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Complex Chaotic Behavior of a Class of Subshift Cellular Automata

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Abstract. A class of parameterized boolean, one-dimensional, bi-infinite cellular automata has been studied and their behavior observed when some parameters of the local function are changed. These automata are equivalent to a particular class of boolean neural networks and the change in the parameters corresponds to a change in the symmetricity of the connection matrix. The purpose is to analyze the different dynamics, beginning with a symmetric connection matrix and moving toward an antisymmetric one. We have observed that simple dynamics corresponds to the symmetric situation, whereas the antisymmetrical case yields more complex behavior. On the basis of these observations, we have identified a new class of cellular automata that is characterized by shiftlike dynamics (simple and complex subshift rules); these cellular automata correspond to the asymmetric situations and they are chaotic dynamical systems.

1. Introduction

Cellular automata (CAs) are dynamical systems with discrete space and time. Discreteness of space means that there is a d-dimensional lattice having a discrete variable that describes the state of each site on the lattice. Discreteness of time means that the state of each site changes at successive steps according to a function of the "neighboring" sites. Moreover, this process happens synchronously for every site on the lattice. Wolfram gave a classification of CAs based on some experimental observations about their dynamical behavior, that is, on the kind of structure emerging from their evolution [22]. In

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the same line, we define here a new class of CAs based on a different kind of observation; our starting point is the analysis of the chaotic properties of the shift map.

The automata we consider are equivalent to a class of bi-infinite neural networks with some restrictions on the connection matrix: every neuron is connected only with itself, with its predecessor, and with its successor in a linear arrangement. Furthermore, the weights of the forward, backward, and self-connection are identical for every neuron, and correspond to three parameters a, b, and c that each characterize a particular net in the class. By analyzing the behavior of the associated CAs as the three parameters are changed, we have identified and studied five different kinds of dynamics. The most interesting of these dynamics consists of CAs that have a shift-like behavior; we have discovered that these rules correspond to subshift of finite type [3, 18]. In particular, we have detected simple and complex subshift of finite type, in which simple subshifts are CA rules that behave like the shift on a closed, invariant subset of configurations Σ_0, while complex subshifts also have a closed and invariant subset Σ_1 or Σ_2 on which they show a more complex shifting behavior (alternating shift or double alternating shift, respectively). The most interesting property of these rules is their chaotic nature. In fact, we have shown that simple subshift rules are chaotic on the set Σ_0, and that alternating and double alternating subshift rules are chaotic on Σ_1 or Σ_2.

In the next section the basic definitions of CAs and the classification of Wolfram are given. In section 3 we introduce a class of bi-infinite neural networks and show how to associate a CA to each net. Section 4 gives the structure of the rule space as the parameters of the networks are changed. In sections 5 and 6 we analyze in detail the simple and complex subshift rules, and finally, in section 7, we draw some conclusions.

2. Basic definitions

In this paper we shall consider only bi-infinite, one-dimensional ($d = 1$) CAs. This means that the sites can be thought of as placed on a straight line.

Definition 2.1. *An infinite, one-dimensional CA is a structure*

$$\mathcal{C} = \langle \mathbf{Z}, G, r, h \rangle$$

where

$\mathbf{Z} = \{\dots, -i, \dots, 0, \dots, i, \dots\}$ *is the set of cells, where* $i \in \mathbf{Z}$ *is the location of cell* i*;*

$G = \{0, 1, \dots, k-1\}$ *is the set of possible states of the cells;*

$r \in \mathbf{N}$ *is the radius of the neighborhood;*

$h : G^{2r+1} \to G$ *is the local function, also called the rule of the automaton.*

Definition 2.2. *A configuration of a CA is a function that specifies a state for each site of the lattice*

$$\underline{x} : \mathbf{Z} \to G$$

and can be represented by a bi-infinite sequence

$$\underline{x} = (\ldots, x_{-m}, x_{-m+1}, \ldots, x_{-1}, x_0, x_{+1}, \ldots, x_{m-1}, x_m, \ldots).$$

In particular, for sake of simplicity, in the sequel we denote by $\rangle x_1, \ldots, x_n \langle$ the periodic configuration $\underline{x} = (\ldots, x_1, \ldots, x_n \mid x_1, \ldots, x_n, \ldots)$ and by $\langle \ldots (x_1, \ldots, x_m) \mid (y_1, \ldots, y_n) \ldots \rangle$ the configuration $\underline{x} = (\ldots, x_1, \ldots, x_m, x_1, \ldots, x_m \mid y_1, \ldots, y_n, y_1, \ldots, y_n, \ldots)$. Thus, the configuration space of the CA is $G^{\mathbf{Z}}$, and the *neighborhood* of a site $i \in \mathbf{Z}$ is the vector

$$(i - r, i - r + 1, \ldots, i - 1, i, i + 1, \ldots, i + r - 1, i + r),$$

that is, the r sites to the left and r sites to the right of i (plus i itself).

The *global function* of the CA

$$\underline{g} : G^{\mathbf{Z}} \to G^{\mathbf{Z}}$$

associates with any configuration $\underline{x} \in G^{\mathbf{Z}}$ the configuration at the next time step:

$$\underline{g}(\underline{x}) = (\ldots, g_{i-1}(\underline{x}), g_i(\underline{x}), g_{i+1}(\underline{x}), \ldots) \in G^{\mathbf{Z}},$$

where $\forall i \in \mathbf{Z}$, its ith component $g_i : G^{\mathbf{Z}} \to G$ specifies the next state of site i according to the rule

$$g_i(\underline{x}) := h(x_{i-r}, x_{i-r+1}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+r-1}, x_{i+r}) \qquad \forall i \in \mathbf{Z}.$$

On CAs the global function $\underline{g}$ gives rise to a *discrete time dynamical system* (DTDS) on the phase space $G^{\mathbf{Z}}$: for every configuration $\underline{x} \in G^{\mathbf{Z}}$ the *positive motion* (or *orbit*) of initial configuration $\underline{x}$ is the mapping

$$\gamma_{\underline{x}} : \mathbf{N} \to G^{\mathbf{Z}},$$

which associates to any $t \in \mathbf{N}$ the configuration at time t, $\gamma_{\underline{x}}(t) := \underline{g}^t(\underline{x}) \in G^{\mathbf{Z}}$. The positive motion of initial state $\underline{x}$ can also be written in sequential notation as

$$\gamma_{\underline{x}} := (\underline{x}, \underline{g}(\underline{x}), \underline{g}^2(\underline{x}), \ldots, \underline{g}^t(\underline{x}), \ldots).$$

Positive motion $\gamma_{\underline{x}}$ is a solution of the difference equation $\underline{x}(t + 1) = \underline{g}(\underline{x}(t))$ with initial condition $\underline{x}(0) = \underline{x}$.

We denote by $\mathrm{Per}_n(\underline{g})$ the set of all the cyclic points of period n, and $\mathrm{Per}(\underline{g}) = \bigcup_n \mathrm{Per}_n(\underline{g})$.

By *elementary CA* we mean the case in which $G = \{0, 1\}$ and $r = 1$. We denote by Σ the configuration space $\{0, 1\}^{\mathbf{Z}}$, and we consider the Tychonoff metric on the space Σ (i.e, $\forall x, y \in \Sigma$) to be

$$d(x, y) = \sum_{i=-\infty}^{\infty} \frac{1}{4^{|i|}} |x_i - y_i|.$$

For boolean CAs of radius r we give the following definitions.

Definition 2.3. *A CA rule $h : \{0,1\}^{2r+1} \to \{0,1\}$ is a shift if, $\forall i \in \mathbf{Z}$, $h(x_{i-r}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+r}) = x_{i+1}$. Trivially the corresponding global function $\underline{g}$ is the shift on the configuration space*

$$\begin{aligned}\underline{g}(\underline{x}) &= \sigma(\ldots, x_{-m}, \ldots, x_{-1}, x_0 \mid x_1, \ldots, x_m, \ldots) \\ &:= (\ldots, x_{-m}, \ldots, x_{-1}, x_0 x_1 \mid x_2, \ldots, x_m, \ldots)\end{aligned}$$

Notice that the component mapping of cell i is

$$g_i(\underline{x}) = \sigma_i(\underline{x}) = x_{i+1}$$

In the elementary case the shift rule is unique and corresponds to rule h_{170}, according to the Wolfram classification scheme [22].

