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Multimodal Deceptive Functions

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Abstract. This paper presents a static analysis of deception in multimodal functions. Deception in a bipolar function of unitation (a function with two global optima and a number of deceptive attractors) is defined, and a set of sufficient conditions relating function values is obtained. A bipolar deceptive function is also constructed from low-order Walsh coefficients. Multimodal functions of bounded deception are formed by concatenating several bipolar deceptive functions. These functions offer a great challenge to global optimization algorithms (including genetic algorithms) because they are deceptive and have a large number of attractors, of which only a few are global optima. These functions also open doors for generalizing the notion of deception, and allow us to better understand the importance of deception in the study of genetic algorithms.

1. Introduction

Deceptive problems raise questions about the fundamental principle of genetic algorithms (GAs). Mechanisms to solve these problems may provide important insights regarding the mechanics of genetic algorithms. To date, the study of deception in GAs has been focused primarily in three different directions: design of deceptive functions [2, 3, 8, 14]; understanding the effect of deception in GA solutions [5, 6, 7, 15, 16, 17, 18] and modification of GAs to solve deceptive problems [4, 10, 11]. In all the aforementioned studies, deception was assumed in a *uniglobal* function—a function with a single global optimum and a single deceptive attractor. It was observed that in a fully deceptive uniglobal function, the deceptive attractor must be the complement of the global optimum [17], and all low-order building blocks favor

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the deceptive attractor. Since low-order building blocks are not instances of the global optima, these functions are difficult to solve using simple tripartite GAs, often converging to the deceptive attractor. The analysis and solution of these uniglobal deceptive problems have helped us understand what problems are hard for GAs, and how GAs can be modified to solve such problems. We take a further step along these lines, and introduce deception in functions with multiple global optima and multiple deceptive attractors.

Like uniglobal deceptive functions, multimodal deceptive functions are designed so that low-order building blocks lead away from all global optima and favor deceptive attractors. Since these functions have a number of deceptive attractors rather than one, the solution may now be attracted to one of many deceptive attractors. In addition to the functions being deceptive, the multimodality of the function space itself may cause difficulty in solving these problems to global optimality. This dual effect of deception and multimodality in such functions is likely to provide a stiff challenge to simple GAs. However, it is for precisely such functions that stochastic optimization methods like GAs are likely candidates for finding any global solutions at all. Due to their population-approach and implicit parallel processing, GAs may even be designed to allow multiple global optimal solutions to coexist in the population, thereby solving multiple optimal solutions simultaneously [12]. In this paper, we introduce the notion of deception in multimodal functions, construct multimodal deceptive functions, and discuss how GAs can be used to solve these problems.

We introduce the notion of deception in multimodal functions by analyzing a *bipolar* function of unitation. (A bipolar function has two global optima and a number of deceptive attractors maximally far apart from both global optima.) A set of sufficient conditions for deception is found for an arbitrary bipolar function of unitation by calculating schema average fitness values. Thereafter, a bipolar deceptive function is constructed from low-order Walsh coefficients. Relations among Walsh coefficients are found in order to construct a bipolar deceptive function and a partially bipolar deceptive function of any order. Multimodal deceptive functions of bounded deception are constructed by concatenating several bipolar deceptive functions. The multimodality of the search space is illustrated by constructing a 12-bit quadrimodal function, and possible extensions to the simple GA are discussed in regard to solving such problems to global optimality.

2. Bipolar deceptive functions

In previous studies of deception, functions with a single global optimum and a single deceptive attractor were considered. Though the deceptive attractor and the local optimum may not be identical in a fully deceptive problem [17], we will consider them to be identical. In this section, we introduce the notion of deception in functions with more than one global optimum, and with more than one deceptive attractor. We call the former class of functions uniglobal functions and the latter class of functions multimodal functions. We define a bipolar function with two global optima and a number of deceptive attractors to find a sufficient condition for deception in a bipolar function. Subsequently, we construct massively multimodal deceptive functions using bipolar deceptive functions.

A bipolar function is defined as having two global optima that are maximally far apart from each other, and a number of deceptive attractors that are maximally far apart from the global optima in the Hamming space. It has been discussed elsewhere [3] that functions of unitation reduce the number of independent function values in a function, thus enabling easier manipulation of function values to find conditions for deception. We consider even-sized bipolar functions of unitation (recognizing that the analysis may be extended to odd-sized bipolar functions). In a bipolar function of unitation of size 2ℓ, there are only $2\ell + 1$ independent function values. The number of independent function values is further reduced by considering a symmetric bipolar function of unitation, in which the function values are symmetric about unitation $u = \ell$. This reduces the number of independent function values to $\ell + 1$. Without loss of generality, we consider a symmetric bipolar function of unitation having two global strings of unitation $u = 0$ and $u = 2\ell$ respectively (maximally far apart from each other), and a number of deceptive attractors of unitation $u = \ell$ (maximally far apart from the global optima). Since there are $\binom{2\ell}{\ell}$ strings of unitation $u = \ell$, the total number of deceptive attractors is $\binom{2\ell}{\ell}$. Any two complimentary strings may also be used as global strings by transforming the function, as suggested elsewhere [8]. One characteristic of such a function is that with increasing problem size, the number of deceptive attractors increases exponentially, but the number of global optima remains two. Thus, for large problem sizes, there are many attractors, of which only two are global optima. The massive multimodality of such functions may cause difficulty in solving them to global optimality. If these functions are deceptive, not only are there many non-global attractors, but low-order schema partitions guide the search away from the global solutions. This makes the functions even more difficult to solve to global optimality.

2.1 Defining bipolar deception

We modify the definition of deception in a uniglobal function to define deception in a bipolar function. In a uniglobal deceptive function, a schema partition is usually defined to be deceptive if the schema containing the deceptive attractor is no worse than any other competing schema in the partition [3]. In a bipolar function, there are two global solutions and a number of deceptive attractors. In all schema partitions of order less than half the problem size, all schemata containing global strings also contain a number of deceptive attractors. In those partitions, there exist some schemata that contain deceptive attractors only. For example, in a six-bit bipolar function, there are two global optima and $\binom{6}{3}$ or twenty deceptive attractors. A schema partition of order two contains four schemata, of which the schema with unitation zero (00****) contains one of the global optima (000000) and the schema with unitation two (11****) contains the other global optimum (111111). Both these schemata, however, contain four different deceptive attractors. For example, the schema of unitation zero contains the following deceptive attractors: 001011, 001101, 001110, and 000111. Schemata of unitation one 10**** and 01**** contain no global optima yet contain six deceptive attractors each. For example, the former schema contains the following deceptive attractors: 100011, 100101, 100110, 101001, 101010, and 101100. Since in any schema partition many schemata contain one or more deceptive attractors, the usual definition of schema partition deception cannot be applied to a bipolar function. Thus, to define deception in a bipolar function, we modify the usual definition of deception, as follows.

Definition 1. *In a bipolar function, a schema partition is defined to be deceptive if the schema (or schemata) containing the maximum number of deceptive attractors is (or are) no worse than other competing schemata.*

Note that this definition reduces to the usual schema partition deception in the uniglobal case. In the case of uniglobal, fully deceptive functions, there is only one global optimum and one deceptive attractor, and the deceptive attractor is the complement of the global optimum [17]. Thus, in any schema partition of such a function, there exists only one schema containing the deceptive attractor. Since all other schemata do not contain the deceptive attractor, the above definition requires that the schema containing the deceptive attractor is no worse than any other schema, which is precisely the definition of schema partition deception in the uniglobal case.

