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An Analysis of Non-Binary Genetic Algorithms with Cardinality 2^v

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1. Introduction

Genetic algorithms provide a stochastic search technique inspired by principles of natural genetics and evolution. They operate through a simulated evolution process on a population of string structures, each of which represents a candidate solution in the search space. Evolution of populations involves two basic steps: (1) a selection mechanism that implements a survival of the fittest strategy, and (2) genetic recombination of the selected high-fitness strings to produce offspring for the new generation. Recombination is effected through the two biologically inspired operators of crossover and mutation. Selection ensures that above-average strings contribute to a greater number of offspring in the next generation (on average).

Genetic algorithms (GAs) are considered suitable for application to complex search spaces and combinatorial optimization problems, where a balance is often sought between full exploitation of the currently known solutions and a robust exploration of the entire search space. GAs provide an effective means for managing this tradeoff. The selection scheme operationalizes exploitation, and the recombination operators effect the exploration of the search space.

A number of researchers have reported success with GAs applied across a wide spectrum of problems, including process control [18], communication

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network design [7], to learning models of consumer choice [20], nonlinear dynamical models of sociological phenomena [11], simulation of rational agents in socio-economic contexts, and scheduling of jobs on a production floor [4, 6, 25, 26, 38, 39].

GAs use an encoding of the search space attributes. The coding scheme is critical to the success of GAs, and traditionally a binary string representation has been used. The use of a binary representation is advocated by the Schema Theorem and the principle of minimal alphabets [15], which maintains that lower cardinality alphabets facilitate higher schemata processing, and thus foster the parallelism that is implicit in genetic processing. As a consequence, most theoretical studies have been conducted in terms of a binary scheme.

Empirical results have shown the value of high cardinality representations. In this paper we provide extensions to a theoretical model of binary-coded genetic search [35] to enable consideration of higher cardinality alphabets. An exact representation of GA search using higher cardinality alphabets is presented.

The organization of the paper is as follows. Section 2 briefly examines the motivation for the study of non-binary GAs. Section 3 then provides an account of the Schema Theorem and the Vose-Liepins framework. The main results of this research are then presented in section 4.

2. The case for non-binary representations

Though the use of a binary representation has been shown to maximize the implicit parallelism inherent in genetic processing, a number of researchers emphasize that higher cardinality alphabets are better suited for practical applications, and have greater utility and intuitive appeal. They hold out numerous successful applications as proof of the power and feasibility of non-binary encoded GAs [5]. Goldberg [13] admits:

> The debate between practitioner and theoretician over this *paradox of real codings* has risen almost to the point of schism. Theoreticians have wondered why practitioners have paid so little attention to the theory, and practitioners have wondered why the theory seems so unable to come to terms with their findings. [13, p. 1]

Most theoretical considerations on genetic search emanate from the Schema Theorem—called the Fundamental Theorem of Genetic Algorithms [15]—which also forms the basis for minimal alphabet arguments and the building block hypothesis. In recent years, however, the use of the Schema Theorem to characterize the actual behavior of GAs has been increasingly questioned. Grefenstette [22] and Grefenstette and Baker [24] re-examine schema analyses that initially cast implications on implicit parallelism. Their arguments call into question certain widely held assumptions in GA theory—the k-armed bandit analogy, and consequent arguments related to the building block hypothesis and quantification of the implicit parallelism exhibited by the genetic search process. Greffenstette [21] elaborates such criticism in applying static schema analyses to study the dynamic behavior of GAs, especially in connection with the study of deceptive problems; Forrest and Mitchell [12] also express concerns with this traditional approach to deception analysis. Mühlenbein [30] expresses strong reservations to schema theorem-based interpretations (the optimality of binary representations arising therefrom), and raises a second objection—that the schema theorem focuses on disruptions caused through recombination, rather than a direct consideration of how increasingly better substrings are formed. See also Vose [37] for a critical appraisal of the schema theorem.

Noting that binary representations are a primary reason for GAs not finding wider acceptance, Antonisse [1] provides an alternative interpretation of schemata that contravenes the minimal alphabet principle, and argues for the use of non-binary discrete alphabets. Wright [40] examines real-coded GAs in terms of schemata analysis and notes advantages over a binary representation. Goldberg [13] also studies high-cardinality alphabets, and proposes a theory suggesting that initial selection pressures reduce high-cardinality alphabets to lower-cardinality virtual alphabets that subsequently undergo processing through genetic operators. He also examines deceptiveness in genetic search when using such higher-cardinality representations.

One approach to considering higher-level representations in genetic search is to interpret binary substrings as primitives of a high-level language [10]. The direct utilization of a non-binary representation scheme, however, provides greater intuitive appeal for practical applications, and a number of researchers have reported success with complex string representations [7, 23]. Grefenstette [23] directly uses a high-level representation, and points out several advantages over binary encodings:

> First, it makes it easier to incorporate existing knowledge, whether acquired from experts or by symbolic inductive learning programs. Second, it is easier to explain the knowledge learned through experience. Third, it may be possible to combine several forms of learning in a single system. [23, p. 343]

Modified operators for high-level representations have been proposed [2, 23]. Some theoretical work with non-binary string encodings has also been reported [1, 13, 40]. Recent work by Vose and Liepins [35] provides a detailed characterization of GA search and a means for a more exact analysis of the behavior of GAs (than allowed by the Schema Theorem and schemata analysis). As with most GA research, however, this work considers only a binary representation. This research aims at using the Vose-Liepins model of GAs to consider higher-cardinality alphabets.

In the next section, we take a deeper look into the theoretical model proposed by Vose and Liepins [35] to explain the behavior of binary encoded GAs.

3. A detailed characterization of GA search behavior: The Vose-Liepins model

A rigorous mathematical formalism for a simple genetic algorithm has been provided by Vose and Liepins [35]. Modeling recombination through crossover and mutation as dispersion operators, and selection as a focusing operator, they obtain a precise characterization of the punctuated equilibrium phenomenon often noticed in genetic search, that is, alternating periods of rapid evolution and generations of relatively stable populations. GAs are considered to be dynamical systems in a high-dimensional Euclidean space, and expected population trajectories are obtained assuming infinite populations. As in many other theoretical studies, Walsh matrices also provide a basis for the analysis of GAs.

Binary strings of a fixed length L are considered, where $N = 2^L$ represents the total number of possible strings. The set of binary strings is also identified with integers from 0 to $N-1$. A vector $s^t \in \mathbb{R}_N$ models the tth generation, with the ith component, s_i^t, being the probability of individual i being chosen for reproduction. Another vector $p^t \in \mathbb{R}^N$ has its ith component equal to the proportion of i in the tth generation. The probability of two strings i and j combining to produce a new string k is represented by $r_{i,j}(k)$. The expected proportion of a string k in the next generation is given by

$$E[p_k^{t+1}] = \sum_{ij} s_i^t s_j^t r_{i,j}(k).$$

Assuming infinite populations, the law of large numbers gives

$$p_k^{t+1} \to E[p_k^{t+1}].$$

For binary recombination through crossover and mutation, the following relationship holds: $r_{i,j}(k \oplus l) = r_{i\oplus k, j\oplus k}(l)$, which gives $r_{i,j}(k) = r_{i\oplus k, j\oplus k}(0)$, where $\oplus$ denotes the exclusive-or operator. Given $r_{i,j}(0)$ for all (i,j) combinations, all the recombination probabilities $r_{i,j}(k)$ are thus obtainable. A matrix M is defined having (i,j)th entry $m_{i,j} = r_{i,j}(0)$. The probability of any two strings i and j recombining, through single-point crossover and uniformly random mutation, to give the string 0 is found to be

$$\begin{aligned} M_{ij} = \frac{(1-\mu)^2}{2} &\left[\eta^{|i|} \left(1 - \chi + \frac{\chi}{L-1} \sum_{k=1}^{L-1} \eta^{-\Delta_{i,j,k}} \right) \right. \\ &\left. + \eta^{|j|} \left(1 - \chi + \frac{\chi}{L-1} \sum_{k=1}^{L-1} \eta^{\Delta_{i,j,k}} \right) \right] \end{aligned}$$

where μ and χ are the mutation and crossover probabilities, respectively, and $\eta = \mu/(1-\mu)$. Here, $|i|$ is the number of 1s in the bit vector corresponding to the integer i, and $\Delta_{i,j,k} = |(2^k - 1) \otimes i| - |(2^k - 1) \otimes j|$, where $\otimes$ denotes the logical-and operator. An operator $\bar{M}$ is defined as

$$\bar{M}(s) = \langle (\sigma_0 s)^T M (\sigma_0 s), \ldots, (\sigma_{N-1} s)^T M (\sigma_{N-1} s) \rangle^T$$

where a^T is the transpose of a and the σ_js denote the permutation:

$$\sigma_j \langle s_0, \ldots, s_{N-1} \rangle^T = \langle s_{j\oplus 0}, \ldots, s_{j\oplus(N-1)} \rangle^T$$

A second operator, F, is a nonnegative diagonal matrix with $F_{i,i} = f(i)$, the objective function value for the string corresponding to the integer i. Then, Fp^t is a vector that points in the same direction as s^t, that is,

$$s^t = \lambda F p^t, \qquad \text{where } \lambda = \frac{1}{\sum_i f(i)p(i)}.$$

Using $\sim$ to denote the equivalence relation defined by $x \sim y$ if $\exists \lambda > 0$ such that $x = \lambda y$, we have $s^t \sim Fp^t$.