The mapping σ is a homeomorphism of Σ; in other words, both σ and $\sigma_R := \sigma^{-1}$ (the *right shift*) are continuous with respect to the metric introduced above. The shift map is a prototypical chaotic dynamical system. Following the classical definition of *deterministic chaos* [7] and some recent results [1], the essential features of chaos for a continuous next state function $\underline{g}$ on some metric state space are *transitivity*, meaning that for all nonempty open subsets U and V of the state space there exists a natural number k such that $f^k(U) \cap V$ is not empty; and *regularity*, meaning the set of periodic points of $\underline{g}$ is a dense subset of the state space. As proved in [1], these two conditions imply the sensitive dependence on initial conditions, which is the main feature of chaotic behavior in dynamical systems.

Definition 2.4. *A CA rule $h : \{0,1\}^{2r+1} \to \{0,1\}$ is an anti-shift if, $\forall i \in \mathbf{Z}$, $h(x_{i-r}, \ldots, x_i, \ldots, x_{i+r}) = \overline{x_{i+1}}$. The associated global function $\underline{g}$ is the anti-shift on the configuration space*

$$\underline{g}(\underline{x}) = \overline{\sigma(\underline{x})}.$$

The anti-shift is a homeomorphism of Σ whose inverse is the right anti-shift $\overline{\sigma_R(\underline{x})}$. Of course, the complex conjugate does not affect the chaotic properties of the shift map. Therefore the anti-shift dynamics is also chaotic.

Definition 2.5. *A CA rule h is a subshift rule if the corresponding global function $\underline{g}$ is such that there exists a closed, invariant subset Σ_0 of Σ (i.e., $g(\Sigma_0) \subseteq \Sigma_0$) on which it is the shift, so $\forall \underline{x} \in \Sigma_0$, $\underline{g}(\underline{x}) = \sigma(\underline{x})$.*

In order to describe the behavior of the particular class of CAs considered in this work, let us give the following definitions.

Definition 2.6. *A CA rule h is an alternating right subshift rule if the corresponding global function $\underline{g}$ is such that there exists a closed, invariant subset Σ_1 of Σ on which it is the alternating right shift, so $\forall \underline{x} \in \Sigma_1$, $\underline{g}^2(\underline{x}) = \sigma_R(\underline{x})$.*

In this case, the dynamics of the automata, for every initial state $\underline{x} \in \Sigma_1$, has the form

$$\gamma_{\underline{x}} = (\underline{x}, g(\underline{x}), \sigma_R(\underline{x}), \sigma_R(\underline{g}(\underline{x})), \sigma_R^2(\underline{x}), \sigma_R^2(\underline{g}(\underline{x})), \ldots).$$

The following sufficient condition is easy to prove.

Proposition 1. *If a CA rule h, with associated global function* $\underline{g}$*, is such that there exists an invariant subset* Σ_1 *of* Σ *such that,* $\forall \underline{x} \in \Sigma_1$*,* $\underline{g}(\underline{x}) = \overline{\underline{x} \vee \sigma_R(\underline{x})}$ *(i.e.,* $\forall i$*,* $h(x_{i-r}, \ldots, x_i, \ldots, x_{i+r}) = \overline{x_i \vee x_{i-1}}$*) and* $x \wedge \sigma_R^2(\underline{x}) \leq \sigma_R(\underline{x})$ *(i.e.,* $\forall i$*,* $x_i \wedge x_{i-2} \leq x_{i-1}$*), then h is an alternating right subshift.*

Definition 2.7. *A CA rule h is a right anti-subshift rule if the corresponding global function* $\underline{g}$ *is such that there exists a closed, invariant subset* Σ_2 *of* Σ *(i.e.,* $g(\Sigma_2) \subseteq \Sigma_2$*) on which it is the right anti-shift, so* $\forall \underline{x} \in \Sigma_2$*,* $\underline{g}(\underline{x}) = \overline{\sigma_R(\underline{x})}$*.*

In this case, for every $\underline{x} \in \Sigma_2$

$$\gamma_{\underline{x}} = (\underline{x}, \overline{\sigma(\underline{x})}, \sigma^2(\underline{x}), \sigma^2(\overline{\sigma(\underline{x})}), \sigma^4(\underline{x}), \sigma^4(\overline{\sigma(\underline{x})}), \ldots).$$

Every anti-shift rule gives rise to a *double alternate shift* dynamic on Σ_2, so $\forall \underline{x} \in \Sigma_2$, $\underline{g}^2(\underline{x}) = \sigma^2(\underline{x})$.

2.1 Wolfram classification of finite cellular automata

In the case of one-dimensional CAs consisting of a finite number of cells, there are many attempts of classification according to their asymptotic behavior [6, 9, 13, 17, 22]. Wolfram has studied CAs extensively and has suggested the following classification [22]:

Class 1: automata that evolve to a unique homogeneous state after a finite transient (static CAs).

Class 2: automata whose evolution leads to a set of separated and simple stable or periodic structures (space-time patterns) (periodic CAs).

Class 3: automata whose evolution leads to aperiodic ("chaotic") space-time patterns (chaotic CAs).

Class 4: automata that evolve into complex patterns that have propagative localized structures, sometimes long-lived (complex CAs).

CAs belonging to the first two classes are the simplest; beginning with any initial configuration they show a simple, periodic behavior. Wolfram has also shown that the set of configurations obtained in the infinite time limit corresponds to a regular language. These systems have a low dependence on initial conditions and a low degree of disorder, which is easy to see from the values assumed by the entropy and propagation speed [22].

Class 3 contains automata that evolve into "chaotic" patterns. Wolfram calls these automata "chaotic" because they show disordered (i.e., nonperiodic) space-time patterns. These automata have a nonzero propagation speed and a nonzero entropy that decreases for a few time steps and reaches an equilibrium value.

Class 4 contains "complex automata," characterized by the presence of propagation structures. These automata show periodic behavior with some initial configurations, whereas with other configurations they appear to evolve into "chaotic patterns." There is no way to understand their dynamics without observing their evolution.

For most Class 3 and 4 CAs, the limit set of configurations contains more complex languages. It seems that CAs belonging to Class 4 are capable of universal computation. Until now, the equivalence between this computational property and the various statistical characterizations of the Class 4 rules has remained an open question, but many observations confirm this hypothesis.

Some examples of elementary CAs belonging to the first three Wolfram classes are shown in the appendix (see Figure A1). None of the elementary CAs belongs to Class 4.

When dealing with finite CAs, it is necessary to introduce boundary conditions to compute the values of the first and the last cells of the lattice. Usually circular CAs or CAs in backgrounds of 0s are considered. On the contrary, in this paper we deal with a particular class of bi-infinite CAs in order to avoid some problems related to these situations.

2.2 The rule space for elementary cellular automata

A CA rule can be represented by a lookup table. For elementary CA rules the rule table is the Boolean vector of length $8 = 2^3$.

Block	Transition
000	f_{000}
001	f_{001}
010	f_{010}
011	f_{011}
100	f_{100}
101	f_{101}
110	f_{110}
111	f_{111}

The *rule space* of elementary CAs can be defined as the pair $\langle R, d_h \rangle$, where $R = \{0,1\}^8$ is the set of 256 boolean vectors of length 8 and d_h is the Hamming distance.

It is interesting to study the differences in the behavior of the CAs as they move across the rule space; in fact, it is possible to detect different regions in this space that contain automata with different characteristics, and to observe the change in the global behavior when moving along a path in the rule space. To help explore the rule space, a parameter λ (which describes the density of 1s in the rules) has been introduced [11]. Varying this parameter from 0 to 0.5, it seems that CA behavior changes from a simple dynamics to a more complex one, and increasing the parameter the behavior passes from a complex to a simple one. In general, it has been observed that CA dynamics go from order to disorder, passing through an intermediate complex behavior (characteristic of Class 4). Packard's study of the rule space for elementary CAs has shown in which regions the Wolfram classes are located [16]; moreover, the connections between rules in the same class and among the different classes are described. There are other techniques for analyzing the rule space, such as Packard's use of mean-field clusters to organize changes in CA dynamics into rules. In this paper we study the rule space of a class of elementary CAs according to yet another criterion.

In the next section we introduce the particular model of neural network that we want to study and show that it is equivalent to a CA model.

3. A class of bi-infinite neural networks

Definition 3.1. *A bi-infinite neural network can be defined as a structure*

$$\mathcal{R} = \langle \mathbf{Z}, G, W, \underline{\tau}, \{f_i : i \in \mathbf{Z}\}\rangle$$

where

$\mathbf{Z} = \{\dots - i, \dots, 0, \dots, i, \dots\}$ *is the set of neurons;*

$G = \{0, 1, \dots, k-1\}$ *is the set of states of the neurons;*

$W = (w_{ij})_{i,j\in\mathbf{Z}}$, $w_{ij} \in \mathbf{R}$, *is the bi-infinite connection matrix, satisfying the condition* $\forall \underline{x} = (x_i)_{i\in\mathbf{Z}} \in G^{\mathbf{Z}}$ *and* $\forall i \in \mathbf{Z}$, $\sum_j w_{ij}x_j$ *is convergent;*

$\underline{\tau} \in \mathbf{R}^{\mathbf{Z}}$ *is the threshold vector; and*

$f_i : \mathbf{R} \to G$ *is the activation function of neuron* i.