In bipolar functions, it is less obvious which schema has the maximum number of deceptive attractors. To find out, we calculate the number of deceptive attractors that are contained in a schema. In a bipolar function of size 2ℓ, a schema partition of order λ contains schemata of unitation varying from zero to λ. Since all deceptive attractors contain ℓ ones and ℓ zeros, it can be shown that an order-λ schema of unitation u contains $\binom{2\ell-\lambda}{\ell-u}$ deceptive attractors. Since this expression is maximum for $u = \lfloor \lambda/2 \rfloor$[1], it is clear that the schema of unitation $u = \lfloor \lambda/2 \rfloor$ contains the maximum number of deceptive attractors. Thus, according to the above definition, a schema partition of order λ is deceptive, if schemata of unitation $u = \lfloor \lambda/2 \rfloor$ are no worse than other schemata in the schema partition.

With this definition of schema partition deception, we define a *bipolar deceptive* function as a function where all schema partitions are deceptive. An order-one schema partition has two schemata containing equal numbers of deceptive attractors. Thus, both schemata in an order-one schema partition have the same fitness. In the remainder of this section, we find a set of sufficient conditions for deception by imposing bipolar deception in all schema partitions.

[1]The operator $\lfloor\ \rfloor$ denotes the *floor* operator that calculates the greatest integer smaller than the operand.

2.2 Deception analysis

We define the fitness of a schema of order λ and unitation u as $f(u, \lambda, 2\ell)$. This schema has λ fixed positions with u ones and $\lambda - u$ zeros, and has $2\ell - \lambda$ don't care positions. Thus, this schema contains strings with unitation varying from u to $2\ell - \lambda + u$. According to this terminology, a string would be represented as $f(u, 2\ell, 2\ell)$; however, we use $f(u)$ to denote the quantity simply. Recognizing that there are $\binom{2\ell-\lambda}{i}$ strings of unitation $i+u$, we obtain the schema fitness in terms of the function value of strings, as follows.

$$f(u, \lambda, 2\ell) = 2^{-(2\ell-\lambda)} \sum_{i=0}^{2\ell-\lambda} \binom{2\ell-\lambda}{i} f(i+u) \tag{1}$$

Since the bipolar function is symmetric about unitation $u = \ell$, we may write that $f(u) = f(2\ell - u)$. Using equation (1) and the symmetry of the function, we observe a number of properties of this function.

Property 1. *For a size-2ℓ bipolar function of unitation with an axis of symmetry at $u = \ell$, schemata in a schema partition of order λ are also symmetric; or, in function notation,*

$$f(u, \lambda, 2\ell) = f(\lambda - u, \lambda, 2\ell) \tag{2}$$

Proof. Using symmetry, and recognizing that $\binom{2\ell-\lambda}{i} = \binom{2\ell-\lambda}{2\ell-\lambda-i}$, we rewrite equation (1) as follows.

$$f(u, \lambda, 2\ell) = 2^{-(2\ell-\lambda)} \sum_{i=0}^{2\ell-\lambda} \binom{2\ell-\lambda}{2\ell-\lambda-i} f(2\ell - u - i)$$

Introducing a new index $j = 2\ell - \lambda - i$, we observe that the limits of the summation remain the same. Writing the right side of the above expression in terms of the new index j proves Property 1. ∎

This property reduces the number of schema competitions to be investigated in a schema partition. It suggests that for bipolar deception we must only compare schemata of unitation $u = \lfloor \lambda/2 \rfloor$ with schemata of unitation $0 \le u \le \lfloor \lambda/2 \rfloor - 1$ in each schema partition of order λ.

Property 2. *For a size-2ℓ bipolar function of unitation with an axis of symmetry at $u = \ell$, the fitness of a schema of order λ (where λ is even) and unitation $\lambda/2$ is the same as the fitness of a schema of order $\lambda + 1$ and unitation $\lambda/2$.*

Proof. We know that the fitness of a schema of order λ and unitation u may be written as the average fitness of schemata of order $\lambda+1$ and of unitations u and $u+1$. For an even value of λ, we obtain

$$f(\lambda/2, \lambda, 2\ell) = [f(\lambda/2, \lambda+1, 2\ell) + f(\lambda/2+1, \lambda+1, 2\ell)]/2$$

Substituting $u = \lambda/2$ in equation (2), we observe that $f(\lambda/2, \lambda+1, 2\ell) = f(\lambda/2+1, \lambda+1, 2\ell)$, which simplifies the right side of the preceding equation to $f(\lambda/2, \lambda+1, 2\ell)$. This proves Property 2. ∎

Property 3. *In a size-2ℓ bipolar function of unitation with an axis of symmetry at $u = \ell$, if a schema partition of order λ (where λ is odd) is deceptive, the schema partition of order $\lambda - 1$ is also deceptive.*

Proof. By Definition 1 and Property 1, a deceptive schema partition of order λ implies that the schema of unitation $u = \lfloor \lambda/2 \rfloor$ has a fitness better than or equal to that of any other competing schemata of unitation $0 \le u \le \lfloor \lambda/2 \rfloor - 1$ in the partition. Without loss of generality, we assume that we are maximizing the function. For odd values of λ, we assume that $\lambda = 2k + 1$. We write deception conditions for a schema partition of order $\lambda - 1$, and express the schema fitness value in terms of fitness of schemata of order λ.

$$f(k, 2k, 2\ell) \ge f(u, 2k, 2\ell)$$

or,

$$f(k, 2k+1, 2\ell) + f(k+1, 2k+1, 2\ell) \\ \ge f(u, 2k+1, 2\ell) + f(u+1, 2k+1, 2\ell)$$

Using Property 1, we observe that two terms in the left side of the second inequality are identical. Using the conditions for deception of a schema partition of order $2k+1$, we obtain $f(k, 2k+1, 2\ell) \ge f(u, 2k+1, 2\ell)$ for $0 \le u \le k$. This proves Property 3. ■

Property 3 suggests that if an odd-order schema partition is deceptive, the immediate lower order schema partition is also deceptive. This reduces the total number of schema partitions to be investigated for deception. Thus, to find conditions for a bipolar deceptive function, we simply consider deception in odd-order schema partitions. Assuming $\lambda = 2k + 1$, we observe that we need to impose the condition that for any schema partition of order $2k + 1$ satisfying $1 \le k \le \ell - 1$[2] the fitness of a schema of unitation k is greater than or equal to that of any other competing schema of unitation $0 \le u \le k - 1$, as follows.

$$f(k, 2k+1, 2\ell) \ge f(u, 2k+1, 2\ell) \tag{3}$$

We assume that function values are non-negative. A schema of unitation u' contains strings of unitation varying from u' to $2\ell - 2k - 1 + u'$:

$$f(u', 2k+1, 2\ell) = 2^{-(2\ell-2k-1)} \sum_{i=u'}^{2\ell-2k-1+u'} \binom{2\ell - 2k - 1}{i - u'} f(i). \tag{4}$$

Since the schema under consideration is an odd-order schema, the summation in the right side of equation (4) involves an even number of terms. Figure 1 shows that strings that are contained in a schema are binomially distributed

[2] Order-one schema partitions are not interesting, since both schemata in an order-one schema partition contain an equal number of deceptive attractors and have identical fitness.