Further, $\bar{M}(s)$ is a vector that has its ith component the expected proportion of i in the next generation, that is, $E[p^{t+1}] = \bar{M}(s^t)$. Assuming infinite populations, we may write $p^{t+1} = \bar{M}(s^t)$.

These operators form the basis of the Vose-Liepins model, and help characterize GA search through the matrices F and M. An exact representation of the limiting behavior of a simple GA, as the population size tends to infinity, is given by the relation

$$s^{t+1} \sim F\bar{M}(s^t).$$

Considering G to be the composition of the operators F and $\bar{M}$, the progression of GA search from one generation to the next is given by the iterates of G, and convergence corresponds to the fixed points of G.

It is shown that the only stable fixed point of F is that associated with the maximum value of the objective function. The fixed points of the quadratic operator $\bar{M}$ have been studied via a matrix M^* having (i,j)th entry $m_{i\oplus j,i}$ related to the differential of $\bar{M}$, and it is proved that if M^* is positive and has its second-largest eigenvalue less than 1/2, then every fixed point of $\bar{M}$ is stable. An explicit expression for the spectrum of M^* has been derived by Koehler [28], and the second-largest eigenvalue is shown to be $1/2 - \mu$. All fixed points of $\bar{M}$ are thus asymptotically stable when the mutation rate is between 0 and 0.5. Furthermore, a conjecture relating to dynamical systems implies that the fixed point is unique, and corresponds to a population with equal proportions of all members. Recombination is thus viewed as a dispersion or diffusion-like operator.

The evolution of populations via the operators F and $\bar{M}$ help explain the punctuated equilibria observed in genetic search. Populations move under the influence of F toward one of the fixed points. If this is not maximally fit, then it is an unstable fixed point, and recombination will ultimately cause a major change in the population and move it toward another fixed point of F.

Nix and Vose [31] derive an exact model of a finite population GA as a Markov chain, and show that for large populations the trajectory of this model follows very closely, and with high probability, that of the infinite population model. With nonzero mutation, a finite population corresponds to an ergodic Markov chain, and thus has some finite probability of visiting every state. The steady-state distribution is, however, shown to concentrate probabilities near the fixed points of the infinite population model (which correspond to local optima). This implies that the GA will move from one local optimum to another in the process of its search. Vose [37] extends this analysis through a geometric interpretation of genetic search trajectories in both the infinite and finite population cases.

4. Non-binary alphabets and the Vose-Liepins framework

The mathematical framework of Vose and Liepins [35] described in section 3 is based on a binary representation scheme. In this section a partial extension of this formalism is developed for genetic algorithms using a non-binary representation. We consider simple genetic search using traditional one-point crossover and uniformly random mutation on fixed-length strings using an alphabet of cardinality 2^v.

A key result of the Vose-Liepins model was based on a conjecture obtained from computational results. This conjectured result has since been proved [28]. In this paper, analogous results for the generalized case of alphabets of cardinality 2^v are provided. Walsh transforms have formed an important part of the theoretical analysis of GAs, and the above framework makes extensive use of Walsh matrices. Our results are also based on an extension of the Walsh matrix definition to cover non-binary representations.

Here we consider the representation to be based on an alphabet of cardinality $K = 2^v$. Though this assumption restricts generalizability to alphabets of arbitrary cardinality, our results nonetheless allow a comparison of properties of binary and non-binary GAs and bring us a step closer to a theoretical basis for analyzing non-binary GAs. Similar restrictions have been considered in Eshelman and Schafer [9].

After defining the notation in the next section, generalized Walsh matrix terms are obtained in section 4.2. Then, in section 4.3, the recombination operator expression is derived. Properties related to the stability of the fixed points of M are given in section 4.4. The expression for the spectrum of M is then derived in section 4.5.

4.1 Notation

We will use the following notation throughout.

K	is the cardinality of the alphabet; $K = 2^v$, for some positive integer v.
L	is the length of the non-binary string.
χ	is the crossover rate.
μ	is the mutation rate (mutation is to a different randomly chosen allele).
η	$\mu/(1-\mu)$.

$d_i(s)$	is the ith digit of the string s.
$\delta(x)$	is 1 if x is nonzero, and 0 otherwise.
$\sum_{i=p}^{q} \delta(d_i(s))$	is the number of nonzero digits in s, from the pth to the qth positions.
$\lvert s\rvert$	is the total number of nonzero digits in the string s.
$\mathrm{rev}(s)$	is the string obtained by taking the digits of s in reverse order.
$\mathrm{wid}(s)$	is the defining length of the string s; that is, the number of digits in s between the outermost nonzero digits; $\mathrm{wid}(0) = 0$.
$\mathrm{del}(x)$	is 1 if $x = 0$, and 0 otherwise; that is, $\mathrm{del}(x) = 1 - \delta(x)$.
$b(s)$	is the binary equivalent of the non-binary string s, obtained by concatenating the v-bit binary representation for every digit in s.
e	is a vector of ones of appropriate length.

4.2 Generalization of Walsh matrix terms

A direct method for computing the Walsh matrix terms in the binary case has been provided by Koehler [28]:

Proposition 1. *For binary strings i and j, the Walsh matrix terms are given by*

$$W_{i,j} = (-1)^{\lvert \mathrm{rev}(i) \otimes j \rvert} = (-1)^{\lvert \mathrm{rev}(j) \otimes i \rvert}.$$

Considering non-binary strings r and s, we have

$$W_{r,s} = (-1)^{\lvert b(r) \otimes \mathrm{rev}(b(s)) \rvert}.$$

Representing the strings by their individual digits, we get

$$W_{r,s} = (-1)^{\left\lvert b\left(\sum_{i=1}^{L} d_i(r) K^{i-1}\right) \otimes \mathrm{rev}\left(b\left(\sum_{i=1}^{L} d_i(s) K^{i-1}\right)\right)\right\rvert}$$

However, the second term in the exponent is

$$\mathrm{rev}\left(b\left(\sum_{i=1}^{L} d_i(s) K^{i-1}\right)\right) = \sum_{i=1}^{L} K^{L-i} \,\mathrm{rev}(b(d_i(s))).$$

Switching the second index size gives

$$\sum_{i=1}^{L} K^{L-i} \,\mathrm{rev}(b(d_i(s))) = \sum_{i=1}^{L} K^{i-1} \,\mathrm{rev}(b(d_{L+1-i}(s))).$$

Thus the Walsh terms may be written as

$$\begin{aligned} W_{r,s} &= (-1)^{\left\lvert b\left(\sum_{i=1}^{L} d_i(r) K^{i-1}\right) \otimes \sum_{i=1}^{L} K^{i-1} \,\mathrm{rev}(b(d_{L+1-i}(s)))\right\rvert} \\ &= (-1)^{\left\lvert \sum_{i=1}^{L} K^{i-1} (b(d_i(r))) \otimes \mathrm{rev}(b(d_{L+1-i}(s)))\right\rvert}. \end{aligned}$$

Since the exponent cannot have negative terms in the sum, we may move the absolute value into the summation. Also, $K^{i-1}b(d_i(r)) = d_i(r)$. We then get

$$\begin{aligned} W_{r,s} &= (-1)^{\sum_{i=1}^{L} |d_i(r)\otimes \mathrm{rev}(d_{L+1-i}(s))|} \\ &= \prod_{i=1}^{L} W_{d_i(r),d_{L+1-i}(s)} \\ &= \prod_{i=1}^{L} W_{d_i(r),d_i(\mathrm{rev}(s))}. \end{aligned}$$

We thus obtain the following theorem:

Theorem 1. *The Walsh matrix terms $W_{r,s}$ for non-binary strings r and s of L digits and cardinality $K = 2^v$ are given by*

$$W_{r,s} = \prod_{i=1}^{L} W_{d_i(r),d_{L+1-i}(s)} = \prod_{i=1}^{L} W_{d_i(r),d_i(\mathrm{rev}(s))}.$$

Note that $d_i(r)$ above represents the ith symbol (digit) of the string r. Considering an alphabet of cardinality K, this term is obtained from the $K \times K$ Walsh matrix, each term of which is given by Proposition 1 (where the binary equivalent of r is used). Note further that the $K^L \times K^L$ Walsh matrix is similar under both the binary and higher-cardinality encodings. The theorem above is thus not a new definition of the Walsh matrix, but rather a means of calculating the Walsh terms suitable for use with higher-cardinality encodings in terms of the bit strings that define each digit of the higher-level encoding.