The network has a bi-infinite number of neurons. Its global activation function is the mapping $g : G^{\mathbf{Z}} \to G^{\mathbf{Z}}$ whose component functions $g_i : G^{\mathbf{Z}} \to G$ are defined as

$$g_i(\underline{x}) = f_i\left(\sum_j w_{ij}x_j - \tau_i\right) \qquad \forall i \in \mathbf{Z}$$

We consider the particular class of binary neural networks (i.e., $G = \{0, 1\}$) whose activation functions are

$$g_i(\underline{x}) := \mathrm{HS}\left(ax_{i-1} + bx_i + cx_{i+1}\right).$$

where the Heavyside function HS is defined as

$$\mathrm{HS}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

y_{-1}	y_0	y_{+1}	$h(y_{-1}, y_0, y_{+1})$
0	0	0	1
0	0	1	HS(c)
0	1	0	HS(b)
0	1	1	HS($b + c$)
1	0	0	HS(a)
1	0	1	HS($a + c$)
1	1	0	HS($a + b$)
1	1	1	HS($a + b + c$)

Table 1: Lookup table for a one-dimensional bi-infinite CA.

In this case, the associated connection matrix has the form

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & b & c & 0 & 0 & 0 & 0 & \dots \\ \dots & a & b & c & 0 & 0 & 0 & \dots \\ \dots & 0 & a & b & c & 0 & 0 & \dots \\ \dots & 0 & 0 & a & b & c & 0 & \dots \\ \dots & 0 & 0 & 0 & a & b & c & \dots \\ \dots & 0 & 0 & 0 & 0 & a & b & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This network has a bi-infinite number of neurons, and each neuron changes its state on the basis of its two adjacent neurons according to a homogeneous activation function. It is easy to see that this net is equivalent to a one-dimensional, bi-infinite CA in which every site evolves according to a rule whose lookup table is shown in Table 1.

Our goal is to observe the behavior of this class of networks by varying the three values a, b, and c. In particular, we are interested in studying the different dynamics beginning with a symmetric weight matrix and moving toward an asymmetric one until the antisymmetric situation is reached. We have studied the subset of the rule space obtained by changing the parameters a, b, and c, trying to detect regions that have unique behaviors.

At first we analyze the case $b < 0$ by varying the three parameters, which yields the three subcases shown in Table 2. Table 2 shows which CA rules correspond to the different values of parameters a, b, and c. In parentheses there are the rule numbers and the number of the smallest equivalent rule under the simple transformations conjugacy and reflection [22]. In the following discussions, for each choice of the parameters we always consider the smallest equivalent rule.

In the next section we describe the structure of the rule space when the parameters are varied.

1. $a > 0$ and $c > 0$

$c > -b$ and $a < -b$	Rule: 10111011 (Rule 187) (eq. 34)
$c > -b$ and $a > -b$	Rule: 11111011 (Rule 251) (eq. 32)
$-(a+b) < c < -b$ and $a < -b$	Rule: 10110011 (Rule 179) (eq. 50)
$-(a+b) < c < -b$ and $a > -b$	Rule: 11110011 (Rule 243) (eq. 34)
$-(a+b) < c$ and $a < -b$	Rule: 00110011 (Rule 51)

2. $a > 0$ and $c < 0$

$c > -a$ and $a < -b$	Rule: 00110001 (Rule 49) (eq. 35)
$-a < c < -(a+b)$ and $a > -b$	Rule: 01110001 (Rule 113) (eq. 43)
$c > -(a+b)$ and $a > -b$	Rule: 11110001 (Rule 241) (eq. 42)
$c < -a$ and $a < -b$	Rule: 00010001 (Rule 17) (eq. 3)
$c < -a$ and $a > -b$	Rule: 01010001 (Rule 81) (eq. 11)

3. $a < 0$ and $c > 0$

$c < -a$ and $c < -b$	Rule: 00000011 (Rule 3)
$c > -a$ and $c < -b$	Rule: 00100011 (Rule 35)
$-b < c < -a$	Rule: 00000011 (Rule 11)
$-a < c < -(a+b)$ and $c > -b$	Rule: 00100011 (Rule 43)
$c > -(a+b)$ and $c > -b$	Rule: 10100011 (Rule 171) (eq. 42)

Table 2: Three subcases for $b < 0$.

4. Structure of the rule space

We have detected five kinds of dynamics:

1. Attracting fixed-point dynamics with a unique isolated cycle of order 2
2. Attracting periodic dynamics
3. Simple subshift rules
4. Complex subshift rules consisting of simple subshift and alternating right subshift
5. Complex subshift rules consisting of simple subshift and double alternating right subshift

Figure 1 shows which rules are obtained by varying parameters a and c, and considering $b < 0$. The y axis represents parameter c, and the x axis represents parameter a. The different regions in the graph correspond to particular values of the three parameters. For example, Rule 43 corresponds to the two choices

$$a > 0,\ b < 0,\ c < 0, \qquad -a < c < -(a+b), \qquad \text{and} \qquad a > -b$$

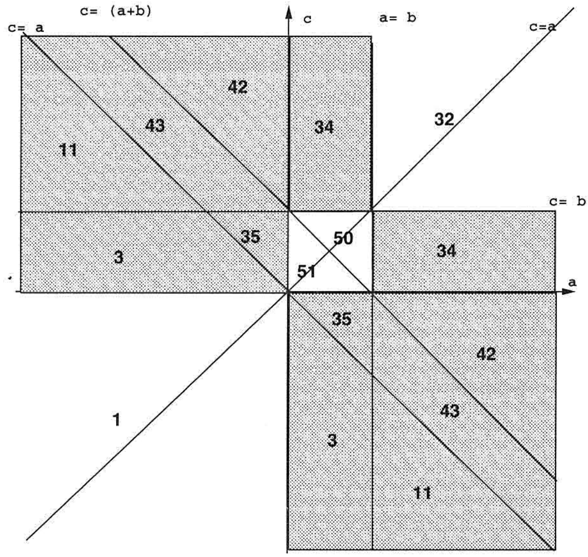

Figure 1: Rule space for $b < 0$.

or

$$a < 0,\ b < 0,\ c < 0, \qquad -a < c < -(a+b), \qquad \text{and} \qquad c > -b.$$

Neural networks with symmetric matrices ($a = c$) correspond to simple CA rules that belong to Classes 1 (h_{32}) and 2 (h_1, h_{50}, h_{51}). For every initial configuration they evolve to a fixed point or to a cyclic state of period two. For a rule in Class 1 there are a unique attracting fixed-point cycle (the null configuration) and a unique isolated cycle (the configurations $\rangle 01 \langle$ and $\rangle 10 \langle$). Moreover, the basin of attraction of the null configuration is the entire phase space minus the two points of the cycle, while for rules in Class 2 the cycles are not isolated. The remaining rule space contains subshift rules ($h_{34}, h_{42}, h_3, h_{35}, h_{11}, h_{43}$).

Rules h_{34} and h_{42} are subshift rules in which the subset Σ_0 (see Definition 2.5) is an attractor. Beginning with any initial configuration, the evolution leads the automata to Σ_0 after one time step (Figure 2).

The other rules ($h_3, h_{35}, h_{11}, h_{43}$), which correspond to the antisymmetrical situation, are more complex in the sense that they divide the configuration space into three subsets: one in which they act as the simple shift rule (Σ_0), one in which they show a more complex shifting behavior (Σ_1 or Σ_2), and one ($\tilde{\Sigma} = \Sigma \setminus (\Sigma_0 \cup \Sigma_i)$) that is not attracted by Σ_0 or Σ_i, and in which $\tilde{\Sigma}$ can be further divided in more complex subsets (Figure 3). Table 3 summarizes the characteristics of the rules for the case $b < 0$.

In the next section the characteristics of the simple subshift rules are analyzed in detail.

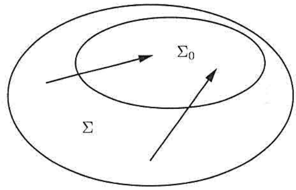

Figure 2: Structure of Σ for simple subshift rules.