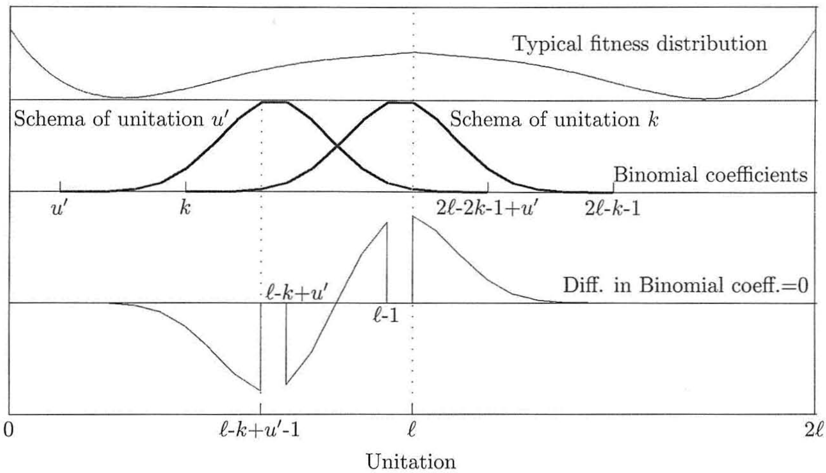

Figure 1: The fitness function and the corresponding binomial coefficients for two schemata are shown. The difference in the binomial coefficients between these two schemata are also shown.

with a maximum occurring at unitation $u = \ell - k + u' - 1$ and $u = \ell - k + u'$. We write the fitness of a schema of unitation k in the same manner:

$$f(k, 2k+1, 2\ell) = 2^{-(2\ell-2k-1)} \sum_{i=k}^{2\ell-k-1} \binom{2\ell - 2k - 1}{i - k} f(i) \tag{5}$$

This schema contains strings of unitation varying from k to $2\ell - k - 1$. The maximum number of terms occur at unitation $u = \ell - 1$ and $u = \ell$ (as shown in figure 1). For the function to be bipolar deceptive, the right side of equation (5) must be greater or equal to the right side of equation (4) for all values of $1 \le k \le \ell - 1$ and $0 \le u' \le k - 1$.

Initially, we confine our analysis to schemata of unitation $1 \le u' \le k - 1$, but we shall subsequently extend this analysis to schema of unitation zero. We observe that equations (4) and (5) are average values of a fitness distribution weighted by a binomial distribution located differently along the unitation axis (as depicted in figure 1). The fitness distribution is symmetric about $u = \ell$, as assumed. Since the deceptive attractor is assumed to be the second-best string, the best string among all strings that are contained in schemata of unitation $1 \le u' \le k - 1$ is the string of unitation ℓ or $\ell - 1$. In equation (4), the maximum value of the binomial coefficient occurs at $u = \ell - k + u' - 1$ and at $u = \ell - k + u'$. In equation (5), it occurs at $u = \ell - 1$ and at $u = \ell$. Defining $d(u', k) = 2^{2\ell-2k-1} [f(k, 2k+1, 2\ell) - f(u', 2k+1, 2\ell)]$, we impose the condition $d(u', k) \ge 0$ for bipolar deception. Using equations (4) and (5), we calculate and plot the difference in the binomial coefficients for each unitation in figure 1. We observe that the total number of negative terms in that distribution is the same as that of the positive terms.

In an earlier study [3], a set of sufficient conditions was obtained for an arbitrary uniglobal function of unitation. Here, we form a symmetric bipolar function with function values satisfying those conditions, and analyze bipolar deception in that function. We rewrite those conditions for a function of size ℓ having global optima at unitation zero and the deceptive attractor—also the second-best string—at unitation ℓ, as follows.

$$\begin{array}{l} \textbf{Primary optimality condition:} \\ \quad f(0) > f(\ell) \\ \textbf{Primary deception condition:} \\ \quad f(\ell) > f(0) + f(1) - f(\ell-1) \\ \textbf{Secondary deception conditions:} \\ \quad f(i) \geq f(j) \quad \text{for } \lceil \ell/2 \rceil \leq i \leq \ell - 1 \text{ and } \ell - i \leq j < i \end{array} \tag{6}$$

The primary optimality condition specifies that the string of unitation zero is better than the string of unitation ℓ. The primary deceptive condition establishes deception in order $\ell - 1$ schemata. The secondary deception conditions imply that a string of unitation $u \geq \lceil \ell/2 \rceil$ is better than any string of unitation as small as $\ell - u$.

We write the quantity $d(u', k)$ as the sum of two quantities—$d_1(u', k)$ for the strings of unitation from $\ell - k + u'$ to $\ell - 1$, and $d_2(u', k)$ for the rest of the strings. We observe that the total number of positive and negative terms in each quantity is the same. Using equations (4) and (5), the positive and negative terms in the quantity $d_1(u', k)$ may be grouped as strings of unitation varying from $\ell - k/2 - u'/2$ to $\ell - 1$, and as strings of unitation varying from $\ell - k + u'$ to $\ell - k/2 - u'/2 - 1$. (This scenario is also shown in figure 1.) The index for the negative terms can be rewritten in terms of the index for the positive terms, as follows.

$$d_1(u', k) = \sum_{i=\ell-k/2-u'/2}^{\ell-1} \left[\binom{2\ell - 2k - 1}{i - k} - \binom{2\ell - 2k - 1}{i - u'} \right] \times [f(i) - f(2\ell - k + u' - 1 - i)] \tag{7}$$

The term inside the first bracket is positive for all values of the summation index, and the secondary deception conditions reveal that the quantity in the second bracket is non-negative for all values of the summation index. Thus, $d_1(u', k)$ is greater than or equal to zero.

To show that $d_2(u', k) \geq 0$, we use the symmetry of the function; reflect the strings of unitation from ℓ to $2\ell - k - 1$ at $u = \ell$; and compare them with strings of unitation varying from u' to $\ell - k + u' - 1$. The number of positive and of negative terms in $d_2(u', k)$ is the same, and for each binomial coefficient in the positive terms there exists an identical binomial coefficient in the negative terms. For each binomial coefficient in the quantity $d_2(u', k)$, we compare the corresponding function values that contribute positive and negative values. For example, we compare the function values of strings of unitation ℓ (positive contribution), with that of unitation $\ell - k + u' - 1$

(negative contribution), and so on. Secondary deception conditions allow the positive terms to be greater than the corresponding negative terms up to unitation $(\ell + k - u' + 1)/2$ for the positive terms. Since no relationship among function values of strings of unitation smaller than $\ell/2$ is imposed in the sufficient conditions, nothing can be concluded, apparently, about terms of unitation smaller than $(\ell + k - u' + 1)/2$ and up to k. However, the binomial coefficients for strings of unitation smaller than $(\ell + k - u' + 1)/2$ and up to k are very small in comparison to the binomial coefficients for strings of unitation greater than or equal to $(\ell + k - u' + 1)/2$ and up to ℓ. In addition, there are more binomial terms for unitation greater than or equal to $(\ell + k - u' + 1)/2$ than there are for unitation smaller than $(\ell+k-u'+1)/2$. On the other hand, the primary deceptive condition suggests that $f(0)$ cannot be arbitrarily large. Thus, the sufficient conditions imply that the net positive quantity for functions satisfying secondary deception conditions cannot be smaller than the net negative quantity. In other words, we may write $d_2(u', k) \geq 0$. Combining our analyses, we may write that $d(u', k) \geq 0$.