4.3 The recombination operator

Considering recombination through simple crossover and mutation, the following relationship has been derived [35]:

$$r_{i,j}(k \oplus l) = r_{i\oplus k, j\oplus k}(l).$$

The probability $r_{i,j}(k)$ of any two binary strings i and j yielding a string k can thus be obtained as

$$r_{i,j}(k) = r_{i,j}(k \oplus 0) = r_{i\oplus k, j\oplus k}(0)$$

and so we only need to be able to compute $r_{i,j}(0)$ in order to get all the recombination probabilities. We note that in the higher-cardinality alphabet case this same relationship is seen to hold by considering binary versions of the strings.

The probabilities $r_{i,j}(0)$ for the non-binary case are derived below. Consider a string r mutating to the string 0. Since mutation is to different randomly chosen allelles, the probability of a nonzero digit of r mutating to the digit 0 is $\mu/(K-1)$. The probability of a 0 digit of r remaining the same is $1-\mu$. The mutation probability on a string r is thus given by

$$\begin{aligned} m(r) &= \prod_{i=1}^{L} \left[\delta(d_i(r)) \frac{\mu}{K-1} + (1-\delta(d_i(r)))(1-\mu) \right] \\ &= \left(\frac{\mu}{K-1} \right)^{|r|} (1-\mu)^{L-|r|} \\ &= \left(\frac{\mu}{1-\mu} \right)^{|r|} \frac{(1-\mu)^L}{(K-1)^{|r|}} \\ &= \left(\frac{\eta}{K-1} \right)^{|r|} (1-\mu)^L. \end{aligned}$$

Now for single-point crossover, let p be the randomly chosen crossover point. Consider two parent strings s and t, and let c_1 and c_2 be the two children resulting from crossover. Then, the number of nonzero digits in c_1 and c_2 are given by

$$|c_1| = |s| - \sum_{i=1}^{p} \delta(d_i(s)) + \sum_{i=1}^{p} \delta(d_i(t))$$

and

$$|c_2| = |t| - \sum_{i=1}^{p} \delta(d_i(t)) + \sum_{i=1}^{p} \delta(d_i(s)).$$

Defining

$$\Delta_{s,t,p} = \sum_{i=1}^{p} \delta(d_i(s)) - \sum_{i=1}^{p} \delta(d_i(t)),$$

we may write

$$|c_1| = |s| - \Delta_{s,t,p}$$

and

$$|c_2| = |t| + \Delta_{s,t,p}.$$

Then considering mutation and crossover together, the probability of c_1 being the string 0 is

$$(1-\chi) \left(\frac{\eta}{K-1} \right)^{|s|} (1-\mu)^L$$

if no crossover actually occurs (with probability $1-\chi$), and it is

$$\frac{\chi}{L-1} \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1} \right)^{|s|-\Delta_{s,t,p}} (1-\mu)^L$$

if crossover does take place. Thus, the probability that the child c_1 is the string 0 is given by

$$(1-\chi)\left(\frac{\eta}{K-1}\right)^{|s|}(1-\mu)^L + \frac{\chi}{L-1}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{|s|-\Delta_{s,t,p}}(1-\mu)^L.$$

Similarly, the probability of c_2 being 0 is

$$(1-\chi)\left(\frac{\eta}{K-1}\right)^{|t|}(1-\mu)^L + \frac{\chi}{L-1}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{|t|+\Delta_{s,t,p}}(1-\mu)^L.$$

Since c_1 and c_2 result with equal probability, we have the following result.

Theorem 2. *The probability of any two strings s and t recombining to yield the string 0 through single-point crossover and uniform random mutation is*

$$r_{s,t}(0) = \frac{(1-\mu)^L}{2}\left[\left(\frac{\eta}{K-1}\right)^{|s|}\left\{(1-\chi) + \frac{\chi}{L-1}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{-\Delta_{s,t,p}}\right\}\right.$$
$$\left. + \left(\frac{\eta}{K-1}\right)^{|t|}\left\{(1-\chi) + \frac{\chi}{L-1}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{\Delta_{s,t,p}}\right\}\right].$$

Vose and Liepins [35] have formalized simple genetic recombination through an operator $\bar{M}$ described in section 3. A related matrix, the twist of M, denoted M^*, plays an important role in their analyses, and is defined below.

Proposition 2 (definition and properties of M^*)

1. $M^*_{i,j} = M_{i\oplus j,i}$.
2. *WM^*W is lower triangular.*

Property (1) in Proposition 2 is obtained for cardinality K in a manner analogous to that for the binary case by noting that since K is assumed to be some power of 2, we may convert any non-binary string to its binary equivalent. The proof of part (2) closely follows that outlined in Vose and Liepins [35] and is given in the appendix.

4.4 Stability of fixed points

The stability of the fixed points of the recombination operator $\bar{M}$ has been studied, and the following result relates stability to the eigenvalues of M^*.

Proposition 3 ([35]) *If the matrix M is positive, then any fixed point of $\bar{M}$ is stable whenever the second largest eigenvalue of M^* is less than 1/2.*

Based on computational results, the authors conjectured that the eigenvalue of M^* would be less than 1/2 if the mutation rate is kept between 0 and .5. An expression for the spectrum of M^* has been obtained in Koehler [28]. We next derive the spectrum of M^* for the non-binary case.

4.5 Spectrum of M^*

4.5.1 Preliminary expression

In obtaining the spectrum of M^*, we make use of the following results derivable from Proposition 2.2 above.

Lemma 1. *Let $C = WM^*W$.*

1. *Then the eigenvalues of M^* are $C_{i,i}/K^L$ for $i = 0, \ldots, K^L - 1$.*
2. *The $C_{i,i}$ values are given by*

$$C_{i,i} = \sum_{j=0}^{K^L-1} W_{j,i} \sum_{k=0}^{K^L-1} M_{j,k} = (WMe)_i.$$

For binary strings ($K = 2$), this result has been proven by Koehler [28]. For the non-binary case, consider the binary equivalent of the strings, and further note that the length of this binary-converted string is K^L, since $K = 2^v$ for some integer v.

The expression for the spectrum of M^* is derived from part (2) of Lemma 1 by first obtaining the row sums of M. We now prove a few identities useful for this derivation.

4.5.2 Some useful identities

Below we give several identities that will be useful throughout the remainder of the paper.

Proposition 4.

1. *The total number of strings of length L having exactly i zeros is*

$$(K-1)^{L-1}\frac{L!}{i!\,(L-i)!} = (K-1)^{L-i}\binom{L}{i}$$

2. *The number of strings of length L having exactly g non-zeros in the first to the pth position (counting positions from the rightmost end of a string) is*

$$\binom{p}{p-g}(K-1)^g K^{L-p}.$$

Proof. If a string has i zeros, then it must have $L - i$ nonzero digits. Now $K-1$ nonzero symbols may be placed in the $L-i$ places in $(K-1)^{L-i}$ ways. Also, the zeros may occur in i places out of L in $L!/[i!\,(L-i)!]$ ways. Hence the first result.

The second result follows from the first. Considering the first p positions only, g non-zeros imply $p-g$ zeros, and thus the number of strings with $p-g$ zeros is

$$\binom{p}{p-g}(K-1)^g.$$

Since the remaining $L-p$ positions may be occupied by any of K digits, we get the desired expression. ∎

The following identities form the sub-expressions in the row sum calculations.

Proposition 5.