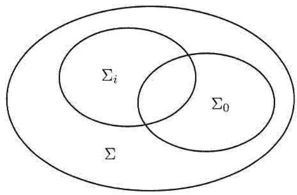

Figure 3: Structure of Σ for complex subshift rules.

Rule	Dynamics
32	Unique attracting fixed point and unique isolated periodic point
1	Periodic
50	Periodic
51	Periodic
34	Simple subshift
42	Simple subshift
3	Simple subshift and alternating right subshift
35	Simple subshift and alternating right subshift
11	Simple subshift and double, alternating right subshift
43	Simple subshift and double, alternating right subshift

Table 3: Rule characteristics for $b < 0$.

5. Subshift rules

A subshift rule can be characterized by the closed, invariant subset Σ_0 in which it behaves like a shift. From another point of view, it can also be characterized by the list of forbidden blocks, that is, the blocks that must not appear in the initial configuration (and thus, by the invariance of Σ_0, in the entire dynamical evolution). It is easy to characterize the subset Σ_0 introduced in Definition 2.5, taking into account definitions of g_i and σ_i. In fact, a configuration $\underline{x} \in \Sigma_0$ iff

$$g_i(\underline{x}) = h(x_{i-r}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+r}) = \sigma_i(\underline{x}) = x_{i+1} \qquad \forall i \in \mathbf{Z}.$$

If we call *admissible* the blocks $(y_{-r}, \ldots, y_0, \ldots, y_{+r}) \in \{0,1\}^{2r+1}$ such that

$$h(y_{-r}, \ldots, y_{-1}, y_0, y_{+1}, \ldots, y_r) = y_{+1},$$

y_{-1}	y_0	y_{+1}	$h_{34}(y_{-1}, y_{+0}, y_{+1})$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

Table 4: Transition table for elementary rule 34.

and *forbidden* the blocks such that

$$h(y_{-r}, \ldots, y_{-1}, y_0, y_{+1}, \ldots, y_r) \neq y_{+1},$$

then the following lemma is easy to prove.

Lemma 1. *A configuration $\underline{x} \in \Sigma_0$ iff $\forall i$, $(x_{i-r} \ldots, x_{i-1}, x_i, x_{i+1}, \ldots x_{i+r})$ is admissible, that is, iff $\underline{x}$ is made (only) of admissible blocks.*

The admissible and forbidden blocks of a CA rule can be quickly found from the transition table of the local rule. For instance, given the transition table (Table 4) for elementary rule 34 we see that the forbidden blocks are $(0, 1, 1)$ and $(1, 1, 1)$. The subset on which rule 34 is a subshift is the set of configurations that do not contain these blocks. It turns out that this is the set of configurations that contain only isolated 1s.

Theorem 1. *The subset Σ_0 is closed and strongly invariant (i.e., $\underline{g}(\Sigma_0) = \Sigma_0$).*

Proof. Let $\underline{x} \in \Sigma \setminus \Sigma_0$. Then, by Lemma 1, there exists in $\underline{x}$ a forbidden block, centered in site i_0 $(x_{i_0-r}, \ldots, x_{i_0}, \ldots, x_{i_0+r})$. Now, let $m_0 \in \mathbf{N}$ be such that $i_0 + r < m_0$ and $m_0 > r - i_0$. Then the open sphere $B_{1/4^{m_0}}(\underline{x})$ consists of all the configurations $\underline{y} \in \Sigma$ such that

$$y_{-m_0} = x_{-m_0}, \ldots, y_0 = x_0, \ldots, y_{m_0} = x_{m_0}.$$

Thus, they also contain the forbidden block $(x_{i_0-r}, \ldots, x_{i_0}, \ldots, x_{i_0+r})$. According to Lemma 1, $\underline{x} \in B_{1/4^{m_0}}(\underline{x}) \subseteq \Sigma \setminus \Sigma_0$, concluding that $\Sigma \setminus \Sigma_0$ is open and consequently Σ_0 is closed.

To show that Σ_0 is strongly invariant, we have to prove that $\underline{g}(\Sigma_0) = \Sigma_0$. First we show that $\underline{g}(\Sigma_0) \subseteq \Sigma_0$. Suppose that $\underline{x} \in \Sigma_0$, that is, that it is made of admissible blocks. Then, since $\forall i \in \mathbf{Z}$

$$\begin{aligned}(g_{i-r}(\underline{x}), \ldots, g_i(\underline{x}), \ldots, g_{i+r}(\underline{x})) &= (\sigma_{i-r}(\underline{x}), \ldots, \sigma_i(\underline{x}), \ldots, \sigma_{i+r}(\underline{x})) \\ &= (x_{i-r+1}, \ldots, x_{i+1}, \ldots, x_{i+r+1})\end{aligned}$$

we find that $\underline{g}(\underline{x})$ is made of admissible blocks also, or $\underline{g}(\underline{x}) \in \Sigma_0$.

Now, we show that $\underline{g}(\Sigma_0) \supseteq \Sigma_0$. Since

$$(\sigma_{i-r}^{-1}(\underline{x}), \ldots, \sigma_i^{-1}(\underline{x}), \ldots, \sigma_{i+r}^{-1}(\underline{x})) = (x_{i-r-1}, \ldots, x_{i-1}, \ldots, x_{i+r-1}) \quad \forall i \in \mathbf{Z}$$

$\sigma^{-1}(\underline{x})$ is made of admissible blocks. Then $\underline{g}$ is a shift on $\sigma^{-1}(\underline{x})$ and

$$\underline{g}\left(\sigma^{-1}(\underline{x})\right) = \sigma\left(\sigma^{-1}(\underline{x})\right) = \underline{x}. \blacksquare$$

An important question that arises is whether, given a set of admissible blocks, the set Σ_0 is not empty. It is possible to give a necessary and sufficient condition to determine whether the set Σ_0 of a CA rule is not empty.

Proposition 2. *The set Σ_0 of a CA rule with n admissible blocks is not empty iff it is possible to find a segment of length $n+1+2r$ (i.e., an element of $\{0,1\}^{n+1+2r}$) made of admissible blocks (where r is the radius of the CA).*

Proof. First we note that if there are no admissible blocks then Σ_0 must be empty. Otherwise, let n be the number of admissible blocks. Every segment of length $n+1+2r$ in any configuration $\underline{x} \in \Sigma_0$ is made of admissible blocks. In particular, there is at least one segment of length $n+1+2r$ made of admissible blocks. This condition is also sufficient. In fact, suppose that we have a segment of length $n+1+2r$ made of admissible blocks

$$(x_{1-r}, \ldots, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+1+r}).$$

This segment contains $n+1$ admissible blocks; however, because there are only n different blocks, two of them must be equal. Suppose they are $(x_{i-r}, \ldots, x_i, \ldots, x_{i+r})$ and $(x_{j-r}, \ldots, x_j, \ldots, x_{j+r})$, with $i < j$. If we repeat the segment $(x_i, \ldots, x_{j-1})$ infinitely we obtain a configuration

$$\ldots, x_{i-r}, \ldots, x_i, \ldots, x_j = x_i, \ldots, x_{j+r} = x_{i+r}, \ldots, x_i, \ldots$$

that is made of admissible blocks. $\blacksquare$

Thus, to check whether Σ_0 is empty for a given CA rule, it is enough to make a list of all the segments of length $n+1+2r$, and then find out if there exists one segment made of admissible blocks.

Example. Rule 34 has 6 admissible blocks, so we need to write all segments of length 9. Among them, segment $(0,0,1,0,0,0,1,0,0)$ is made of admissible blocks, so we are certain that Σ_0 is not empty. Because the admissible block $(0,1,0)$ is repeated twice,

$$(0, \overbrace{0,1,0}^{i}, 0, \overbrace{0,1,0}^{j}, 0),$$

the admissible block between the ith and $(j-1)$st places is $(1,0,0,0)$; by repeating indefinitely we build a configuration of Σ_0:

$$\ldots,0,1,0,0,0,1,0,0,0,\ldots$$

The analysis of the dynamics of rule 34 can be completed by showing that every configuration falls into Σ_0 after one step, that is, the CA is definitely a shift. In fact, if we look at the transition table

x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	$g_{i-1}(\underline{x})$	$g_i(\underline{x})$	$g_{i+1}(\underline{x})$
0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	1
0	0	0	1	0	0	1	0
0	0	0	1	1	0	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	1	1	0	0	0	0
1	1	1	1	1	0	0	0

we see that the blocks that appear after one step are admissible. We now put together all the results about the subshift elementary rules.