We now compare schemata of unitation k with schemata of unitation zero in all schema partitions of order $1 \leq k \leq \ell - 1$. A schema of unitation zero contains only one string of unitation zero. Thus, using the schema fitness expressions given in equation (4), we combine the function value for the string of unitation zero with only one term of the function value for the string of unitation one, and write separately:

$$\begin{aligned} & 2^{2\ell-2k-1} f(0, 2k+1, 2\ell) \\ & = [f(0) + f(1)] + 2(\ell - k - 1) f(1) + \sum_{i=2}^{2\ell-2k+1} \binom{2\ell - 2k - 1}{i - 2} f(i) \end{aligned} \tag{8}$$

We rewrite equation (5) in the same manner, except that in this case, we separate one term each of the function values of unitation $\ell - 1$ and ℓ:

$$\begin{aligned} & 2^{2\ell-2k-1} f(k, 2k+1, 2\ell) \\ & = [f(\ell-1) + f(\ell)] + \left[\binom{2\ell - 2k - 1}{\ell - k} - 1\right] [f(\ell-1) + f(\ell)] \\ & \quad + \sum_{i=k; i\neq \ell, \ell-1}^{2\ell-k-1} \binom{2\ell - 2k - 1}{i - k} f(i) \end{aligned} \tag{9}$$

The primary deception condition establishes that the quantity inside the first bracket of equation (9) must be greater than the quantity inside the first bracket of equation (8). The analysis carried out in the previous paragraphs follows for the rest of the terms in both equation (8) and (9). Thus, any odd-order schema partition is deceptive in a bipolar function satisfying the sufficient conditions. Because an odd-order schema deception implies deception in the immediately smaller even-order schema partition, the argument follows for all permissible schema partitions, and the function is bipolar deceptive.

3. Construction of bipolar deceptive functions

We have shown that the sufficient conditions for an arbitrary deceptive function of unitation of size ℓ with one global optimum and one deceptive attractor are sufficient for maximal deception in a bipolar symmetric function of size 2ℓ with identical function values in the range $0 \le u \le \ell$. We define a parameter *folded unitation*, $e = \mid u - \ell \mid$, and observe that the bipolar symmetric function expressed in folded unitation may be identically represented by the function of unitation with one global optimum and one deceptive attractor.

Thus, a bipolar deceptive function may be easily constructed from a uniglobal function of unitation by using the idea of folded unitation. We assume that the uniglobal, fully deceptive function is represented by $g(u)$ with unitation u varying from zero to ℓ, and that the function has a global solution at $u = \ell$ and a deceptive attractor at $u = 0$. We then construct a bipolar deceptive function $f(u)$ requiring $f(u) = g(e)$. The bipolar function $f(u)$ has two global optimal strings of $u = 0$ and $u = 2\ell$, and $\binom{2\ell}{\ell}$ deceptive attractors of $u = \ell$. The analysis in subsection 2.2 shows that if the uniglobal function of unitation $g(u)$ is fully deceptive, the bipolar function of unitation $f(u)$ constructed from $g(u)$ is bipolar deceptive.

In the following, we construct a bipolar deceptive function of unitation from a uniglobal, fully deceptive trap function.

3.1 A folded-trap function

The condition of deception of a uniglobal trap function is found elsewhere [2]. A trap function is a function of unitation with a global optimum and a deceptive attractor located maximally far apart from the global optimum. Without loss of generality, we consider that the global optimum and the deceptive attractor are strings of unitation ℓ and zero respectively, and have function values equal to b and a respectively. The function value reduces as the unitation increases, and the string of unitation z has a function value equal to zero. The function value increases thereafter until the function value is b at $u = \ell$. We write this function as follows.

$$g(u) = \begin{cases} \frac{a}{z}(z-u), & \text{if } u \le z \\ \frac{b}{\ell - z}(u - z), & \text{otherwise} \end{cases} \tag{10}$$

We construct a 2ℓ-bit, symmetric bipolar function of unitation $f(u)$ (a folded-trap function), by using the function in equation (10), $g(u)$, and imposing $f(u) = g(e)$. Elsewhere [3], a deception condition for a fully deceptive trap function has been found by using the conditions of equation (6). Using the same conditions, we obtain a sufficient condition for the 2ℓ-bit, bipolar deceptive folded-trap function, $f(u)$:

$$\frac{a}{b} > \frac{2 - 1/(\ell - z)}{2 - 1/z} \tag{11}$$

Figure 2 shows a 10-bit folded-trap function with $a = 0.95$, $b = 1.00$, and $z = 3$. These parameter values satisfy the above condition. The schema

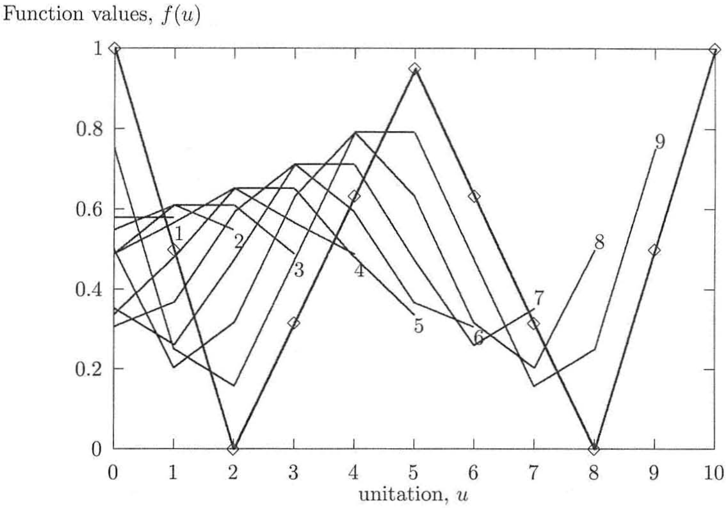

Figure 2: A 10-bit folded-trap function plotted versus unitation. The schema average function values for schemata of order one to nine are also shown. The function is bipolar deceptive.

average function values of schemata of order one to nine are also shown. The figure shows that all schema partitions are bipolar deceptive.

We have shown a way to construct a bipolar deceptive function from a uniglobal, fully deceptive function. We extend this analysis to find a set of sufficient conditions for an arbitrary bipolar function.

3.2 An arbitrary bipolar function

In this subsection, we rewrite the conditions of equation (6) for an arbitrary bipolar function. In an arbitrary bipolar function, all function values of strings of a particular unitation may not be equal; therefore, we modify the conditions using the maximum and minimum function values of strings of identical folded unitation. For example, the primary optimality condition requires that $f(0)$ be greater than $f(\ell)$. There is one string of unitation zero and there are $\binom{2\ell}{\ell}$ strings of unitation ℓ. In order to impose the primary optimality condition for all strings of unitation ℓ, we require that $f(0)$ be greater than the maximum function value of strings of unitation ℓ. Using this principle, we may rewrite primary and secondary deception conditions. Assuming that the global optima are strings of unitation 0 and 2ℓ and that deceptive attractors (which are also the second-best strings in the search space) are strings of unitation ℓ, we denote $\min f(u)$ and $\max f(u)$ to be the minimum and maximum function values of strings of folded unitation u,[3]

[3]In other words, $\min f(u)$ is the minimum function value of all strings of unitation u and $2\ell - u$, and $\max f(u)$ is the maximum function value of all strings of unitation u and $2\ell - u$.

and using the conditions of equation (6), we obtain sufficient conditions for a bipolar deceptive function, as follows.

$$\begin{aligned}
&\textbf{Primary optimality conditions:}\\
&\qquad f(0) > \max f(\ell), \quad \min f(\ell) > \max f(\ell-1)\\
&\textbf{Primary deception condition:}\\
&\qquad \min f(\ell) > f(0) + \max f(1) - \min f(\ell-1)\\
&\textbf{Secondary deception conditions:}\\
&\qquad \min f(i) \geq \max f(j)\\
&\qquad\qquad \text{for } \lceil \ell/2 \rceil \leq i \leq \ell-1 \text{ and } \ell-i \leq j < i
\end{aligned} \tag{12}$$

Since the deceptive attractors are the second-best strings, the primary optimality condition contains the additional condition that $\min f(\ell) > \max f(\ell-1)$. It is interesting to note that these conditions do not imply any restrictions on the minimum value of strings of folded unitation varying from one to $\lfloor \ell/2 \rfloor$. The conditions of equation (12) are used to construct an arbitrary bipolar deceptive function in the appendix.