1. $\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{|t|} = (1-\mu)^{-L}$.
2. $\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(t))} = K^{L-p}(1-\mu)^{-p}$.
3. $\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=p+1}^{L}\delta(d_i(t))} = K^{p}(1-\mu)^{p-L}$.
4. $$\sum_{t=0}^{K^L-1}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{|s|-\Delta_{s,t,p}}$$
$$= \left(\frac{\eta}{K-1}\right)^{|s|}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{-\sum_{i=1}^{p}\delta(d_i(s))} K^{L-p}(1-\mu)^{-p}.$$
5. $$\sum_{t=0}^{K^L-1}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{|t|+\Delta_{s,t,p}} = \sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(s))} K^{p}(1-\mu)^{p-L}.$$

Proof. We first prove (1). We have

$$\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{|t|} = n_0\left(\frac{\eta}{K-1}\right)^{0} + n_1\left(\frac{\eta}{K-1}\right)^{1} + \cdots + n_L\left(\frac{\eta}{K-1}\right)^{L}$$

where n_i = number of strings with i nonzero digits. So,

$$\begin{aligned}\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{|t|} &= m_L\left(\frac{\eta}{K-1}\right)^{0} + m_{L-1}\left(\frac{\eta}{K-1}\right)^{1} + \cdots + m_0\left(\frac{\eta}{K-1}\right)^{L}\\ &= \sum_{i=0}^{L}\left(\frac{\eta}{K-1}\right)^{i} m_{L-i}\end{aligned}$$

where m_i = number of strings with i zeros. Using Proposition 4, part (1), gives

$$\begin{aligned}\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{|t|} &= \sum_{i=0}^{L}\left(\frac{\eta}{K-1}\right)^{i}(K-1)^i\frac{L!}{(L-i)!\,i!}\\ &= \sum_{i=0}^{L}\left(\frac{\mu}{1-\mu}\right)^{i}\frac{L!}{(L-i)!\,i!}\\ &= (1-\mu)^{-L}\sum_{i=0}^{L}\mu^i(1-\mu)^{L-i}\binom{L}{i}\\ &= (1-\mu)^{-L}.\end{aligned}$$

The last step results from noting that the binomial terms sum to unity.

Now consider (2). We have

$$\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(t))} = \left(\frac{\eta}{K-1}\right)^0 t_p^0 + \cdots + \left(\frac{\eta}{K-1}\right)^p t_p^p$$

where t_p^i is the number of strings t having i nonzero digits from the first through the pth position. Then, using Proposition 4, part (2), we have

$$\begin{aligned}\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(t))} &= \sum_{g=0}^{p}\binom{p}{g}(K-1)^g K^{L-p}\left(\frac{\eta}{K-1}\right)^g \\ &= \sum_{g=0}^{p}\binom{p}{g}K^{L-p}\left(\frac{\mu}{1-\mu}\right)^g \\ &= K^{L-p}(1-\mu)^{-p}\sum_{g=0}^{p}\binom{p}{g}\mu^g(1-\mu)^{p-g}.\end{aligned}$$

Since the last binomial term sums to unity, we have the desired result.

Relation (3) is obtained similarly. We have

$$\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=p+1}^{L}\delta(d_i(t))} = \sum_{h=0}^{L-p} t_{p,L}^h\left(\frac{\eta}{K-1}\right)^h$$

where $t_{p,L}^h$ gives the number of strings t with h non-zeros from the $(p+1)$st to the Lth position. Using Proposition 4, part (2), we find that

$$\begin{aligned}\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=p+1}^{L}\delta(d_i(t))} &= \sum_{h=0}^{L-p}(K-1)^h\binom{L-p}{h}K^p\left(\frac{\eta}{K-1}\right)^h \\ &= K^p\sum_{h=0}^{L-p}\binom{L-p}{h}\left(\frac{\mu}{1-\mu}\right)^h \\ &= K^p(1-\mu)^{-(L-p)}\sum_{h=0}^{L-p}\binom{L-p}{h}\mu^h(1-\mu)^{(L-p)-h} \\ &= K^p(1-\mu)^{p-L}.\end{aligned}$$

We now consider the expression in part (4). We know that

$$\begin{aligned}&\sum_{t=0}^{K^L-1}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{|s|-\Delta_{s,t,p}} \\ &\quad = \left(\frac{\eta}{K-1}\right)^{|s|}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{-\sum_{i=1}^{p}\delta(d_i(s))}\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(t))} \\ &\quad = \left(\frac{\eta}{K-1}\right)^{|s|}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{-\sum_{i=1}^{p}\delta(d_i(s))} K^{L-p}(1-\mu)^{-p}\end{aligned}$$

using Proposition 5, part (2).

Similarly, considering part (5), we have

$$\sum_{t=0}^{K^L-1}\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{|t|+\Delta_{s,t,p}}$$
$$=\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(s))}\sum_{t=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=p+1}^{L}\delta(d_i(t))}$$
$$=\sum_{p=1}^{L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(s))}K^p(i-\mu)^{p-L}. \blacksquare$$

We now consider another set of useful identities that relate to Walsh matrix terms in the non-binary case.

Proposition 6.

1. $\sum_{x=0}^{K^L-1} W_{r,x} = K^L$ *if* $|r|=0$; 0 *otherwise.*
2. $\sum_{x=0}^{K^p-1} W_{r,x} = K^p$ *if* $\sum_{i=L-p+1}^{L}\delta(d_i(r))=0$; 0 *otherwise.*

Proof. Consider part (1). Expressing the Walsh terms in product form (Theorem 1), we have

$$\sum_{x=0}^{K^L-1} W_{r,x} = \sum_{x=0}^{K^L-1}\left(\prod_{i=1}^{L} W_{d_i(r),d_i(\mathrm{rev}(x))}\right).$$

Since summation over x and its reverse are equivalent, the above expression is equivalent to

$$\sum_{x=0}^{K^L-1}\left(\prod_{i=1}^{L} W_{d_i(r),d_i(x)}\right).$$

Now note that r and x are strings of length L. Consider the digits of these strings as $r_1,\ldots,r_L$ and $x_1,\ldots,x_L$, respectively. Then in the expression

$$\sum_{x=0}^{K^L-1}\left(\prod_{i=1}^{L} W_{d_i(r),d_i(x)}\right)$$

there are K^L terms. Combine these terms in groups of K, such that x_1 varies from 0 to $K-1$ within one group, and all other x_is remain fixed. There will be K^{L-1} such groups, and each group sum is of the form

$$W_{r_L,x_L}\ldots W_{r_2,x_2}(W_{r_1,0}+W_{r_1,1}+\cdots+W_{r_1,K-1})$$
$$=\begin{cases} KW_{r_L,x_L}\ldots W_{r_2,x_2} & \text{if } r_1=0\\ 0 & \text{if } r_i\neq 0.\end{cases}$$

If $r_1 = 0$, then consider the K^{L-2} groups where x_2 varies from 0 to $K-1$. That is, consider

$$KW_{r_L,x_L}\dots W_{r_3,x_3}(W_{r_2,0}+W_{r_2,1}+\dots+W_{r_2,K-1}) = \begin{cases} 0 & \text{if } r_2 \neq 0 \\ K^2 W_{r_L,x_L}\dots W_{r_3,x_3} & \text{if } r_1 = r_2 = 0 \end{cases}.$$

Continuing in this manner, we find that if $r_1 = r_2 = \dots = r_{L-1} = 0$, then the remaining K groups where x_L varies from 0 to $K-1$ have the form

$$K^{L-1}(W_{r_L,0}+W_{r_L,1}+\dots+W_{r_L,K-1}) = \begin{cases} 0 & \text{if } r_L \neq 0 \\ K^L & \text{if } r_L = r_{L-1} = \dots = r_1 = 0 \end{cases},$$

which proves the first result.

Now consider part (2), namely,

$$\sum_{x=0}^{K^p-1} W_{r,x} = \sum_{x=0}^{K^p-1}\left(\prod_{i=0}^{L} W_{d_i(r),d_i(\text{rev}(x))}\right).$$

Since the lower $L-p$ digits of rev(x) are 0s, the Walsh terms $W_{d_i(r),d_i(\text{rev}(x))} = 1$ for these digits. Thus, the expression is

$$\begin{aligned}\sum_{x=0}^{K^p-1} W_{r,x} &= \sum_{x=0}^{K^p-1}\left(\prod_{i=L-p+1}^{L} W_{d_i(r),d_i(\text{rev}(x))}\right) \\ &= \sum_{x=0}^{K^p-1}\left(W_{r_L,x_1}W_{r_{L-1},x_2}\dots W_{r_{L-p+1},x_p}\right) \\ &= \begin{cases} K^p & \text{if } r_{L-p+1} = \dots = r_{L-1} = r_L = 0 \\ 0 & \text{otherwise} \end{cases}.\end{aligned}$$

Using steps similar to those used to obtain part (1) above gives the desired expression. ∎

The following identities are necessary for obtaining the spectrum (in section 4.5.4) from the row sums expression (derived in the next section).