Proposition 3. *Elementary rule 34 is a subshift over the (nonempty) set of configurations Σ_0 that do not contain the forbidden blocks $(0,1,1)$ and $(1,1,1)$. Every configuration in Σ goes to Σ_0 in one step.*

An analogous result holds for rule 42.

Proposition 4. *The elementary rule 42 is a subshift over the (nonempty) set of configurations Σ_0 that do not contain the forbidden block $(1,1,1)$. Every configuration in Σ goes to Σ_0 in one step.*

A subshift over an invariant subset Σ_0 can also be specified by giving two objects ([2, 3, 18]):

1. a positive integer n, and
2. an $n \times n$ matrix (M) with entries in $\{0,1\}$.

Using these two objects it is possible to build a finite set $\mathcal{A}$ with n elements, called the *alphabet* of the subshift, and a special subset Σ_0 of the bi-infinite sequences of elements of $\mathcal{A}$:

$$\Sigma_0 := \{\underline{x} = (\ldots, x_{-1}, x_0, x_{+1}, \ldots) : \\ \forall t \in \mathbf{Z},\ x_t \in \mathcal{A},\ \text{and}\ M(x_t, x_{t+1}) = 1\}.$$

Σ_0 consists of all the admissible sequences of elements of M, called the *transition matrix*, a sequence $\underline{x}$ being admissible if $M(x_t, x_{t+1}) = 1$ for each $t \in \mathbf{Z}$, that is, if the pair $x_t x_{t+1}$ may appear as adjacent symbols in the sequence. The subshift can also be described by giving the list of *forbidden blocks*, the

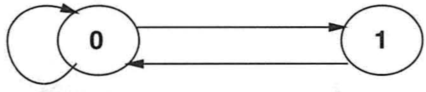

Figure 4: Subshift rule 34.

sequences of symbols that cannot appear consecutively in a configuration $\underline{x} \in \Sigma_0$. Another way of describing a subshift is by a directed graph (V, E) that has n vertices $v_1, \ldots, v_n$ and whose edges are described by matrix M.

It is easy to see that this kind of description corresponds to the other definitions given above that characterize the subshift rules over the subset Σ_0.

Definition 5.1. *A transition matrix M is called* irreducible *if, for every couple $0 \le i, j \le n$, there exists an integer $k = k(i,j) > 0$ such that $(M^k)_{ij} \neq 0$. This means that there is a sequence of edges from any vertex to any other vertex:*

$$\forall v_i, v_j \in V \, \exists k = k(i,j) \text{ such that } (v_i, v_{i+1}) \in E, \\ (v_{i+1}, v_{v_i+2}) \in E, \ldots, (v_{i+k}, v_j) \in E.$$

In [4] an algorithmic procedure is given in order to obtain in a reduced form the transition matrix associated with a one-dimensional CA rule. The matrix presented in the following are constructed according to this procedure.

Example. In the case of elementary rule 34, the transition matrix for $n = 2$ is the following:

$$M_{34} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can define the alphabet $\mathcal{A} = \{0, 1\}$, whose corresponding graph is shown in Figure 4. In this case Σ_0 contains only isolated 1s.

Example. For elementary rule 42 with $n = 4$, the alphabet has four symbols $\mathcal{A} = \{q_0, q_1, q_2, q_3\}$, where the symbols correspond to the strings $q_0 = 11$, $q_1 = 10$, $q_2 = 01$, and $q_3 = 00$. The corresponding graph is shown in Figure 5, and matrix M_{42} is

$$M_{42} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

In this case, the one-entry in the matrix between the symbol q_0 and q_2 means that in the admissible, bi-infinite configuration the string 11 can be

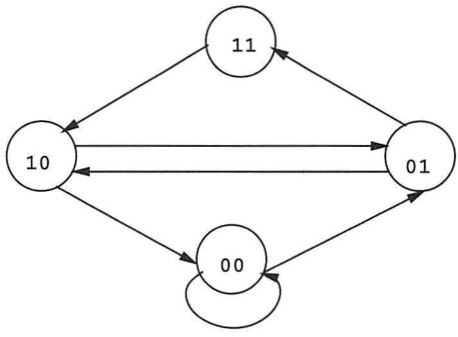

Figure 5: Subshift rule 42.

followed by the string 01. In the examples that follow we will denote by q_0, q_1, q_2, and q_3 the strings 11, 10, 01, and 00, respectively.

From Definition 5.1 it is easy to see that elementary rules 34 and 42 are described by an irreducible matrix. In [19] the following theorem and lemma are proved.

Theorem 2. *If the matrix M associated to a subshift over a subset Σ_0 is irreducible, then the subshift with domain Σ_0 has (1) a countable infinity of periodic orbits, (2) an uncountable infinity of nonperiodic orbits, and (3) a dense orbit.*

Lemma 2. *Suppose M is an irreducible matrix. Then, given any $a, b \in G$ there exist a $k = |G|$ and an admissible string $s_1, \ldots s_k$ such that $a, s_1, \ldots s_k, b$ is an admissible string of length $k+2$.*

On the basis of the previous observation and Lemma 2, we give the following theorem.

Theorem 3. The subshift rules with irreducible transition matrix M are chaotic over the closed, invariant subset Σ_0.

Proof. In order to prove that a subshift rule $\underline{g}$ is chaotic over a set Σ_0, we want to prove that, relative to Σ_0:

1. $\underline{g}$ is transitive, and
2. $\mathrm{Per}_{\Sigma_0}(\underline{g})$, the set of periodic points in Σ_0, is dense in Σ_0.

Since A is irreducible, it follows from the previous observations that the subshift has a dense orbit, and thus it is transitive. Now, we want to prove that $\forall \underline{x} \in \Sigma_0$, $\forall \epsilon$, $\exists y \in \mathrm{Per}_{\Sigma_0}(\underline{g})$ such that $y \in B_\epsilon(\underline{x})$.

Let us consider a fixed configuration $\underline{x} \in \Sigma_0$ ($\underline{x} = (\ldots, x_{-m}, \ldots, x_{-1} \mid x_0, x_1, \ldots x_m, \ldots)$). Then for every ϵ, we want to build a periodic configuration

$\underline{y} \in \Sigma_0$ that belongs to $B_\epsilon(\underline{x})$. For any ϵ, let us consider an m such that $\epsilon < 1/4^m$ and an admissible string $(s_1, \ldots, s_k)$ with the properties of Lemma 2. Then we build the periodic configuration

$$\underline{y} = (\ldots, s_1, \ldots s_k, y_{-m}, \ldots, y_{-1} \mid y_0, y_1, \ldots y_m, s_1, \ldots s_k,$$
$$y_{-m}, \ldots, y_{-1}, y_0, y_1, \ldots y_m, \ldots)$$

in such a way that $x_i = y_i$ for $-m < i < m$. In this way $\underline{y} \in B_\epsilon(\underline{x})$. Because (from Lemma 2) $y_m, s_1, \ldots s_k, y_{-m}$ is an admissible string, it follows that $\underline{y}$ is made only of admissible strings, so $\underline{y} \in \Sigma_0$. ∎

The structure of Σ for the simple subshift rules is shown in Figure 2. In the next section the characteristics of the complex subshift rules are analyzed in detail.

6. Complex subshift rules

In this section we analyze the complex subshift elementary CA rules $\underline{g}_3$, $\underline{g}_{35}$, $\underline{g}_{11}$, and $\underline{g}_{45}$. In order to study their dynamics we introduce the following definition.

Definition 6.1. *For a CA with local rule $h : \{0,1\}^{2r+1} \mapsto \{0,1\}$ we define $h^{[2]} : \{0,1\}^{4r+1} \to \{0,1\}$ in the following way:*

$$h^{[2]}(y_{-2r}, \ldots, y_0, \ldots, y_{2r}) =$$
$$h(h(y_{-2r}, \ldots, y_{-r}, \ldots, y_0), \ldots, h(y_0, \ldots, y_r \ldots, y_{2r}))$$

6.1 Simple subshift and alternating right subshift

Rules $\underline{g}_3$ and $\underline{g}_{35}$ are both simple and alternating subshift. Using the same technique used in the previous section, it is possible to prove the following proposition.

