can be modeled by probabilistic relationships.

It appears that the terminal model, applied to Newtonian dynamics, can provide a mathematical formalization of the fact that dynamical parameters cannot be observed with infinite precision in real physical systems; hence, all the equilibrium points are actually terminal. This formalization makes Newtonian dynamics irreversible, and it provides a formal mathematical explanation for the appearance of random solutions in chaotic systems.

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