Proposition 7.

1. $$\sum_{d_i(s)=0}^{K-1} W_{d_i(s),d_{L-i+1}(r)}\left(\frac{\eta}{K-1}\right)^{\delta(d_L(s))} = \begin{cases} 1-\frac{\eta}{K-1} & \text{if } d_{L-i+1}(r) \neq 0 \\ 1+\eta & \text{if } d_{L-i+1} = 0 \end{cases}$$

2. $$\sum_{x=0}^{K^p-1} W_{r,x}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(x))} = \left(1-\frac{\eta}{K-1}\right)^{\sum_{i=L-p}^{L}\delta(d_i(r))}(1+\eta)^{p-\sum_{i=L-p}^{L}\delta(d_i(r))}$$

3. $$\sum_{y=0}^{K^{L-p}-1} W_{r,yK^p} = \begin{cases} K^{L-p} & \text{if } \sum_{i=1}^{L-p}\delta(d_i(r)) = 0 \\ 0 & \text{otherwise} \end{cases}$$

4. $$\sum_{y=0}^{K^{L-p}-1} W_{r,yK^p}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p}\delta(d_i(y))} = \left(1-\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p}\delta(d_i(r))}(1+\eta)^{L-p-\sum_{i=1}^{L-p}\delta(d_i(r))}.$$

Proof. First consider part (1). We have

$$\begin{aligned}
&\sum_{d_i(s)=0}^{K-1} W_{d_i(s),d_{L-i+1}(r)}\left(\frac{\eta}{K-1}\right)^{\delta(d_i(s))} \\
&= W_{0,d_{L-i+1}(r)}\left(\frac{\eta}{K-1}\right)^{0}+\left(\frac{\eta}{K-1}\right)\left[W_{1,d_{L-i+1}(r)}+\cdots+W_{K-1,d_{L-i+1}(r)}\right] \\
&= \begin{cases} 1+\frac{\eta}{K-1}(-1) & \text{if } d_{L-i+1}(r)\neq 0 \\ 1+\frac{\eta}{K-1}(K-1) & \text{if } d_{L-i+1}(r)=0 \end{cases}.
\end{aligned}$$

The above expression results from noting that the Walsh coefficients are terms from the $K\times K$ Walsh matrix, and that the row sums of the Walsh matrix, leaving out the first column, equal $K-1$.

Now consider part (2). We have

$$\begin{aligned}
&\sum_{x=0}^{K^p-1} W_{r,x}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(x))} \\
&= \sum_{x=0}^{K^p-1}\left(\frac{\eta}{K-1}\right)^{\delta(d_1(x))}\left(\frac{\eta}{K-1}\right)^{\delta(d_2(x))}\cdots\left(\frac{\eta}{K-1}\right)^{\delta(d_p(x))} \\
&\quad \left[W_{d_1,(x),d_L(r)}W_{d_2,(x),d_{L-1}(r)}\cdots W_{d_L(x),d_1(r)}\right] \\
&= \sum_{x=0}^{K^p-1}\left\{\left[\left(\frac{\eta}{K-1}\right)^{\delta(d_1(x))}W_{d_1(x),d_1(\mathrm{rev}(r))}\right]\right. \\
&\quad \left[\left(\frac{\eta}{K-1}\right)^{\delta(d_2(x))}W_{d_2(x),d_2(\mathrm{rev}(r))}\right]\cdots \\
&\quad \left[\left(\frac{\eta}{K-1}\right)^{\delta(d_p(x))}W_{d_p(x),d_p(\mathrm{rev}(r))}\right] \\
&\quad \left.\left[W_{d_{p+1}(x),d_{p+1}(\mathrm{rev}(r))}\cdots W_{d_L(x),d_L(\mathrm{rev}(r))}\right]\right\}
\end{aligned}$$

However, the last bracketed expression equals 1, since the $(p+1)$st to Lth digits of x are 0 (since $x=0$ to K^p-1). Then, using part (1), our expression becomes

$$\begin{aligned}
&= \left(1+\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(\mathrm{rev}(r)))}(1+\eta)^{L-\sum_{i=1}^{p}\delta(d_i(\mathrm{rev}(r)))} \\
&= \left(1+\frac{\eta}{K-1}\right)^{\sum_{i=L-p}^{L}\delta(d_i(r))}(1+\eta)^{p-\sum_{i=L-p}^{L}\delta(d_i(r))}.
\end{aligned}$$

Next, considering part (3),

$$\sum_{y=0}^{K^{L-p}-1} W_{r,yK^p} = \sum_{y=0}^{K^{L-p}-1} \left(\prod_{i=1}^{L} W_{d_i(r),d_i(\mathrm{rev}(yK^p))} \right)$$

note that in yK^p, the lower (i.e., the rightmost) p digits of y are 0 (multiplication here is modulo K). Thus, as y varies from 0 to $K^{L-p}-1$, yK^p varies from K^p to K^L-1. That is, the lower p digits of yK^p are zero. Hence, writing $yK^p = q$, the above expression becomes

$$\sum_{y=0}^{K^{L-p}-1} W_{r,yK^p} = \sum_{q=K^p}^{K^L-1} \left(\prod_{i=1}^{L} W_{d_i(r),d_i(\mathrm{rev}(q))} \right).$$

Note that $\mathrm{rev}(q)$ has its upper p digits equal to 0, so we need consider only the lower $L-p$ digits of r. Letting r_1 be the string r with upper p digits 0, we get

$$\begin{aligned}\sum_{y=0}^{K^{L-p}-1} W_{r,yK^p} &= \sum_{q=K^p}^{K^L-1} \left(\prod_{i=1}^{L-p} W_{d_i(r_1),d_i(\mathrm{rev}(q))} \right) \\ &= \sum_{q=0}^{K^{L-p}-1} \left(\prod_{i=1}^{L-p} W_{d_i(r_1),d_i(q)} \right).\end{aligned}$$

Using Proposition 6, part (1), we then get

$$\sum_{y=0}^{K^{L-p}-1} W_{r,yK^p} = \begin{cases} K^{L-p} & \text{if all digits of } r_1 \text{ are } 0 \\ 0 & \text{otherwise} \end{cases}.$$

Finally, consider part (4). The left-hand side of the expression is

$$\begin{aligned}&\sum_{y=0}^{K^{L-p}-1} W_{r,yK^p} \left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p}\delta(d_i(y))} \\ &= \sum_{y=0}^{K^{L-p}-1} \left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p}\delta(d_i(r))} \left[\prod_{i=1}^{L} W_{d_i(r),d_{L+i-1}(yK^p)}\right] \\ &= \sum_{y=0}^{K^{L-p}-1} \left(\frac{\eta}{K-1}\right)^{\delta(d_1(y))} \left(\frac{\eta}{K-1}\right)^{\delta(d_2(y))} \cdots \left(\frac{\eta}{K-1}\right)^{\delta(d_{L-p}(y))} \\ &\quad \left[W_{d_1(r),d_1(\mathrm{rev}(yK^p))} \cdots W_{d_{L-p}(r),d_{L-p}(\mathrm{rev}(yK^p))}\right] \\ &\quad \left[W_{d_{L-p+1}(r),d_{L-p+1}(\mathrm{rev}(yK^p))} \cdots W_{d_L(r),d_L(\mathrm{rev}(yK^p))}\right]\end{aligned}$$

The last bracketed expression equals 1, since $d_{L-p+1}(\mathrm{rev}(yK^p))$ to $d_L(\mathrm{rev}(yK^p))$ constitute the upper p digits of $\mathrm{rev}(yK^p)$; that is, the lower p digits of yK^p, which are all 0s. Hence the expression becomes

$$\sum_{y=0}^{K^{L-p}-1} W_{r,yK^p} \left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p}\delta(d_i(y))}$$

$$= \left[\sum_{d_1(y)=0}^{K-1} \left(\frac{\eta}{K-1} \right)^{\delta(d_1(y))} W_{d_1(yK^p),d_1(\mathrm{rev}(r))} \right] \cdots$$
$$\left[\sum_{d_{L-p}(y)=0}^{K-1} \left(\frac{\eta}{K-1} \right)^{\delta(d_{L-p}(y))} W_{d_{L-p}(yK^p),d_{L-p}(\mathrm{rev}(r))} \right].$$

Using Proposition 7, part (1), this equals

$$= \left(1 - \frac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p} \delta(d_i(r))} (1+\eta)^{L-p-\sum_{i=1}^{L-p} \delta(d_i(r))}. \blacksquare$$