4. Bipolar deceptive functions using Walsh coefficients

We have found a set of sufficient conditions for a bipolar deceptive function of unitation. We may also construct a bipolar deceptive function of unitation using low-order Walsh coefficients, following the method used elsewhere [8] to construct a uniglobal deceptive function. Constructing a function from Walsh coefficients provides an easier way to calculate schema fitness values, thereby simplifying the deception analysis. We took the harder way first, to obtain a general set of sufficient conditions for deception. In this section, we find a relationship among low-order Walsh coefficients in order to construct a deceptive bipolar function of unitation, and use the sufficient conditions of equation (6) to obtain the same relationship for bipolar deception.

We use the notation of Section 3 to represent a schema fitness value and a function value, and assume that all Walsh coefficients of the same order are identical. The fitness of a schema of order λ and unitation u is written as follows [8].

$$f(u,\lambda,2\ell) = \sum_{i=0}^{\lambda} w_i \psi_i'(u,\lambda) \tag{13}$$

where w_i is the Walsh coefficient for order i, and the term ψ_i' represents the sum of all the evaluations of order i Walsh functions defined for any schema of unitation u and order λ:

$$\psi_i'(u,\lambda) = \sum_{j=0}^{i} (-1)^j \binom{u}{j} \binom{\lambda-u}{i-j} \tag{14}$$

A function of size 2ℓ may be constructed from equation (13) by substituting $\lambda = 2\ell$. Recognizing that we are interested in a symmetric bipolar function with an axis of symmetry at $u = \ell$, and using the above equation, we observe the following property for any function of unitation.

Property 4. *If in a function of unitation of size 2ℓ all odd-order Walsh coefficients are zero, the function is symmetric about unitation ℓ.*

Proof. Using equation (14) and substituting $\lambda = 2\ell$, it is a straightforward matter to show that $\psi_i'(2\ell - u, 2\ell) = (-1)^i \psi_i'(u, 2\ell)$. Using this equation, we calculate the difference in function values $d(u) = f(u) - f(2\ell - u)$:

$$d(u) = 2 \sum_{i=1,3,\ldots}^{2\ell-1} w_i \psi_i'(u, 2\ell)$$

Setting all odd-order Walsh coefficients to be zero, we obtain $d(u) = 0$, implying that $f(u) = f(2\ell - u)$ for all values of $0 \le u \le \ell - 1$. Thus, the function is symmetric about $u = \ell$. ∎

We assume that only zeroth, second, and fourth order Walsh coefficients are nonzero. The order-zero Walsh coefficient represents the average of all function values. Using equation (13), we write the schema average fitness of a schema of order λ and unitation u in terms of three Walsh coefficients (w_0, w_2, and w_4):

$$f(u, \lambda, 2\ell) = w_0 + w_2 \left[\binom{\lambda - u}{2} - u(\lambda - u) + \binom{u}{2} \right] \tag{15}$$

$$+ w_4 \left[\binom{\lambda - u}{4} - u\binom{\lambda - u}{3} + \binom{u}{2}\binom{\lambda - u}{2} - \binom{u}{3}(\lambda - u) + \binom{u}{4} \right]$$

4.1 Optimality and deception conditions

In this subsection, we find optimality and deception conditions for a bipolar deceptive function in terms of three Walsh coefficients. We then find a set of these coefficients which satisfies all optimality and deception conditions.

The bipolar function is symmetric at unitation $u = \ell$. Thus, using Property 1, we need to consider only ℓ optimality conditions. The optimality conditions impose that $f(0) > f(u)$ for $1 \le u \le \ell$. Substituting function values from equation (15) and simplifying, we obtain

Optimality conditions:

$$w_2 > -w_4 \left[u^2 - 2u\ell + 2\ell^2 - 3\ell + 2 \right] / 3 \quad \text{for } 1 \le u \le \ell \tag{16}$$

The deception conditions may be obtained by imposing the condition that a schema of order λ and unitation $\lfloor \lambda/2 \rfloor$ is no worse than any other competing schema. It has been shown in the previous section that the deception in all odd-order schema partitions implies bipolar deception in a function. Thus, we impose deception in odd-order schema partitions only. Assuming $\lambda = 2k+1$, we impose the conditions for deception: $f(k, 2k+1, 2\ell) \ge f(u, 2k+1, 2\ell)$ for $1 \le k \le \ell - 1$ and $0 \le u \le k - 1$. Using equation (15) and simplifying, we obtain

Deception conditions:

$$w_2 \le -w_4 \left[(2k - 2u)^2 - 8k - 4u + 4 \right] / 12 \quad \text{for } 1 \le k \le \ell - 1, \; 0 \le u \le k - 1 \tag{17}$$

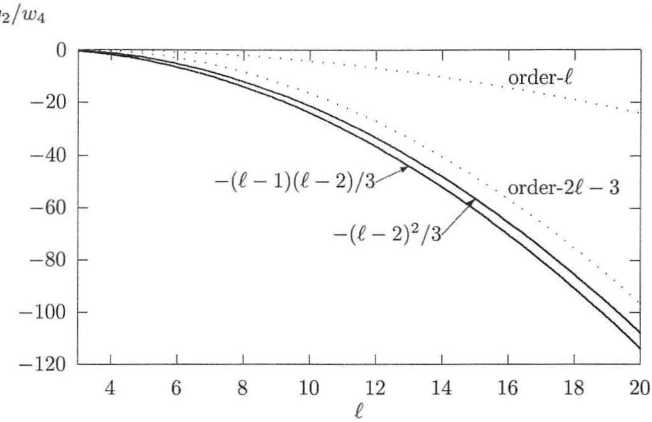

Figure 3: The bound on the ratio w_2/w_4 is plotted with ℓ. Any value of the ratio between the two solid lines constitutes a bipolar deceptive function of unitation. The upper bounds for this ratio are also shown for partially bipolar deceptive functions of unitation to order $2\ell-3$ and ℓ. The lower limit is the same in all cases.

The parameters w_2 and w_4 can take any algebraic sign. First, we assume that w_4 is positive. Considering all conditions in equation (16), we observe that the critical condition occurs for $u=\ell$. Substituting $u=\ell$ in equation (16), we obtain $w_2/w_4 > -(\ell-1)(\ell-2)/3$. Furthermore, we observe from equation (17) that the critical condition occurs for $k=\ell-1$ and $u=0$. Substituting these values in equation (17), we obtain $w_2/w_4 \leq -(\ell-2)^2/3$. Combining these two conditions, we write the conditions for bipolar deception of a function of unitation as follows.

$$-(\ell-1)(\ell-2)/3 < w_2/w_4 \leq -(\ell-2)^2/3. \tag{18}$$

Since w_4 is assumed positive, the above condition implies that w_2 is negative. Next, we consider that w_4 is negative. Finding the critical conditions of equations (16) and (17), we observe that there are no w_2 and w_4 that satisfy these critical conditions. Thus, a bipolar function as given in equation (15) with negative w_4 is not possible. Figure 3 shows the lower and upper bound of the ratio w_2/w_4 in solid lines. The figure depicts that the range of permissible values of this ratio is small.

