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A Phase Diagram for Elementary Cellular Automata

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Abstract. We construct a phase diagram for the possible dynamics of one-dimensional, two-state, three-neighbor cellular automaton rules, using a new parameter (in conjunction with the previously known activity parameter). The new parameter estimates the average sensitivity of rule outcome to small changes in neighborhood configuration.

1. Introduction

Cellular automata (CAs) are useful as a paradigm for the study of complex dynamics in spatially extended systems with local interactions. Numerous types of behavior have been observed, leading to several classifications following different criteria: spatiotemporal patterns [1, 2]; the action of rules on n-step Markov measures [3]; and properties of the CA's limiting behavior [4, 5] and state-transition graphs [6, 7].

In this paper we address the closely related question of the *ordering* of the space of rules. We consider the family of simplest one-dimensional deterministic rules, with two states per site ($k = 2$) and one neighbor on each side ($r = 1$), typically referred to as *elementary* CAs. In particular, we define and use a new parameter that is obtainable directly from the description of a rule, to construct a two-parameter phase diagram of elementary CAs. This parameter is independent of the activity parameter λ that appears in the literature [8–11].

This study is complementary to those of Li and Packard [2] and of Langton and coworkers [8–11], which we review in the following section. We define the new sensitivity parameter in section 3, and show the phase diagrams in section 4. We discuss the results in section 5.

2. The structure of the CA rule space

For elementary CA rules, there are $2^3 = 8$ neighborhoods, and $2^8 = 256$ rules. After left–right and zero–one transformations are considered, 88 independent rules remain. Li and Packard [2] conducted an interesting study of this family of rules, which contains three major contributions. First, they proposed a refinement of the four original classes of behavior defined by Wolfram [1]. The

five Li-Packard Classes are: N (null), homogeneous fixed points, equivalent to Wolfram Class 1; F, nonhomogeneous fixed points; P, periodic global states; L, locally chaotic regions separated by domain walls; and C (chaotic), roughly equivalent to Wolfram Class 3. Wolfram Class 4 rules may fit into different Li-Packard Classes, according to the criterion that is used. In this paper we primarily consider the Li-Packard Classes.

The second contribution of [2] was the exploration of the structure of rule space, seen as an 8-dimensional hypercubic representation of the state-transition table (STT), which is a list of the outcomes of the 8 possible neighborhoods. In this space, two rules are adjacent if their STTs differ by only one bit. Li and Packard found a significant degree of clustering according to class, and a reasonable (but far from perfect) correlation between proximity in rule space and similarity of behavior. In addition, Li and Packard identified hot bits in the STT: the outputs of neighborhoods (000) and (111) are particularly important predictors of behavior—this was their third contribution. When these outputs are (0,1), rules tend to be N, F, or P; when they are (1,0), rules tend to be P; and when they are (0,0), rules go from N to C. The importance of hot bits is explained in [2] in terms of the mean-field maps of the rules.

It would be desirable to order rules in terms of simple global properties of the STT, rather than in very high-dimensional spaces: 27 dimensions for $k = 3$, $r = 1$; and 32 dimensions for $k = 2$, $r = 2$. An important rule-based parameter has been used by Langton and coworkers [8–11]. There appears to be a correlation between dynamical disorder and the activity parameter λ (defined as the fraction of neighborhoods with an active (nonzero) output). However, numerical observations of simultaneous first- and second-order transitions to chaos as λ increases, and general fuzziness in activity-complexity plots, suggest that at least a second parameter is required to fully describe the rule space (see especially [10]). We propose such a parameter in the following section. The effects are particularly dramatic for elementary CAs, where λ is almost useless as a predictor of behavior.

3. The sensitivity parameter

A simple explanation for the fact that λ works (when given enough neighbors and dimensions), is that in most cases it should be a reasonable predictor of the asymptotic density of active sites. By a simple combinatorial argument, when $\lambda \sim 1/2$, more global states are available, and the evolution is more complex; this argument ignores questions of broken ergodicity and accessibility of states. As λ moves away from $1/2$, fewer states are available, until $\lambda = 0$ or 1, at which points only homogeneous states become accessible. There are clear exceptions to this argument, caused by strong correlations such as gliders or domain formation. A notorious example is the identity rule (known as elementary rule 204), which is very simple although $\lambda = 1/2$. We note that values λ, $1 - \lambda$ are considered to be dynamically equivalent; therefore, activity plots are usually folded about $\lambda = 1/2$.