4.5.3 Row sums of M

The elements of M are

$$r_{s,t}(0) = \frac{(1-\mu)^L}{2} \left[\left(\frac{\eta}{K-1} \right)^{|s|} \left(1 - \chi + \frac{\chi}{L-1} \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1} \right)^{-\Delta_{s,t,p}} \right) \right.$$
$$\left. + \left(\frac{\eta}{K-1} \right)^{|t|} \left(1 - \chi + \frac{\chi}{L-1} \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1} \right)^{\Delta_{s,t,p}} \right) \right].$$

To obtain the row sums of M, we sum the above expression from $t = 0$ to $K^L - 1$. Now,

$$\sum_{t=0}^{K^L-1} \frac{(1-\mu)^L}{2} \left[\left(\frac{\eta}{K-1} \right)^{|s|} (1-\chi) + \left(\frac{\eta}{K-1} \right)^{|t|} (1-\chi) \right]$$
$$= K^L \frac{(1-\mu)^L}{2} \left(\frac{\eta}{K-1} \right)^{|s|} (1-\chi)$$
$$+ \frac{(1-\mu)^L}{2} (1-\chi) \sum_{t=0}^{K^L-1} \left(\frac{\eta}{K-1} \right)^{|t|}$$
$$= K^L \frac{(1-\mu)^L}{2} \left(\frac{\eta}{K-1} \right)^{|s|} (1-\chi) + \frac{(1-\chi)}{2}$$

using Proposition 5, part (1).

Also, combining Proposition 5, parts (4) and (5), we get

$$\frac{(1-\mu)^L}{2} \frac{\chi}{L-1} \left[\sum_{t=0}^{K^L-1} \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1} \right)^{|s|-\Delta_{s,t,p}} + \sum_{t=0}^{K^L-1} \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1} \right)^{|t|-\Delta_{s,t,p}} \right]$$
$$= \frac{(1-\mu)^L}{2} \frac{\chi}{L-1} \left[\sum_{p=1}^{L-1} \left(\frac{\eta}{K-1} \right)^{|s|-\sum_{i=1}^{p} \delta(d_i(s))} K^{L-p} (1-\mu)^{-p} \right.$$
$$\left. + \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1} \right)^{\sum_{i=1}^{p} \delta(d_i(s))} K^p (1-\mu)^{p-L} \right]$$
$$= \frac{\chi}{2(L-1)} \left[\sum_{p=1}^{L-1} \left(\frac{\eta}{K-1} \right)^{|s|-\sum_{i=1}^{p} \delta(d_i(s))} (K - K\mu)^{L-p} \right.$$

$$
\begin{aligned}
&+ \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p} \delta(d_i(s))} (K-K\mu)^p \Bigg] \\
&= \frac{\chi}{2(L-1)} \left[\sum_{h=1}^{L-1} \left(\frac{\eta}{K-1}\right)^{|s|-\sum_{i=1}^{L-h} \delta(d_i(s))} (K-K\mu)^h \right. \\
&\left. + \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p} \delta(d_i(s))} (K-K\mu)^p \right] \\
&= \frac{\chi}{2(L-1)} \left(\sum_{p=1}^{L-1} (K-K\mu)^p \left[\left(\frac{\eta}{K-1}\right)^{|s|-\sum_{i=1}^{L-p} \delta(d_i(s))} \right.\right. \\
&\left.\left. + \left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p} \delta(d_i(s))} \right] \right) \\
&= \frac{\chi}{2(L-1)} \sum_{p=1}^{L-1} (K-K\mu)^p G(p,s)
\end{aligned}
$$

where

$$
G(p,s) = \left(\frac{\eta}{K-1}\right)^{|s|-\sum_{i=1}^{L-p} \delta(d_i(s))} + \left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p} \delta(d_i(s))}
$$

Then combining the two expressions obtained above, we get the expression for the row sums:

$$
\begin{aligned}
\sum_{t=0}^{K^L-1} r_{s,t}(0) = \frac{(1-\chi)}{2} &+ K^L \frac{(1-\mu)^L}{2} \left(\frac{\eta}{K-1}\right)^{|s|} (1-\chi) \\
&+ \frac{\chi}{2(L-1)} \sum_{p=1}^{L-1} (K-K\mu)^p G(p,s).
\end{aligned}
$$

4.5.4 Expression for the Spectrum

The following terms comprise prominent sub-expressions in the derivation of the spectrum.

Proposition 8.

1. $\displaystyle \sum_{s=0}^{K^L-1} W_{r,s} \left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L} \delta(d_i(s))} = \left(1 - \frac{\eta}{K-1}\right)^{|r|} (1+\eta)^{L-|r|}.$

2. $\displaystyle \sum_{s=0}^{K^L-1} W_{r,s} \left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p} \delta(d_i(s))}$

$$
= \begin{cases} K^{L-p} \left(1 - \frac{\eta}{K-1}\right)^{|r|} (1+\eta)^{p-|r|} & \text{if } \sum_{i=1}^{L-p} \delta(d_i(r)) = 0 \\ 0 & \text{otherwise} \end{cases}.
$$

3. $$\sum_{s=0}^{K^L-1} W_{r,s}\left(\frac{\eta}{K-1}\right)^{|s|-\sum_{i=1}^{p}\delta(d_i(s))} = \begin{cases} K^p\left(1-\frac{\eta}{K-1}\right)^{|r|}(1+\eta)^{L-p-|r|} & \textit{if } \sum_{i=L-p+1}^{L}\delta(d_i(r))=0 \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. First consider part (1). The left expression is

$$\sum_{s=0}^{K^L-1} W_{r,s}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L}\delta(d_i(s))}$$
$$= \sum_{s=0}^{K^L-1}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L}\delta(d_i(s))}\left[\prod_{i=1}^{L} W_{d_i(s),d_{L-i+1}(r)}\right]$$
$$= \left[\sum_{d_1(s)=0}^{K-1} W_{d_1(s),d_L(r)}\left(\frac{\eta}{K-1}\right)^{\delta(d_1(s))}\right]$$
$$\left[\sum_{d_2(s)=0}^{K-1} W_{d_2(s),d_L(r)}\left(\frac{\eta}{K-1}\right)^{\delta(d_2(s))}\right]\cdots$$
$$\left[\sum_{d_L(s)=0}^{K-1} W_{d_L(s),d_L(r)}\left(\frac{\eta}{K-1}\right)^{\delta(d_L(s))}\right].$$

Using Proposition 4, part (1), this expression becomes

$$= \left(1-\frac{\eta}{K-1}\right)^{|r|}(1+\eta)^{L-|r|}.$$

Now considering part (2), the left expression may be written as

$$\sum_{s=0}^{K^L-1} W_{r,s}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(s))}$$
$$= \sum_{y=0}^{K^{L-p}-1}\sum_{x=0}^{K^p-1} W_{r,x+yK^p}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(s))}$$
$$= \sum_{y=0}^{K^{L-p}-1} W_{r,yK^p}\sum_{x=0}^{K^p-1} W_{r,x}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{p}\delta(d_i(s))}.$$

Using Proposition 4, part (2), for the second summation above gives

$$= \sum_{y=0}^{K^{L-p}-1} W_{r,yK^p}\left[\left(1-\frac{\eta}{K-1}\right)^{\sum_{i=L-p}^{L}\delta(d_i(r))}(1+\eta)^{p-\sum_{i=L-p}^{L}\delta(d_i(r))}\right].$$

Using the result of Proposition 4, part (4), this becomes

$$= \begin{cases} K^{L-p}\left(1-\frac{\eta}{K-1}\right)^{\sum_{i=L-p}^{L}\delta(d_i(r))}(1+\eta)^{p-\sum_{i=L-p}^{L}\delta(d_i(r))} \\ 0, \text{ otherwise} \end{cases}.$$

But if

$$\sum_{i=1}^{L-p} \delta(d_i(r)) = 0,$$

then

$$\sum_{i=L-p}^{L} \delta(d_i(r)) = \sum_{i=1}^{L} \delta(d_i(r)) = |r|.$$

Thus, we get

$$= \begin{cases} K^{L-p}\left(1 - \dfrac{\eta}{K-1}\right)^{|r|}(1+\eta)^{p-|r|} & \text{if } \sum_{i=1}^{L-p} \delta(d_1(r)) = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Finally, consider part (3). The left expression is

$$\begin{aligned} &\sum_{s=0}^{K^L-1} W_{r,s}\left(\frac{\eta}{K-1}\right)^{|s|-\sum_{i=1}^{p}\delta(d_i(s))} \\ &= \sum_{s=0}^{K^L-1} W_{r,s}\left(\frac{\eta}{K-1}\right)^{\sum_{i=p+1}^{L}\delta(d_i(s))} \\ &= \sum_{y=0}^{K^{L-p}-1}\sum_{x=0}^{K^p-1} W_{r,x+yK^p}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p}\delta(d_i(y))} \\ &= \sum_{x=0}^{K^p-1} W_{r,x}\left[\sum_{y=0}^{K^{L-p}-1} W_{r,y+yK^p}\left(\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p}\delta(d_i(y))}\right]. \end{aligned}$$