In figure 4, we have plotted a bipolar deceptive 10-bit function of unitation. In addition to the function values, fitness of schemata in all schema partitions of order one to nine are also shown. In this function, the lower and upper limit of the ratio w_2/w_4 are -4 and -3, respectively. The function is constructed with $w_0 = 0.4350960$, $w_2 = -0.0248397$, and $w_4 = 0.0080128$, so that the ratio w_2/w_4 is -3.1 and all function values are nonnegative. The figure depicts that the schema of unitation equal to half of its order is no worse than any other schema in the partition. However, both schemata in the

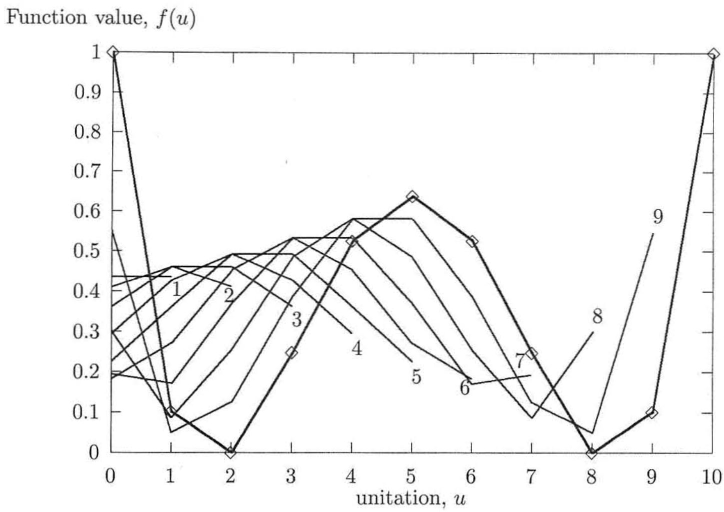

Figure 4: A 10-bit bipolar deceptive function is plotted with unitation. This function is constructed with $w_0 = 0.4350960$, $w_2 = -0.0248397$, and $w_4 = 0.0080128$. Fitness of schemata of all schema partitions is also shown.

order-one schema partition have the same fitness value. It may be observed that this function satisfies all properties of a bipolar function, as previously discussed.

4.2 Partially bipolar deceptive functions

The equation (17) may be used to find conditions for a partially bipolar deceptive function of unitation to any order. A function is defined to be *partially* bipolar deceptive to an order p, if all schema partitions of order less than p are deceptive. For a partially bipolar deceptive function, the optimality conditions must hold. Deception conditions may be found depending on the parity of p. If p is odd, equation (17) must be used for all values of $2k+1 < p$. It is interesting to note that the right side of equation (17) varies quadratically with k. Denoting the right side of equation (17) by $r(u,k)$, we obtain $r(u,k+1) - r(u,k) = -w_4[4(k-u)-3]/12$. Since $u < k$, the right side of that inequality is monotonically decreasing with respect to k. Thus, the critical condition occurs for $2k+1 = p$. Substituting $k = (p-1)/2$ and $u = 0$ in equation (17) and using optimality conditions, we obtain the following bound on the ratio w_2/w_4 for a partially bipolar deceptive function of unitation to an odd order p.

$$-(\ell-1)(\ell-2)/3 < w_2/w_4 \leq -(p-3)^2/12 \qquad (19)$$

If p is even, a similar analysis may be performed using equation (15). Assuming $p = 2k$, we obtain the general condition of deception, $w_2 \leq w_4(4k^2 +$

$4u^2 - 8ku - 12k + 8)/12$. For a partially bipolar deceptive function to order p, the critical case occurs for $2k = p$ and $u = 0$. The bound of the ratio w_2/w_4 for a partially bipolar deceptive function of unitation to an even order p is

$$-(\ell - 1)(\ell - 2)/3 < w_2/w_4 \leq -(p-4)(p-2)/12. \tag{20}$$

The upper bounds of w_2/w_4 for partially bipolar deceptive functions to order $2\ell - 3$ and ℓ are shown in figure 3. The lower bound is governed by the critical optimality condition and holds for a partially bipolar deceptive function to any order. Thus, as the order of deception increases, the range of the ratio w_2/w_4 for a partially bipolar deceptive function decreases.

4.3 Deception from sufficient conditions

We can use sufficient conditions found in the Section 2.2 (equation (6)) to investigate deception in the function given in equation (15). There are three conditions to be satisfied.

Primary optimality condition: Imposing the condition $f(0) > f(\ell)$ in equation (15), we obtain the inequality $w_2/w_4 > -(\ell-1)(\ell-2)/3$, which is the lower bound of w_2/w_4 found previously.

Primary deception condition: Imposing the primary deception condition, we obtain the inequality $w_2/w_4 < -(\ell-2)^2/3$, which is the upper bound of w_2/w_4 found previously.

Secondary deception conditions: In order to investigate the secondary deception conditions, we first calculate the unitation u^* for which the function is the minimum, by calculating the slope of the function and by setting the slope to zero. Since the function is quartic to the unitation, the derivative of the function is cubic, and there are three values of unitation for which the slope is zero. We know that one of them is at $u = \ell$ and that the other two lie symmetrically on either side of $u = \ell$. Differentiating equation (15) once with respect to u, we obtain

$$\frac{\partial f(u)}{\partial u} = 4(u-\ell)\left[3w_2 + w_4[(u-\ell)^2 - 3\ell + 2]\right] \tag{21}$$

This equation reveals that one of the roots of the equation $\partial f(u)/\partial u = 0$ is $u^* = \ell$, and other two roots are symmetrically placed about $u = \ell$. These two roots are $u^* = \ell \pm \sqrt{(1.5\ell - 1) - 1.5w_2/w_4}$. Substituting the bounds of w_2/w_4 from equation (18), we observe that minimum $u^* \approx 0.292\ell$. Since there is no change in slope for function values in the intervals $[0, u^*]$ and $[u^*, \ell]$, we need to compare only the schemata of unitation $\ell - u$ to the schemata of unitation u. Calculating the difference in function values $f(u) - f(\ell - u)$, and setting the difference to be greater than or equal to zero, we obtain

$$w_2/w_4 \leq -[\ell^2 - 3\ell + 2 - 2u(\ell - u)]/3 \tag{22}$$

Secondary deception conditions require this for all values of u in the interval $\lceil \ell/2 \rceil \leq u \leq \ell - 1$. The critical condition occurs for $u = \ell - 1$. Substituting this value of u, we obtain $w_2/w_4 \leq -(\ell-1)(\ell-4)/3$. The right side of this inequality is always greater than $-(\ell-2)^2/3$. Thus, secondary deception conditions are always satisfied.

Thus, the bipolar function obtained from the low-order Walsh coefficients given in equation (15) is bipolar deceptive. These calculations have shown how the set of sufficient conditions for deception of equation (6) may be used to find deception conditions or investigate deception in any bipolar function.

5. Multimodal functions from bipolar functions

Previously, we derived conditions among problem parameters to construct a bipolar deceptive function. In this section, we describe a way to construct multimodal functions of bounded deception from bipolar deceptive functions, and discuss the difficulty of solving these problems.

A multimodal function of bounded deception may be constructed by concatenating a number of bipolar deceptive functions. An ℓ-bit deceptive multimodal function to order k may be obtained by adding m bipolar deceptive functions of size k:

$$F(x) = \sum_{i=1}^{m} f_i(x_i) \tag{23}$$

where x_i's are substrings of size k such that $\cup_{i=1}^{m} x_i = x$. Functions f_i are bipolar deceptive functions of size k. Without loss of generality, we assume that all subfunctions are the same, and x_i's are nonoverlapping substrings. Because each bipolar function has two global attractors and $\binom{k}{\lfloor k/2 \rfloor}$ deceptive attractors, there are a total of $[2 + \binom{k}{\lfloor k/2 \rfloor}]^m$ attractors, of which only 2^m are global attractors. As the size and the number of bipolar functions increase, the number of deceptive attractors increases exponentially. Figure 5 shows a quadrimodal deceptive function constructed from two six-bit bipolar deceptive folded-trap functions in which $a = 0.7$, $b = 1.0$, and $z = 2$.[4] Since these values satisfy the condition of equation (11), the folded-trap function with two global strings 000000 and 111111 is bipolar deceptive. This function is then transformed using a technique suggested elsewhere [8] to construct a double-trap function with global strings 010101 and 101010. The left half of Figure 5 shows the location of all attractors in the decoded parameter space of two subfunctions. For each subfunction, there are $\binom{6}{3}$ or 20 deceptive attractors, each of which is three bits away from both global strings. Thus, there are a total of $(20 + 2)^2$ or 484 attractors, of which only four are global attractors. The global attractors are shown by the largest filled circles in the

[4]Note that six bits is the minimal size bipolar deceptive function allowed by our definition.