Proposition 5. *Rule 3 is a subshift over the (nonempty) set of configurations Σ_0 that do not contain the forbidden blocks* (000), (011), (101), *and* (111). *There is at least one configuration in Σ that is not attracted by Σ_0.*

Proof. We only prove the second part of the proposition because the first part can be easily proved in analogy with the cases previously discussed. We want to show that $\exists \underline{x} \in \Sigma$, $\lim_{t\to\infty} \underline{g}^t(\underline{x}) \notin \Sigma_0$. We can consider, for instance, the configuration $x = \langle \ldots (0) \mid \underline{(1)} \ldots \rangle$. It is easy to see that it is a periodic point, thus $\lim_{t\to\infty} \underline{g}^t(\underline{x}) \notin \Sigma_0$. ∎

We can also describe subshift rule 3 by means of its corresponding graph and transition matrix. For $n = 3$ we have $\mathcal{A} = \{q_1, q_2, q_3\}$. The transition matrix is

$$M_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

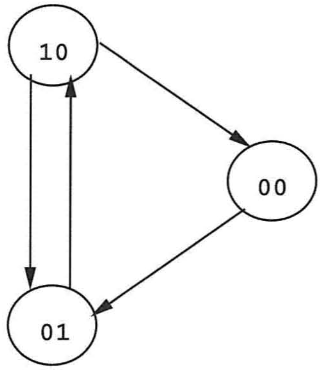

Figure 6: Subshift rule 35.

It is easy to see that M_3 is not irreducible.

The following proposition holds for rule 35.

Proposition 6. *Rule 35 is a subshift over the (nonempty) set of configurations* Σ_0 *that do not contain the forbidden blocks* (000), (011), *and* (111). *There is at least one configuration in* Σ *that never goes to* Σ_0.

Describing rule 35 by means of its corresponding graph and transition matrix, we have $n = 3$ and $\mathcal{A} = \{q_1, q_2, q_3\}$, and the transition matrix has the form

$$M_{35} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

M_{35} is irreducible and the corresponding graph is shown in Figure 6.

In analogy with Σ_0, it is easy to characterize the subset Σ_1 introduced in Definition 2.6:

$$\Sigma_1 = \{x \in \Sigma : \underline{g}^2(\underline{x}) = \sigma_R(\underline{x})\}.$$

In fact, a configuration $x \in \Sigma_1$ iff, $\forall i \in \mathbf{Z}$,

$$(g^2)_i(x) = h^{[2]}(x_{i-2r}, \ldots, x_i, \ldots, , x_{i+2r}) = (\sigma_R)_i(x) = x_{i-1}.$$

If we call *admissible* the blocks $(y_{-2r}, \ldots, , y_0, \ldots, y_{+2r}) \in \{0,1\}^{4r+1}$ such that

$$h^{[2]}(y_{-2r}, \ldots, y_0, \ldots, y_{+2r}) = y_{-1},$$

and *forbidden* the blocks such that

$$h^{[2]}(y_{-2r}, \ldots, y_0, \ldots, y_{+2r}) \neq y_{-1},$$

then the following lemma is easy to prove.

Lemma 3. *A configuration $\underline{x} \in \Sigma_1$ iff, $\forall i$,*

$$(x_{i-2r}, \ldots, x_i, \ldots, x_{i+2r}) \quad \text{and} \quad (g_{i-2r}(\underline{x}), \ldots, g_i(\underline{x}), \ldots, g_{i+2r}(\underline{x}))$$

are admissible, that is, iff $\underline{x}$ and $\underline{g}(\underline{x})$ are made of admissible blocks.

Theorem 4. *The subset Σ_1 is closed and strongly invariant (i.e., $\underline{g}(\Sigma_1) = \Sigma_1$).*

Proof. $\underline{x} \in \Sigma \setminus \Sigma_1$ iff there exists either a forbidden block centered in i_0 $(x_{i_0-2r}, \ldots, x_{i_0}, \ldots, x_{i_0+2r})$, or a forbidden block centered in i_1 $(g_{i_1-2r}(\underline{x}), \ldots, g_{i_1}(\underline{x}), \ldots, g_{i_1+2r}(\underline{x}))$. Choosing $m_0 \in \mathbf{N}$ such that $m_0 > \max\{(i_0+2r, i_1+2r\}$ and $m_0 > \min\{2r - i_0, 2r - i_1\}$ and proceeding as in Theorem 1, we can conclude that Σ_1 is closed.

Let $\underline{x} \in \Sigma_1$. Then, by Lemma 1,

$$(g_{i_1-2r}(\underline{x}), \ldots, g_{i_1}(\underline{x}), \ldots, g_{i_1+2r}(\underline{x})) \qquad \forall i$$

is admissible. Moreover,

$$\begin{aligned}
&(g^2_{i_1-2r}(\underline{x}), \ldots, g^2_{i_1}(\underline{x}), \ldots, g^2_{i_1+2r}(\underline{x})) \\
&\quad = ((\sigma_R)_{i_1-2r}(\underline{x}), \ldots, (\sigma_R)^2_{i_1}(\underline{x}), \ldots, (\sigma_R)^2_{i_1+2r}(\underline{x})) \\
&\quad = (x_{i_1-2r-1}, \ldots, x_{i_1-1}, \ldots, x_{i_1+2r-1}),
\end{aligned}$$

which is an admissible block, so $\underline{g}(\underline{x}) \in \Sigma_1$.

Let $\underline{x}$ be an admissible configuration; then $\sigma_L(\underline{x})$ is also an admissible configuration. Since $\underline{g} \circ \sigma_L = \sigma_L \circ \underline{g}$ and $\underline{g}(\underline{x})$ is admissible (owing to Lemma 3), $\underline{g}(\sigma_L(\underline{x})) = \sigma_L(\underline{g}(\underline{x}))$ is admissible. We can conclude that

$$\underline{g}(\underline{g}(\sigma_L(\underline{x}))) = \underline{g}^2(\sigma_L(\underline{x})) = \underline{x}$$

with $\underline{g}(\sigma_L(\underline{x})) \in \Sigma_1$. ∎

The admissible and forbidden blocks of an elementary CA rule can be found quickly from the transition table of $h^{[2]}(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$ using the same procedure described above. In the same way it is possible to determine whether the subset Σ_1 is empty.

Using this technique to detect the forbidden blocks for Σ_1, we obtain, in the case of rule 3, that they are $b_1 = (10100)$, $b_2 = (10101)$, $b_3 = (10110)$, and $b_4 = (10111)$. The following proposition is then easy to prove.

Proposition 7. *The blocks $b_1 = (10100)$, $b_2 = (10101)$, $b_3 = (10110)$, and $b_4 = (10111)$ are forbidden iff the block (101) is forbidden.*

Proof. Suppose b_1, b_2, b_3, and b_4 are forbidden. It follows that the block (101) is also forbidden, because it cannot appear in any configuration $x \in \Sigma_1$ followed by the couples 00, 01, 10, or 11.

Conversely, if (101) is forbidden, then $x \in \Sigma_1$ does not contain (101) followed by the couple 00, 01, 10, or 11. Hence b_1, b_2, b_3, and b_4 are forbidden. ∎

As a consequence of Lemma 2 and Proposition 7, the following proposition holds.

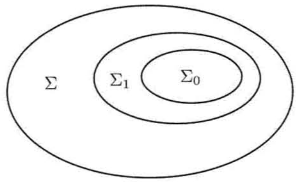

Figure 7: Structure of Σ for rule h_3.

Proposition 8. *Rule 3 is an alternating right subshift over the (nonempty) set of configurations Σ_1 that do not contain the forbidden block (101). Every configuration goes to Σ_1 in one step.*

The structure of Σ for rule 3 is shown in Figure 7.

The results for rule 35 can be proved easily using the same technique.

Proposition 9. *Rule 35 is an alternating right subshift over the (nonempty) set of configurations Σ_1 that do not contain the forbidden block* (101). *There is at least one configuration in Σ that does not go to Σ_1 in a finite number of steps.*

We give an example of a configuration that does not go to Σ_1 in a finite number of steps. The configuration $\underline{x} = \langle \ldots 1010 \mid 000000 \ldots \rangle$, which can be written as

$$\underline{x} = \langle \ldots (10) \mid (0) \ldots \rangle,$$

is such that $\forall t \in \mathbf{N}$, $\underline{g}^t(\underline{x}) \notin \Sigma_1$ (but $\lim_{t\to\infty} \underline{g}^t(\underline{x}) \in \Sigma_1$).

Theorem 5. *An alternating subshift rule whose transition matrix corresponding to $\underline{g}^2$ is irreducible is chaotic over the closed invariant subset Σ_1.*

Proof. Because $\underline{g}^2(\underline{x}) = \sigma(\underline{x})$ for any $\underline{x} \in \Sigma_1$, and the transition matrix corresponding to $\underline{g}^2$ is irreducible, it follows from Theorem 3 that $\underline{g}^2$ is chaotic on Σ_1, and thus has a dense orbit. If $\underline{g}^2$ has a dense orbit, then $\underline{g}$ also has a dense orbit, so $\underline{g}$ is transitive.