Using Proposition 4, part (4), this becomes

$$\begin{aligned} &\sum_{s=0}^{K^L-1} W_{r,s}\left(\frac{\eta}{K-1}\right)^{|s|-\sum_{i=1}^{p}\delta(d_i(s))} \\ &= \sum_{x=0}^{K^p-1} W_{r,x}\left[\left(1-\frac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p}\delta(d_i(r))}(1+\eta)^{L-p-\sum_{i=1}^{L-p}\delta(d_i(r))}\right]. \end{aligned}$$

Now using the identity of Proposition 6, part (2), we have

$$\begin{aligned} &\sum_{s=0}^{K^L-1} W_{r,s}\left(\frac{\eta}{K-1}\right)^{|s|-\sum_{i=1}^{p}\delta(d_i(s))} \\ &= \begin{cases} K^p\left(1-\dfrac{\eta}{K-1}\right)^{\sum_{i=1}^{L-p}\delta(d_i(r))}(1+\eta)^{L-p-\sum_{i=1}^{L-p}\delta(d_i(r))} & \\ \qquad\qquad \text{if } \displaystyle\sum_{i=L-p+1}^{L}\delta(d_i(r)) = 0 & \\ 0 \qquad\qquad \text{otherwise} & \end{cases}. \end{aligned}$$

But if

$$\sum_{i=1}^{L-p+1} \delta(d_i(r)) = 0,$$

we get

$$\sum_{i=1}^{L-p} \delta(d_i(r)) = \sum_{i=1}^{L} \delta(d_i(r)) = |r|.$$

Hence, the result. ∎

The expression for the spectrum may now be obtained using the above results. Let S_s denote the sth row sum of M. Then from the expression for the row sums of M derived in section 4.5.3, we have

$$\begin{aligned}\sum_s W_{r,s} S_s = {} & \frac{(1-\chi)}{2} \sum_s W_{r,s} \\ & + K^L \left(\frac{1-\mu^L}{2}\right) (1-\chi) \sum_s W_{r,s} \left(\frac{\eta}{K-1}\right)^{|s|} \\ & + \frac{\chi}{2(L-1)} \sum_{p=1}^{L-1} (K - K\mu)^p \sum_s W_{r,s} G(p,s)\end{aligned}$$

First consider the last term. We have

$$\begin{aligned}& \sum_{p=1}^{L-1} (K - K\mu)^p \sum_s W_{r,s} G(p,s) \\ & = \sum_{p=1}^{L-1} (K - K\mu)^p \left[K^{L-p} \left(1 - \frac{\eta}{K-1}\right)^{|r|} (1+\eta)^{p-|r|} \operatorname{del}\left(\sum_{i=1}^{L-p} \delta(d_i(r))\right) \right. \\ & \qquad \left. + K^{L-p} \left(1 - \frac{\eta}{K-1}\right)^{|r|} (1+\eta)^{p-|r|} \operatorname{del}\left(\sum_{i=L-p+1}^{L} \delta(d_i(r))\right) \right] \\ & = \sum_{p=1}^{L-1} K^p (1-\mu)^p k^{L-p} \left(1 - \frac{\eta}{K-1}\right)^{|r|} (1+\eta)^{p-|r|} \left[\operatorname{del}\left(\sum_{i=1}^{p} \delta(d_i(r))\right) \right. \\ & \qquad \left. + \operatorname{del}\left(\sum_{i=p+1}^{L} \delta(d_i(r))\right) \right] \\ & = K^L (1 - K\mu)^{|r|} \sum_{p=1}^{L-1} \left[\operatorname{del}\left(\sum_{i=1}^{p} \delta(d_i(r))\right) + \operatorname{del}\left(\sum_{i=p+1}^{L} \delta(d_i(r))\right) \right].\end{aligned}$$

When $r = 0$, this expression equals $K^L 2(L-1)$. When $r > 0$, the last summation equals $n_{\text{right}}(1+0) + \operatorname{wid}(r) + n_{\text{left}}(0+1)$, where

n_{right} = number of trailing 0s in r,
n_{left} = number of leading 0s, leaving out the leftmost digit, in r,

and wid(r) is the same as the defining length of r.

Thus, we have

$$\sum_{p=1}^{L-1}(K-K\mu)^p\sum_s W_{r,s}G(p,s) = K^L(1-K\mu)^{|r|}[L-1-\mathrm{wid}(r)].$$

So, when $r=0$,

$$\sum_s W_{r,s}S_s = \frac{(1-\chi)}{2}K^L + K^L\frac{(1-\mu)^L}{2}(1-\chi)(1+\eta)^L + \frac{\chi}{2(L-1)}K^L 2(L-1).$$

Then by Lemma 1, part (2), the eigenvalue of M^* corresponding to $r=0$ is obtained by dividing the above expression by K^L, which gives

$$\frac{1-\chi}{2}+\frac{1-\chi}{2}+\chi = 1.$$

When $r>0$,

$$\begin{aligned}\sum_s W_{r,s}S_s &= K^L\left(\frac{1-\mu^L}{2}\right)(1-\chi)\left(1-\frac{\eta}{K-1}\right)^{|r|}(1+\eta)^{L-|r|}\\ &\quad+\frac{\chi}{2(L-1)}K^L(1-K\mu)^{|r|}(L-1-\mathrm{wid}(r))\\ &= K^L\frac{(1-\chi)}{2}\left(1-\frac{K\mu}{K-1}\right)^{|r|}\\ &\quad+\frac{\chi}{2(L-1)}K^L(1-K\mu)^{|r|}(L-1-\mathrm{wid}(r)).\end{aligned}$$

Again, by Lemma 1, part (2), the eigenvalues of M^* are obtained by dividing this expression by K^L giving

$$\frac{1-\chi}{2}\left(1-\frac{K\mu}{K-1}\right)^{|r|}+\frac{\chi}{2(L-1)}(1-K\mu)^{|r|}(L-1-\mathrm{wid}(r)),$$

which is decreasing in $|r|$ and in wid(r) when $\mu<1/K$.

The second-largest eigenvalue corresponds to $r=1$ and is

$$\frac{1-\chi}{2}\left(1-\frac{K\mu}{K-1}\right)+\frac{\chi}{2(L-1)}(1-K\mu)(L-1).$$

This simplifies to

$$\frac{1}{2}\left[1-\frac{K}{K-1}\mu-\chi K\mu\left(\frac{K-2}{K-1}\right)\right],$$

which is less than $1/2$ for $\mu<1/K$.

Thus, the second-largest eigenvalue of M^* is less than $1/2$ for $\mu<1/K$. By Proposition 3, this implies that for our non-binary, higher-cardinality representations, the fixed points of the recombination operator $\bar{M}$ are stable irrespective of the crossover rate when $\mu<1/K$.

5. Summary and future directions

The theoretical model proposed by Vose and Liepins [35] provides a detailed characterization of the search behavior of binary encoded GAs. Formalizing the genetic operators of crossover and mutation as dispersion operators, and fitness-proportionate selection as a focusing operator, they provide a precise description of the punctuated equilibria that are typically observed in genetic search experiments. A crucial conjecture relating to the asymptotic stability of the fixed points of the search operators has since been proved by Koehler [28]. In this paper, we have extended these results for GAs employing higher cardinalities of 2^v that include Koehler's [28] result as a special case.

As in other studies [9], the analysis presented here considers alphabets with cardinality restricted to powers of 2. The results obtained allow comparison of binary versus higher-cardinality string encodings. A complete generalization to alphabets of arbitrary cardinality is currently being pursued.

Walsh functions have been invaluable in the theoretical analyses of binary GAs [16, 17]. This paper provides an extension of the Walsh matrix terms when considering higher-cardinality representations. This, together with other identities derived in the process of our analysis, should prove useful in the study of non-binary GAs. Analysis of modified crossover and mutation operators and of deceptiveness in non-binary GAs are important areas for future research.