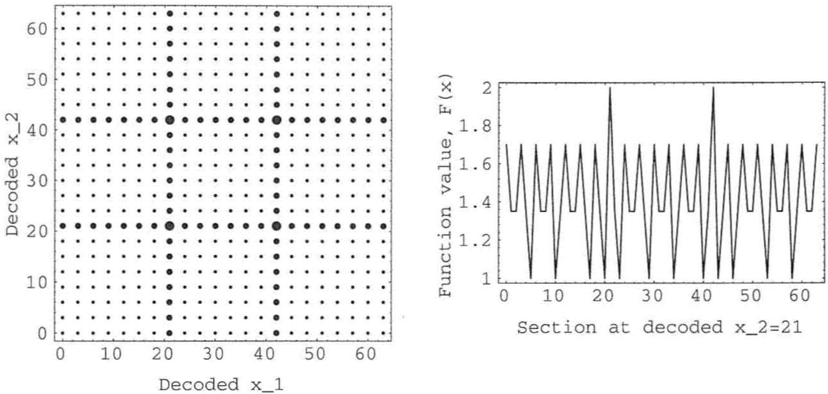

Figure 5: A 12-bit quadrimodal function is shown. The figure captures the multimodality of the function by showing all attractors of the top three function values. Function values for a fixed substring 010101 in the second subfunction are also shown.

figure. The next-best solutions are strings with one global substring and one deceptive substring. There are a total of 4×20 or 80 such, each of which is shown by a medium-sized filled circle in the figure. The remaining 20×20 or 400 optima consist of one deceptive attractor from each subfunction, and are shown by the smallest filled circles. If more subfunctions are added, the numbers of deceptive attractors and global optima increase exponentially. For example, with three subfunctions, there would be a total of 22^3 or 10,648 attractors, of which only 2^3 or 8 would be global attractors. The right half of Figure 5 shows the 12-bit function $F(x)$ for a fixed substring 010101 in the second subfunction, as x_1 is varied from zero to 63 in its decoded parameter. All 20 deceptive attractors and two global attractors are shown.

These functions offer a stiff challenge to any optimization method simply because of their multimodality. Many traditional optimization methods will perform poorly in these functions because there are many optima, and most of them are not global. Many restarts may be necessary to find a global solution. In stochastic search techniques (such as genetic algorithms), deception, along with multimodality, may cause difficulty in finding the global solution(s) of such functions. Because low-order building blocks do not guide the search in the right direction, the combination of the low-order building blocks may not form the necessary higher-order building blocks. This may cause GAs to converge to one of the deceptive attractors. However, this problem may be solved if proper population sizing is adopted [9]. Because all schema partitions of order lower than the subfunction size (k) are deceptive, GAs must play at least 2^k-armed bandits in order to solve these problems successfully, and there must be enough room for GAs to do so. Furthermore, the population must be large enough to statistically evaluate a building block correctly and detect the smallest difference in schema fitness in the midst of collateral noise [13]. With such a population size, the schemata containing the global solutions may win a schema partition competition and continue to be expressed in the population. With multiple copies in the population, recombination operators should be able to mix these building blocks together to form a global solution.

Multimodal deceptive functions offer a greater challenge to GAs if the objective is not to find one global optimum but to find all (or as many as possible) global optima simultaneously. GAs with a niching technique have been found to create stable subpopulations around distinct optima in multimodal function optimization [1, 12]. Niching is introduced by degrading an individual's objective function value by a factor dependent on the number of neighbors measured in some space. Similar techniques may be used to form stable subpopulations around each global optimal solution. One characteristic of this type of niching is that if it is tuned so that two optima do not participate in degrading each other's objective function value, two separate niches will be formed around each optima. In that case, the steady-state number of individuals in a niche is directly proportional to its objective function value. On the other hand, if two optima participate in degrading each other's objective function value, then only the optimum with higher objective function value remains at the steady-state. This apparently may cause difficulty in the solution. Setting a small niche size may allow distinct niches to be formed around each optimum. Because the number of attractors are many, this may require a large population size to accommodate all niches in the population. In contrast, if a large niche size is used, multiple global optima may be included in a single niche. This may cause competition among global optima and may result in the solution of only a few global optima. Other niching methods may be necessary to alleviate these problems. Nevertheless, single or multiple global solutions using GAs would be a first step towards solving this difficult class of functions, and may provide important insights into the complex mechanics of genetic algorithms.

6. Conclusions

The idea of deception has been extended to multimodal functions. A bipolar function has been defined as a function with two global attractors and a number of deceptive attractors maximally far apart from the global attractors. The deception in a bipolar deceptive function has been defined, and a set of sufficient conditions for a bipolar deceptive function has been found. It has been observed that a bipolar function may be constructed from a uniglobal, fully deceptive function by the notion of *folded* unitation.

A bipolar deceptive function of unitation has also been constructed from low-order Walsh coefficients. It has been observed that non-zero Walsh coefficients of order zero, two, and four are sufficient to construct a bipolar deceptive function of unitation. Relations among these Walsh coefficients have also been found for a partially bipolar deceptive function of unitation to any order. The set of sufficient conditions found previously has been used to find the same relationship among these Walsh coefficients for a bipolar deceptive function of unitation.

Multimodal functions of bounded deception have been constructed from bipolar deceptive functions. These functions offer a stiff challenge to optimization methods because of their multimodality. For genetic algorithms, the difficulty comes not only from the multimodality, but also from the inherent deception of the functions. However, with proper population sizing, GAs may be able to solve these problems to global optimality. Furthermore, GAs with a niching technique may be used to find all or a number of global optimal solutions simultaneously. It is for precisely such functions, rather than simple unimodal functions, that population-based stochastic search methods such as genetic algorithms can outperform traditional methods of global optimization. Furthermore, the attempt to find one or more global solutions using GAs may improve our understanding of genetic algorithms.

The door has been opened for generalizing the definition of deception for application to more general types of problems. The attempt to define deception in a more general framework may give us a broader picture of the importance of deception in the study of genetic algorithms.

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References

[1] K. Deb, "Genetic Algorithms in Multimodal Function Optimization," MS Thesis and TCGA Report No. 89002 (Tuscaloosa: University of Alabama, The Clearinghouse for Genetic Algorithms, 1989).

[2] K. Deb and D. E. Goldberg, "Analyzing Deception in Trap Functions," IlliGAL Report No. 91009 (Urbana: University of Illinois, Illinois Genetic Algorithms Laboratory, 1991).

[3] K. Deb and D. E. Goldberg, "Sufficient Conditions for Deceptive and Easy Binary Functions," IlliGAL Report No. 92001 (Urbana: University of Illinois, Illinois Genetic Algorithms Laboratory, 1992).