Now we want to prove that $\forall \underline{x} \in \Sigma_1$ and $\forall \epsilon$, $\exists y \in B_\epsilon(\underline{x}) \cap \mathrm{Per}(\underline{g})$. Let us consider a configuration $\underline{x} \in \Sigma_1$ ($\underline{x} = (\ldots, x_{-m}, \ldots, x_{-1}, x_0, x_1, \ldots x_m, \ldots)$), where m is such that $1/4^m < \epsilon$. We consider an admissible string $(s_1, \ldots, s_k)$ with the properties of Lemma 2, and we build a configuration $\underline{y} \in \mathrm{Per}(\underline{g}^2) = \mathrm{Per}(\sigma)$,

$$\underline{y} = ()s_1, \ldots s_k, y_{-m}, \ldots, y_{-1}, y_0, y_1, \ldots y_m(),$$

where $)x_1, \ldots, x_n($ means the bi-infinite repetition of the string $x_1, \ldots, x_n$. Because $(y_m, s_1, \ldots s_k, y_{-m})$ is an admissible string (from Lemma 2), it follows that $\underline{y}$ is made of admissible strings only, so $\underline{y} \in \Sigma_1$ is periodic with period $n = k + 2m + 1$.

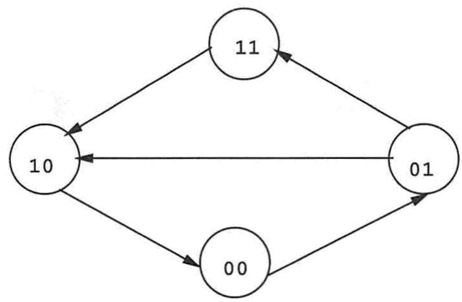

Figure 8: Subshift rule 11.

Let us consider $\underline{g}(\underline{y})$. Because $\underline{g}(\underline{y}) = \overline{\underline{y}} \wedge \overline{\sigma_R(\underline{y})}$, it is easy to see that, if $\underline{y} \in \mathrm{Per}_n(\underline{g}^2)$, then $\underline{g}(\underline{y}) \in \mathrm{Per}_n(\underline{g}^2) = \mathrm{Per}_n(\sigma)$. In fact, $\underline{y} \in \mathrm{Per}(\underline{g}^2)$ has the form

$$\underline{y} = ()s_1, \dots s_k, y_{-m}, \dots, y_{-1}, y_0, y_1, \dots y_m().$$

Since $\underline{g}(\underline{y}) = \overline{\underline{y}} \wedge \overline{\sigma_R(\underline{y})}$, $\underline{g}(\underline{y})$ has the form

$$)(\overline{s_1} \wedge \overline{y_m}), (\overline{s_2} \wedge \overline{s_1}), \dots, (\overline{y_{-m}} \wedge \overline{s_k}), \dots, (\overline{y_m} \wedge \overline{y_{m-1}})(.$$

Thus $\underline{g}(\underline{y})$ is still periodic and belongs to $\mathrm{Per}_n(\underline{g})$. Because $\underline{y}$ is such that $y_i = x_i$, $\forall i$ with $-m < i < m$, it follows that $\underline{y} \in B_\epsilon(\underline{x})$. ∎

6.2 Simple subshift and double alternating right subshift

Rules $\underline{g}_{11}$ and $\underline{g}_{43}$ are both simple and alternating subshift.

Proposition 10. *Rule 11 is a subshift over the (nonempty) set of configurations Σ_0 that does not contain the forbidden blocks* (000), (101), *and* (111). *There is at least one configuration in Σ that never goes to Σ_0 (for instance, the configuration*)0011()*.*

The proof is similar to the one for rule 3 shown previously. Describing rule 11 by means of its corresponding graph and transition matrix, we have $n = 4$ and $\mathcal{A} = \{q_0, q_1, q_2, q_3\}$, and the transition matrix M_{11} has the form

$$M_{11} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

M_{11} is irreducible and the corresponding graph is shown in Figure 8.

For rule 43 the following proposition holds.

Proposition 11. *Rule 43 is a subshift over the (nonempty) set of configurations Σ_0 that do not contain the forbidden blocks (000) and (111). There is at least one configuration in Σ that never goes to Σ_0 (for instance, the configuration $\rangle 1100 \langle$).*

Describing rule 43 by means of its corresponding graph and transition matrix, we have $n = 4$ and $\mathcal{A} = \{q_0, q_1, q_2, q_3\}$, and the transition matrix M_{43} has the form

$$M_{43} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

M_{43} is irreducible.

Now we characterize the subset Σ_2 in analogy to the characterization of subset Σ_0. The subset Σ_2 introduced in Definition 2.7 is such that a configuration $x \in \Sigma_2$ iff

$$g_i^2(x) = h^{[2]}(x_{i-2r}, \ldots, x_i, \ldots, x_{i+2r}) = \sigma_{iR}^2(x) = x_{i-2} \qquad \forall i \in \mathbf{Z}.$$

If we call *admissible* the blocks $(y_{-2r}, \ldots, y_0, \ldots, y_{+2r}) \in \{0,1\}^{4r+1}$ such that

$$h^{[2]}(y_{-2r}, \ldots, y_0, \ldots, y_{2r}) = y_{-2},$$

and *forbidden* the blocks such that

$$h^{[2]}(y_{-2r}, \ldots, y_0, \ldots, y_{2r}) \neq y_{-2},$$

then we can prove the following lemma.

Lemma 4. *A configuration $\underline{x} \in \Sigma_2$ iff $\forall i$,*

$$(x_{i-2r}, \ldots, x_i, \ldots, x_{i+2r}) \quad \text{and} \quad (g_{i-2r}(\underline{x}), \ldots, g_i(\underline{x}), \ldots, g_{i+2r}(\underline{x}))$$

are admissible, that is, iff $\underline{x}$ and $\underline{g}(\underline{x})$ are made of admissible blocks.

Theorem 6. *The subset Σ_2 defined above is closed and strongly invariant (i.e., $\underline{g}(\Sigma_2) = \Sigma_2$).*

Proof. We observe that Σ_2 can be described by the form

$$\Sigma_2 : \{x \in \Sigma : \underline{g}(x) = \sigma_R(\overline{x})\}.$$

To show that Σ_2 is strongly invariant we have to prove that $\underline{g}(\Sigma_2) = \Sigma_2$. First we show that $\underline{g}(\Sigma_2) \subseteq \Sigma_2$. Suppose that $x \in \Sigma_2$, so it is made of admissible blocks. Then, because

$$\begin{aligned}(g_{i-r}(x), \ldots, g_i(x), \ldots, g_{i+r}(x)) &= ((\sigma_R)_{i-r}(\overline{x}), \ldots, (\sigma_R)_i(\overline{x}), \ldots, (\sigma_R)_{i+r}(\overline{x})) \\ &= (\overline{x_{i-r-1}}, \ldots, \overline{x_{i-1}}, \ldots, \overline{x_{i+r-1}}) \qquad \forall i \in \mathbf{Z},\end{aligned}$$

it is also easy to see that $\underline{g}(x) = (\overline{x_{i-r}}, \ldots, \overline{x_i}, \ldots, \overline{x_{i+r}})$ is made of admissible blocks, that is, $\underline{g}(x) \in \Sigma_2$.

Now we show that $\underline{g}(\Sigma_2) \supseteq \Sigma_2$. Because

$$\begin{aligned}((\sigma_R)_{i-r}^{-1}(\overline{x}), \ldots, (\sigma_R)_i^{-1}(\overline{x}), \ldots, (\sigma_R)_{i+r}^{-1}(\overline{x})) \\ = (\overline{x_{i-r+1}}, \ldots, \overline{x_{i+1}}, \ldots, \overline{x_{i+r+1}}) \qquad \forall i \in \mathbf{Z},\end{aligned}$$

${\sigma_R}^{-1}(\overline{x})$ is made of admissible blocks. Then $\underline{g}$ is a shift on $\sigma^{-1}(x)$ and

$$\underline{g}\left(\sigma^{-1}(x)\right) = \sigma\left(\sigma^{-1}(x)\right) = x. \blacksquare$$

The admissible and forbidden blocks of an elementary CA rule can be found quickly from the transition table of $h^{[2]}(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$ using the same procedure that we described in Section 3. In the same way it is possible to check whether the subset Σ_2 is empty. Using the technique described above to detect the forbidden blocks for Σ_2, we obtain in the case of rule 11 that they are $b_1 = (10110)$, $b_2 = (10111)$, $b_3 = (01000)$, $b_4 = (01001)$, $b_5 = (01010)$, and $b_6 = (01011)$. We can prove the following.