The obtained higher-cardinality representation model also allows the generalization of the Markov chain model of genetic search provided in Nix and Vose [31]. Aytug and Koehler [3] use this Markov chain model to obtain bounds on the run-time complexity of binary GAs. Suzuki [33] analyzes elitist selection through a Markov chain analysis and obtains bounds on the probability that the optimal string is attained in a given number of generations. Our results also allow the extension of such analyses to the non-binary case.

Appendix

The proof of the second part of Proposition 2 follows that for the binary case. Consider

$$\begin{aligned}(WM^*W)_{i,j} &= \sum_{k_3} W_{i,k_3} \sum_{k_4} M^*_{k_3,k_4} W_{k_4,j} \\ &= \sum_{k_3} W_{i,k_3} \sum_{k_4} M_{k_3 \oplus k_4, k_3} W_{k_4,j}.\end{aligned}$$

Now $C = WMW$ gives $N^2 M = WCW$ (N being K^L), so $M \propto WCW$. The above expression may thus be written as

$$(WM^*W)_{i,j} \propto \sum_{k_3,k_4} W_{i,k_3} \sum_{k_1,k_2} W_{k_3 \oplus k_4, k_1} C_{k_1,k_2} W_{k_2,k_3} W_{k_4,j}.$$

Combining the second and third Walsh terms and rearranging gives

$$(WM^*W)_{i,j} \propto \sum_{k_1,k_2,k_3} W_{i,k_3} W_{k_1 \oplus k_2, k_3} C_{k_1,k_2} \sum_{k_4} W_{k_1,k_4} W_{k_4,j}.$$

Since the last summation terms equals N when $j = k_1$, and 0 otherwise, we have

$$(WM^*W)_{i,j} \propto \sum_{k_2} C_{j,k_2} \sum_{k_3} W_{i,k_3} W_{j \oplus k_2, k_3}.$$

Again, noting that the last summation equals N when $i = j \oplus k_2$, and 0 elsewhere, we get

$$(WM^*W)_{i,j} \propto C_{j,i \oplus j}. \tag{A.1}$$

Now consider $C = WMW$. In the expression for M (Theorem 2), combining the μ and χ terms as suitable constants c_1 and c_2 allows us to write

$$\begin{aligned} M_{s,t} &= c_1 \left(\frac{\eta}{K-1}\right)^{|s|} + c_2 \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1}\right)^{(|s|-\Delta_{s,t,p})} \\ &\quad + c_1 \left(\frac{\eta}{K-1}\right)^{|t|} + c_2 \sum_{p=1}^{L-1} \left(\frac{\eta}{K-1}\right)^{(|t|-\Delta_{s,t,p})} \\ &= f(|s|) + \sum_{P=1}^{L-1} h(|s| - \Delta_{s,t,p}) + f(|t|) + \sum_{P=1}^{L-1} h(|t| + \Delta_{s,t,p}) \end{aligned}$$

where f and h are functions appropriately defined. We may thus write

$$\begin{aligned} c_{i,j} = \sum_{k_1,k_2} W_{i,k_1} W_{k_2,j} \Bigg[& f(|k_1|) + f(|k_2|) + \sum_{k_3=1}^{L-1} h(|k_1| - \Delta_{k_1,k_2,k_3}) \\ & + \sum_{k_3=1}^{L-1} h(|k_2| + \Delta_{k_1,k_2,k_3}) \Bigg]. \end{aligned}$$

The indices here are chosen to match the proof for the binary case [35].

Now the term

$$\sum_{k_1,k_2} W_{i,k_1} W_{k_2,j} f(|k_1|) = \sum_{k_1} W_{i,k_1} f(|k_1|) \sum_{k_2} W_{k_2,j}$$

where the last term equals 0 when $j > 0$. Similarly,

$$\sum_{k_1,k_2} W_{i,k_1} W_{k_2,j} f(|k_2|) = 0 \text{ when } i > 0.$$

Thus, considering $i, j > 0$, we have

$$\begin{aligned} c_{i,j} &= \sum_{k_1,k_2} W_{i,k_1} W_{k_2,j} \sum_{k_3=1}^{L-1} h(|k_1| - \Delta_{k_1,k_2,k_3}) \\ &\quad + \sum_{k_1,k_2} W_{i,k_1} W_{k_2,j} \sum_{k_3=1}^{L-1} h(|k_2| + \Delta_{k_1,k_2,k_3}). \end{aligned} \tag{A.2}$$

Consider the second sub-expression in (A2) and let

$$k_4 = |k_2| + \Delta_{k_1,k_2,k_3}.$$

Note that k_4 represents the number of non-zeros in an offspring of recombination. Calling this offspring k_5 (i.e., $k_4 = |k_5|$), the strings k_1 and k_2 can be expressed in terms of k_5. With this change of indices, the second sub-expression of (A2) above can be written as

$$\sum_{k_3=1}^{L-1} \sum_{k_4=0}^{L} h(k_4) \sum_{\substack{k_1,k_2 \\ |k_2|+\Delta_{k_1,k_2,k_3}=k_4}} W_{i,k_1} W_{k_2,j}$$
$$= \sum_{k_3=1}^{L-1} \sum_{k_4=0}^{L} h(|k_4|) \sum_{\substack{k_5=0 \\ |k_5|=k_4}}^{K^L-1} \sum_{k_6=0}^{(K^{k_3}-1)} \sum_{k_7=0}^{(K^L-k_3-1)}$$
$$W_{i,(k_5 \bmod K^{k_3} \oplus k_7 K^{k_3})} W_{(K^{k_3} \lfloor k_5 K^{-k_3} \rfloor \oplus k_6),j}$$
$$= \sum_{k_3=1}^{L-1} \sum_{k_4=0}^{L} h(|k_4|) \sum_{\substack{k_5=0 \\ |k_5|=k_4}}^{K^L-1} W_{i,(k_5 \bmod K^{k_3})} W_{K^{k_3} \lfloor k_5 K^{-k_3} \rfloor,j}$$
$$\sum_{k_6=0}^{(K^{k_3}-1)} W_{k_6,j} \sum_{k_7=0}^{(K^{L-k_3}-1)} W_{i,k_7 K^{k_3}}.$$

Now, using Proposition 6, part (2), we have

$$\sum_{k_6=0}^{K^{k_3}-1} W_{k_6,j} = K^{k_3} \text{ if } \sum_{p=L-k_3+1}^{L} \delta(d_p(j)) = 0, \text{ and 0 otherwise.}$$

Next consider

$$\sum_{k_7=0}^{K^{L-k_3}-1} W_{i,k_7 K^{k_3}} = \sum_{k_7=0}^{K^{L-k_3}-1} \left(\prod_{p=0}^{L} W_{d_p(i),d_p(\mathrm{rev}(k_7 K^{k_3}))} \right)$$

and let $\mathrm{rev}(k_7 K^{k_3}) = r$. Note that r has its upper k_3 digits equal to 0, so the Walsh terms are equal for $p = (L - k_3 + 1)$ to L. The expression above can thus be written as

$$\sum_{k_7=0}^{K^{L-k_3}-1} W_{i,k_7 K^{k_3}} = \sum_{r=0}^{K^{L-k_3}-1} \left(\prod_{p=0}^{L-k_3} W_{d_p(i),d_p(\mathrm{rev}(r))} \right)$$
$$\sum_{k_7=0}^{K^{L-k_3}-1} W_{i,k_7,K^{k_3}} = K^{L-k_3} \text{ if } \sum_{p=0}^{L-k_3} \delta(d_p(i)) = 0, \text{ and 0 otherwise.}$$

Using Proposition 6, part (1), the second sub-expression of (A2)

$$\sum_{k_1,k_2} W_{i,k_1} W_{k_2,j} \sum_{k_3=1}^{L-1} h(|k_2| + \Delta_{k_1,k_2,k_3})$$

equals 0, except where the upper k_3 digits of j are 0 and where the lower $L - k_3$ digits of i are 0. Thus, the expression takes nonzero values only where $j < \gcd(i, N)$. Similarly, the other sub-expression of (A2)

$$\sum_{k_1,k_2} W_{i,k_1} W_{k_2,j} \sum_{k_3=1}^{L-1} h(|k_1| - \Delta_{k_1,k_2,k_3})$$

is nonzero only for $i < \gcd(j, N)$.

Now, returning to (A1), since $C_{i,j} = 0$ when $i \geq \gcd(j, N)$ and $j \geq \gcd(i, N)$, it remains to be shown that $j > i$ implies $C_{j,i \oplus j} = 0$. This is obtained exactly as for the binary case (Vose and Liepins, 1991) and so the proof is complete.

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