[4] L. J. Eshelman, "The CHC Adaptive Search Algorithm: How to Have Safe Search When Engaging in Nontraditional Genetic Recombination," pages 265–283 in *Foundations of Genetic Algorithms*, edited by G. Rawlins (San Mateo: Morgan Kaufmann, 1991).

[5] D. E. Goldberg, "Simple Genetic Algorithms and the Minimal Deceptive Problem," pages 74–88 in *Genetic Algorithms and Simulated Annealing*, edited by L. Davis (London: Pitman, 1987).

[6] D. E. Goldberg, "Genetic Algorithms and Walsh Functions: Part I, A Gentle Introduction," *Complex Systems*, **3** (1989) 129–152.

[7] D. E. Goldberg, "Genetic Algorithms and Walsh Functions: Part II, Deception and Its Analysis," *Complex Systems*, **3** (1989) 153–171.

[8] D. E. Goldberg, "Construction of High-Order Deceptive Functions Using Low-Order Walsh Coefficients," IlliGAL Report No. 90002 (Urbana: University of Illinois, Illinois Genetic Algorithms Laboratory, 1990).

[9] D. E. Goldberg, K. Deb, and J. H. Clark, "Genetic Algorithms, Noise, and the Sizing of Populations," IlliGAL Report No. 91010 (Urbana: University of Illinois, Illinois Genetic Algorithms Laboratory, 1991).

[10] D. E. Goldberg, K. Deb, and B. Korb, "Messy Genetic Algorithms Revisited: Studies in Mixed Size and Scale," *Complex Systems*, **4** (1990) 415–444.

[11] D. E. Goldberg, B. Korb, and K. Deb, "Messy Genetic Algorithms: Motivation, Analysis, and First Results," *Complex Systems*, **3** (1989) 493–530.

[12] D. E. Goldberg and J. Richardson, "Genetic Algorithms with Sharing for Multimodal Function Optimization," pages 41–49 in *Proceedings of the Second International Conference on Genetic Algorithms* (1987).

[13] D. E. Goldberg and M. Rudnick, "Genetic Algorithms and the Variance of Fitness," *Complex Systems*, **5** (1991) 265–278.

[14] G. E. Liepins and M. D. Vose, "Representational Issues in Genetic Optimization," *Journal of Experimental and Theoretical Artificial Intelligence*, **2**(2) (1990) 4–30.

[15] M. Mitchell and S. Forrest, "What Is Deception Anyway? And What Does It Have to Do with GAs? Some Concerns Inspired by the Tanese Functions" (unpublished manuscript, 1991).

[16] M. D. Vose, "Generalizing the Notion of Schema in Genetic Algorithms," *Artificial Intelligence* (in press).

[17] D. Whitley, (1991a). "Fundamental Principles of Deception in Genetic Search," pages 221–241 in *Foundations of Genetic Algorithms*, edited by G. Rawlins (San Mateo: Morgan Kaufmann, 1991).

[18] D. Whitley, "Deception, Dominance, and Implicit Parallelism in Genetic Search" (unpublished manuscript, 1991).

Appendix

A.1 An arbitrary bipolar deceptive problem

In an arbitrary bipolar deceptive function, we recognize that the strings of identical unitation may not have identical function value. In order to design an arbitrary bipolar deceptive function, we first choose the minimum ($f_{\min}$)

and maximum ($f_{\max}$) function value of strings of every unitation so that they satisfy the conditions of equation (12). Thereafter, we assign the other strings a random function value in the range $[f_{\min}, f_{\max}]$.

We design a six-bit arbitrary bipolar deceptive function with the following minimum and maximum function value of strings of different unitation.

Unitation						
0 and 6	1 and 5		2 and 4		3	
	$f_{\min}$	$f_{\max}$	$f_{\min}$	$f_{\max}$	$f_{\min}$	$f_{\max}$
1.0	0.0	0.3	0.5	0.8	0.9	0.9

It is a straightforward matter to show that above values satisfy the conditions of equation (12). There are two global optima (strings 000000 and 111111) and $\binom{6}{3}$ or 20 deceptive attractors (all strings of unitation three). The function values for all 2^6 or 64 strings are shown in the following listing.

000000	1.000	011000	0.619	100101	0.900	101110	0.624
		100001	0.792	100110	0.900	110011	0.722
000001	0.057	100010	0.539	101001	0.900	110101	0.676
000010	0.162	100100	0.520	101010	0.900	110110	0.800
000100	0.018	101000	0.560	101100	0.900	111001	0.553
001000	0.252	110000	0.509	110001	0.900	111010	0.617
010000	0.279			110010	0.900	111100	0.793
100000	0.094	000111	0.900	110100	0.900		
		001011	0.900	111000	0.900	011111	0.269
000011	0.500	001101	0.900			101111	0.300
000101	0.645	001110	0.900	001111	0.723	110111	0.283
000110	0.599	010011	0.900	010111	0.612	111011	0.000
001001	0.655	010101	0.900	011011	0.574	111101	0.110
001010	0.787	010110	0.900	011101	0.625	111110	0.227
001100	0.675	011001	0.900	011110	0.785		
010001	0.725	011010	0.900	100111	0.694	111111	1.000
010010	0.709	011100	0.900	101011	0.614		
010100	0.678	100011	0.900	101101	0.678		

The schema average fitness values of a number of schemata are calculated and shown in the following table.

```
                                               Schemata containing maximum number of
                                                     deceptive attractors
Order-two schema partitions --------------------------------------------------------
00**** (4)  0.605     11**** (4)  0.618      01**** (6)  0.705      10**** (6)  0.676

Order-three schema partitions ------------------------------------------------------
000*** (1)  0.485     111*** (1)  0.525      001*** (3)  0.724      010*** (3)  0.713
                                             011*** (3)  0.697      100*** (3)  0.667
                                             101*** (3)  0.684      110*** (3)  0.711
```

Order-four schema partitions											
0000**	(0)	0.430	0111**	(1)	0.645	0011**	(2)	0.800	0101**	(2)	0.773
0001**	(1)	0.540	1011**	(1)	0.625	0110**	(2)	0.748	1001**	(2)	0.754
0010**	(1)	0.648	1101**	(1)	0.665	1010**	(2)	0.743	1100**	(2)	0.758
0100**	(1)	0.653	1110**	(1)	0.517						
1000**	(1)	0.581	1111**	(0)	0.533						

Order-five schema partitions											
00000*	(0)	0.529	01111*	(0)	0.527	00011*	(1)	0.750	10001*	(1)	0.719
00001*	(0)	0.331	10111*	(0)	0.462	00101*	(1)	0.843	10010*	(1)	0.710
00010*	(0)	0.331	11011*	(0)	0.542	00110*	(1)	0.788	10011*	(1)	0.797
00100*	(0)	0.453	11101*	(0)	0.309	00111*	(1)	0.812	10100*	(1)	0.730
01000*	(0)	0.502	11110*	(0)	0.451	01001*	(1)	0.804	10101*	(1)	0.757
10000*	(0)	0.443	11111*	(0)	0.614	01010*	(1)	0.789	10110*	(1)	0.789
						01011*	(1)	0.756	11000*	(1)	0.705
						01100*	(1)	0.759	11001*	(1)	0.811
						01101*	(1)	0.737	11010*	(1)	0.888
						01110*	(1)	0.762	11100*	(1)	0.726

For every schema the number of deceptive attractors that it contains is shown in parentheses, followed by its average fitness value. The two rightmost columns show the schemata containing the maximum number of deceptive attractors. The table shows that in any schema partition the schemata containing the maximum number of deceptive attractors are no worse than any schemata that do not contain the maximum number of deceptive attractors. The schema average fitness of other schema partitions may also be calculated to show that the function is bipolar deceptive.