Proposition 12. *The blocks* $b_1 = (10110)$, $b_2 = (10111)$, $b_3 = (01000)$, $b_4 = (01001)$, $b_5 = (01010)$, *and* $b_6 = (01011)$ *are forbidden iff the blocks* (101) *and* (010) *are forbidden.*

Proof. Suppose that blocks b_1, b_2, b_3, b_4, b_5, and b_6 are forbidden. From b_2, b_3, b_4, b_5, and b_6 it is easy to see that $(0, 1, 0)$ cannot be followed by 00, 01, 10, or 11, which means that $(0, 1, 0)$ is forbidden. From b_1 and b_2 it is easy to see that (1011) is also forbidden; but because (1010) is also forbidden (as an immediate consequence of the fact that (010) is forbidden), we conclude that (101) can be followed by neither 0 nor 1, which means that it is forbidden.

Conversely, suppose (101) and (010) are forbidden. It is immediately clear that b_1, b_2, b_3, b_4, b_5, and b_6 are all forbidden because each of them contains one of the forbidden blocks ((101) or (010)). $\blacksquare$

As a consequence of Lemma 3 and Proposition 12, the following proposition holds.

Proposition 13. *Rule 11 is a double alternating right subshift over the (nonempty) set of configurations* Σ_2 *that do not contain the forbidden blocks* (101) *and* (010). *There is at least one configuration in* Σ *that never goes to* Σ_2 *(for instance, the configuration* $\rangle 100 \langle$*).*

Proof. The proof follows directly from the fact that $\forall \underline{x} \in \Sigma_2$ rule 11 can be expressed as $\underline{g}_{11}(\underline{x}) = \overline{\sigma_R(\underline{x})}$. $\blacksquare$

Proposition 14. *Rule 11 is an alternating right subshift over the (nonempty) set of configurations* Σ_1 *that do not contain the forbidden blocks* (011), (1000), *and* (1010). *There is at least one configuration in* Σ *that never goes to* Σ_1 *(for instance, the configuration* $\rangle 0011 \langle$*).*

Similar results for rule 43 can be proved using the same technique.

Proposition 15. *Rule 43 is a double alternating right subshift over the (nonempty) set of configurations Σ_2 that do not contain the forbidden blocks* (0100), (1011), (1101), (00101), *and* (10010). *There is at least one configuration in Σ that never goes to Σ_2 (for instance, the configuration $\langle \ldots (100) \mid (0) \ldots \rangle$).*

We now state the following result, which can be proved similarly to the case of the alternating right subshift.

Theorem 7. *Double alternating subshift rules that have an irreducible transition matrix are chaotic over the closed, invariant subset Σ_2.*

The evolution of subshift rule 11 is shown in the appendix. Figure A2 is the space-time diagram of the CA starting with an initial configuration $\in \Sigma_0$ (simple left subshift). Figure A3 is the space-time diagram of the same rule starting with an initial configuration in Σ_2 (double alternating right subshift).

7. Conclusions and open problems

Analyzing the structure of the rule subspace of Figure 1, we have observed that CAs have a simple behavior in correspondence to the symmetric region ($a = c$), whereas, in the antisymmetric case ($a = -c$) a complex subshift behavior has been detected (asymmetric region). In the rest of the rule subspace the CAs show a simple subshift behavior. Summarizing our results, we can divide the rule space in 3 regions:

1. The symmetric region with periodic rules,
2. The asymmetric region consisting of simple subshift rules, and
3. The antisymmetric region consisting of complex subshift rules.

It is interesting to note that the periodic rules are distributed along the line of symmetry, and the complexity increases with an increase in asymmetry.

For the simple subshift rules, the subset Σ_0 is an attractor, and all the initial configurations will eventually fall into it; in the particular cases we have considered, they fall into Σ_0 after only one step. In the complex subshift rules we observe also the set Σ_i ($i = 1, 2$) on which the rules have a more complex behavior; in this case the set $\tilde{\Sigma} = \Sigma \setminus (\Sigma_0 \cup \Sigma_i)$ is in general not attracted by an invariant subset of configurations. We are now investigating the subspace $\tilde{\Sigma}$ for the complex subshift rules, and we are trying to give a precise characterization of this subspace. To date, the precise description of this space is still an open question, but our study suggests that the characterization of $\tilde{\Sigma}$ can be refined and different behaviors can be identified. For example, we have detected three subsets with different dynamics:

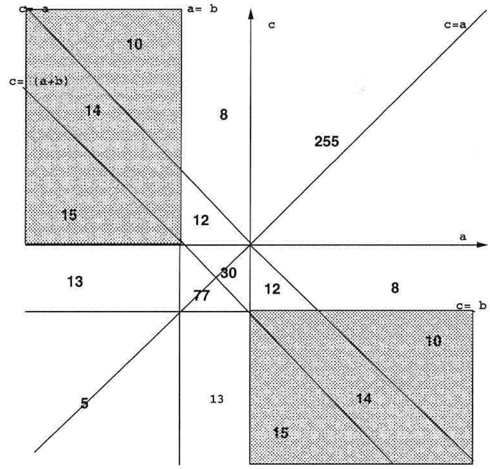

Figure 9: Rule space for $b > 0$.

1. a subset attracted by Σ_i,
2. a subset attracted by Σ_0, and
3. an invariant subset $\Sigma_I \subseteq \tilde{\Sigma}$.

We have proved that simple subshift rules that have an associated irreducible transition matrix have the classical properties of chaotic dynamical systems on Σ_0, and the same holds for the alternating and double alternating subshift on Σ_i.

For sake of completeness we include the rule space for $b > 0$ (Figure 9). In this case, the rule subspace has a simpler structure. The differences between the two cases merit further study.

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Appendix

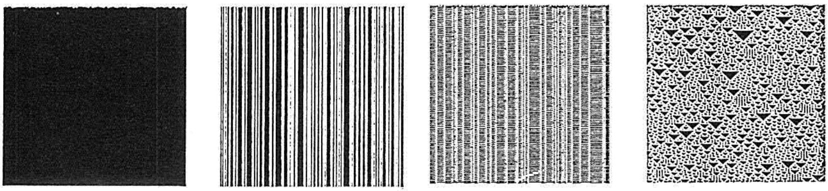

Figure A1: Space-time diagrams of some CAs and Wolfram classifications. Left to right: rules 8 (Class 1), 12, 23 (Class 2), and 18 (Class 3).

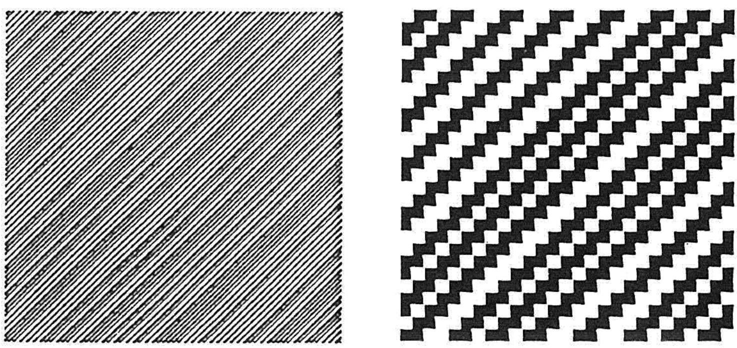

Figure A2: A simple left subshift. Space-time diagram of rule 11 with an initial configuration in Σ_0 (left), and magnification of a portion of it (right).

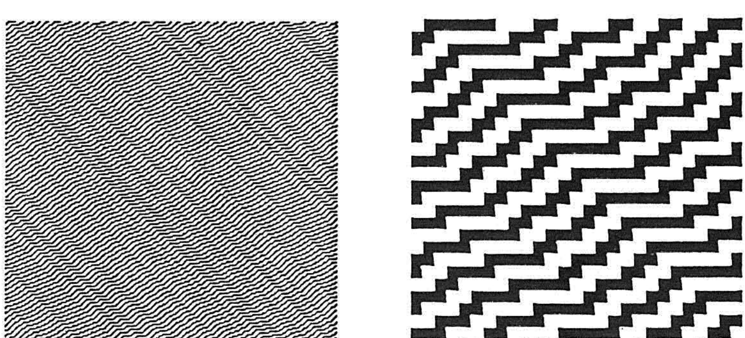

Figure A3: A double alternating right subshift. Space-time diagram of rule 11 with an initial configuration in Σ_2 (left), and magnification of a portion of it (right).