An independent parameter that could be used to discriminate between rules with equal λ should take into account the *sensitivity* of rules to changes in one or more sites. Wolfram Classes 3 and 4 are known to be much more sensitive than Classes 1 and 2 [12]. We define an average sensitivity parameter μ (to the same order of approximation used to define λ) as the average over all sites of all neighborhoods of change in output induced by change in site value, as follows.

$$\mu = \frac{1}{nm} \sum_n \sum_m \left| \frac{\partial s_o}{\partial s_m} \right|,$$

where the sums are over all possible neighborhood configurations n and over the states s_m of all m bits in each neighborhood configuration, and where s_o is the output of neighborhood configuration n.

The quantity $|\partial s_o / \partial s_m| \in (0, 1)$ is related to the Boolean derivative for CAs [13], and to the discrete metric introduced by Robert [14] for discrete iterations. While Vichniac [13] hinted at the relation between the Boolean derivative and complexity, the overall parameter related to rule behavior that we introduce in this paper is new. It is true that Wolfram [1] and Packard [15] have used the *measured* average spreading rate resulting from a single site change. However, their parameter (γ) is *not* obtainable a priori from the STT, and it is used as a *measure*, not a *predictor*, of complexity. Two bounds for μ are easily verified: $0 \leq \mu \leq 1/2$, and $\lambda/3 \leq \mu \leq \lambda$ (the latter only for elementary CAs). It is possible to have a rule with high λ and fairly low μ, but not the reverse. In the same way that λ estimates the asymptotic density of active sites, μ estimates the sensitivity to small configurational changes, if all neighborhoods are equally probable. In some respects μ is fairly crude, as it does not differentiate between central and peripheral bits. For example, μ for the identity rule is fairly high, even though the rule does not allow transmission of information.

We show values of λ and μ for the 88 independent elementary CA rules in Table 1. These values will be used to construct phase diagrams in the following section. We use the standard notation for rule numbering [1], which is to give the decimal equivalent of the STT.

4. The phase diagrams

We have used Table 1 in conjunction with [2] to construct Figure 1, which shows schematically the parameter domains for each Li-Packard Class. The space of activity–sensitivity values contains 11 points, which is more than activity alone (5), but fewer than the STT rule space (256). As discussed in Section 3, the parameter space has been folded about $\lambda = 1/2$. We note that, while some overlap occurs, the domains for each class are nicely convex and connected. Near $\lambda = 1/2$, disorder seems to increase with sensitivity. *All* elementary rules have been included; some violate the definition of λ, which requires a quiescent state, $f(000) = 0$. This is corrected in Figure 2.

$(\lambda,\ \mu)$	Rules
(0,0)	0
(1/8, 1/8)	1, 2, 4, 8, 32, 128
(1/4, 1/6)	3, 5, 10, 12, 34, 136, 160
(1/4, 1/4)	6, 9, 18, 24, 33, 36, 40, 72, 130, 132
(3/8, 5/24)	7, 11, 13, 14, 19, 35, 42, 50, 76, 138, 140, 162, 168, 200
(3/8, 7/24)	25, 26, 28, 37, 38, 44, 56, 62, 74, 94, 110, 122, 152, 164
(3/8, 3/8)	22, 41, 73, 104, 134, 146
(1/2, 1/6)	15, 51, 170, 204
(1/2, 1/4)	23, 27, 29, 43, 46, 58, 77, 78, 126, 142, 172, 178, 184, 232
(1/2, 1/3)	30, 45, 54, 57, 60, 90, 106, 108, 154, 156
(1/2, 1/2)	105, 150

Table 1: Activity (λ) and sensitivity (μ) parameters, and independent elementary CA rules with the given values of λ and μ.

To clarify the diagram, we eliminate from it all nonquiescent rules and a few exceptional rules, as follows.

- *Chaotic rules with low μ (18, 126)*: according to [2], as both rules have $f(000) = f(111) = 0$, their return maps are single-humped and they tend to be chaotic. Indeed, simulations show the early formation of disordered triangular patterns, characteristic of Class C.
- *Null rule with high λ (40)*: for this rule $f(011) = f(101) = 1$. It is seen that initial conditions of the form $(011)_n$ shift, while all others erode to 0.
- *Fixed-point rules with low λ (2, 4)*: in the first case, the only surviving bits at $t = 1$ come from 001 strings in the initial condition; these shift to the left thereafter. In the second case, only 010 in the initial condition survives and remains.
- *Fixed-point rule with high μ (104)*: $f(011) = f(101) = f(110) = 1$. Strings 0110 in the initial condition are invariant; all other strings eventually evolve to zeros, leaving stripes of width 2—hence Class F.
- *Periodic rule with high μ (134)*: $f(001) = f(010) = f(111) = 1$. Strings of consecutive 1s followed by a 0 in the initial condition are eroded from the right by 0s. The remaining 1s oscillate between being isolated or in groups of two, producing patterns that shift to the left with speed 1/2.

The remaining independent rules are shown schematically in an activity–sensitivity diagram in Figure 2. The segregation of rules by class is quite satisfactory. As in Figure 1, near $\lambda = 1/2$ the sensitivity parameter appears

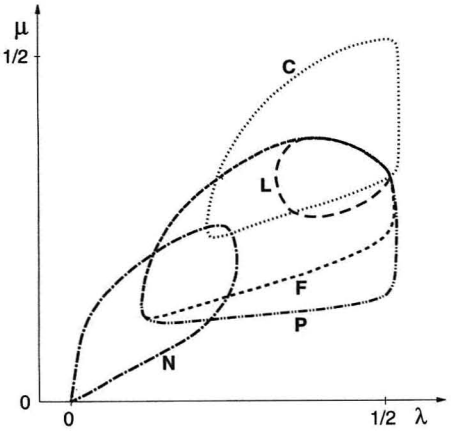

Figure 1: Phase diagram for *all* independent elementary CAs. The axes are activity (λ) and sensitivity (μ). Classes are as defined by Li and Packard [2]. We have included rules which do not have a quiescent state, in violation of the usual definition of λ.

to be quite useful in discriminating between classes, especially between N, F, and C. The domain for Class L is not shown, for clarity.

Two more diagrams, not shown here, were considered. Legal rules (quiescent with left–right symmetry) did not yield additional insights. A diagram indicating the domains of Wolfram Classes shows them segregating very nicely. The domain of Class 1 corresponds to Class N in Figure 1, Class 3 corresponds to C, and Class 2 corresponds roughly to the union of F and P. There are overlapping regions between Classes 1 and 2 and between 2 and 3, and only one "triple point" at $\lambda = 1/2$, $\mu = 1/4$, which can be explained in terms of the exceptional rules discussed above.

In conclusion, the phase diagrams in this Section confirm, to a great extent, our expectations, based on numerical observations (see, for example, Figure 1 in [2], and Figure 13 in [10]).

5. Discussion

The approximate prediction and ordering of rule behavior in terms of simple properties of the STT has been an important open problem in the theory of CAs. Previous studies have considered either the clustering of rules in a high-dimensional space, or one-parameter transitions from order to chaos. (We note that the usefulness of λ recently has been called into question [16].)

The present study is an extension and complement to previous studies; we have introduced a new parameter (μ) which estimates the average sensitivity of rules to small configurational changes. We have shown that in the space of the activity [8] and sensitivity parameters, rules cluster very nicely according to either their Wolfram Class [1] or their Li-Packard Class [2]. We have

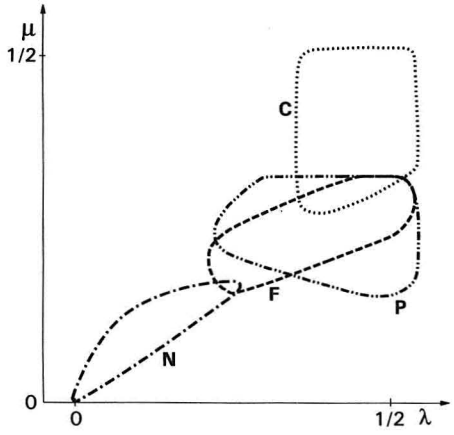

Figure 2: Revised phase diagram of rules for elementary CAs, after nonquiescent and exceptional rules have been removed. L rules are not shown, for simplicity.

considered only elementary CAs in this paper; we should point out that a similar phase diagram for totalistic $k = 2$, $r = 2$ rules [15] explains the numerical observation that both first- and second-order transitions to chaos can occur over the same range of the activity parameter.

Given the profound complexity of CAs, it is clear that such simple parameters as those used in this paper are of limited value. This is especially true for elementary rules, which are known to be highly correlated. For example, many features of the evolution of particular initial conditions are decided over the first few time steps; this cannot be taken into account by a priori parameters.

We expect that the two-parameter phase diagrams will be sharper in higher dimensions, and that sensitivity studies of the order-to-chaos transition will help in elucidating the fuzziness that is observed when λ is used.

Finally, we realize that μ is a very crude parameter, which still allows some overlap between classes. One possible improvement would be to give different weights to sensitivity to changes in the central versus the peripheral bits, or somehow to separate the effects of information generation and transmission. A more phenomenological approach, which loses the predictive aspect of a priori parameters, would be to modify any of the proposed versions of the μ parameter to reflect the numerically observed frequency of neighborhoods (e.g., as in Packard's γ).

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