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Parallel Chip Firing on Digraphs

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Abstract. Given some multidigraph, a *state* is any distribution of some chips on its vertices. We transform this initial state step by step. Every vertex checks whether it is able to send one chip through every outgoing arc. If it can, it does; otherwise it does not send any chip. All vertices check and send in parallel. Finally, at every vertex all incoming chips are added to the remaining chips. This transformation on the set of states is iterated.

If the digraph and the total number of chips are finite, then we finally arrive at some periodic configuration. Here we investigate how these periodic configurations depend on the digraph and the total number of chips. There is a sharp contrast in the behavior for Eulerian digraphs (where the in-degree of each vertex equals its out-degree) and non-Eulerian digraphs.

1. Introduction

We define multidigraphs as directed graphs $D = (V, A)$ having multiplicities on the arcs, that is, with some mapping $\mu : A \to \mathcal{N}^*$ from the arc set onto the set of positive integers. The *out-neighborhood* $N^+(x)$ of a vertex x is the set of all vertices y with $xy \in A$. The *out-degree* $d^+(x)$ of x is $\sum_{xy \in A} \mu(xy)$. *In-neighborhoods* and *in-degrees* $d^-(x)$ are defined in the same way. We assume throughout that D is *locally finite*, meaning that $d^+(x)$ and $d^-(x)$ are finite for every vertex x. *Eulerian* multigraphs have the property that every vertex has equal in- and out-degree. Figure 1 shows two multidigraphs, the right one Eulerian. Undirected multigraphs can be viewed as symmetric multidigraphs without multiple arcs, or as special Eulerian multidigraphs.

A *state* is any mapping $f : V \to \mathcal{N}$. Informally, it can be viewed as any distribution of some chips on the vertices of the graph. For any state f let the *activity* $a_f(x)$ of a vertex x be defined as 1 if $f(x) \geq d^+(x)$ and as 0 otherwise; we call these vertices x having $a_f(x) = 1$ *active* and the others *passive*.

Given any state, we derive a new state $\Phi(f)$ using the rule

$$\Phi(f)(x) := f(x) - a_f(x)d^+(x) + \sum_{y \in N^-(x)} a_f(y)\mu(yx).$$

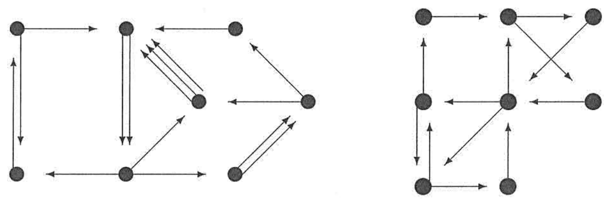

Figure 1: Two multidigraphs, the right one Eulerian.

This transformation can be interpreted as follows. On every vertex x there are $f(x)$ chips. Every vertex wants to send one chip to every out-neighbor. If this is not possible (since $f(x) < d^+(x)$), it resigns and sends nothing. Otherwise the vertex is active, and sends these chips. The chips arrive at their destinations, and we have a new state. Note that the total number of chips $N = \sum_{v\in V} f(v)$ does not change during the transformation. It is obvious that loops do not play a great role in the process. If there is a loop xx we get a similiar process by deleting the loop and deleting $\mu(xx)$ of the chips that lie on x, so we assume in the following that there are no loops.

We now iterate this transformation. Define, as usual, $\Phi^n(f)$ as $\Phi(\Phi^{n-1}(f))$ for $n \geq 2$, and $\Phi^1(f) := \Phi(f)$. To simplify notation, we write $\Phi^n f$ instead of $\Phi^n(f)$, and $\Phi^n f(x)$ instead of $(\Phi^n(f))(x)$. A state f is *periodic* if $\Phi^p f = f$ for some integer n; the smallest such integer is called the *period* $per(f)$ of f. Instead of "periodic with period k" we say k*-periodic*, and instead of "1-periodic" we say *fixed*.

In general there may be no periodic states, even for a finite number N of chips. The simplest example is the one-way infinite directed path, with one chip on some vertex. However, for finite D, the total number of states with a fixed (finite) number N of chips is finite. In this case, every initial state eventually becomes periodic.

Bitar and Goles investigated this parallel chip firing process on finite, undirected trees, and showed that only period 1 or 2 occur in that case [2]. In this paper we investigate the periodic states in general (finite) multidigraphs. More precisely, we ask how the possible period lengths depend on the total chip number N, given a fixed finite multidigraph. It turns out that *Eulerian* multidigraphs show a very specific behavior; most notably the resulting pattern between possible periods and N is symmetric. For non-Eulerian multidigraphs, the basis of the null space of the Laplacian of the multidigraph is an important tool for excluding certain numbers as periods. Moreover, it is possible to show that the period is not bounded by some polynomial in the size of the digraph. Note that all undirected multigraphs (that is, symmetric multidigraphs) are Eulerian multi*di*graphs.

The corresponding chip firing *games* of graphs or digraphs have been investigated in [1, 3, 4, 5, 6, 9]. In these games, at any state only one active vertex fires, and we have the choice of which active vertex to choose.

2. Active or passive vertices

2.1 Forever active or passive vertices

Let some state be given. A vertex is called *forever active* or *forever passive* if it is active or passive, respectively, in all future states.

Remark 2.1. *In periodic states all in-neighbors of every forever-passive vertex are forever passive.*

Proof. Assume there is such an arc xy, where y is forever passive but x is not. Since our state f is periodic, there are infinitely many states $\Phi^n f$ where x is active, that is, x fires a chip toward y. y never loses chips; nevertheless, at any time there are fewer than $d^+(y)$ chips at y. By the assumption $d^+(y)$ is finite, a contradiction. ∎

In this paper we shall concentrate on strongly connected digraphs because the problem reduces to this case in the following sense. The *condensation* of the digraph D has all strong components of D as vertices, and an arc from vertex x to vertex y if and only if there is some arc from some vertex of the component corresponding to x toward some vertex in the component corresponding to y. A *sink component* is a strong component that forms a sink in the condensation of D.

Proposition 2.2. *Let the total number of chips $N = \sum_{x \in V} f(x)$ be finite. In every periodic state, all vertices in nonsink components are forever passive.*

Proof. Choose any strong component that is no sink component. This means that there must be some vertex $x \in V(Q)$ and some arc $xy \in A$ such that there is no directed path from y to x. Let B denote the set of all vertices from which there is some directed path toward x, surely $y \notin B$. Since there is no arc from $V \setminus B$ toward B, this set B never gains chips. Since the total number $\sum_{z \in B} f(z)$ of chips on B is finite, this implies that for a periodic state f, there should not be any loss of chips on B. This implies that x must be totally forever passive. Remark 2.1 implies that all vertices in B, and in particular in Q, must be forever passive, and it suffices to treat chip firing on strongly connected digraphs. ∎

The periodic states on acyclic digraphs can be characterized as follows.

Corollary 2.3. *Only period 1 occurs in acyclic multidigraphs for finite N. The fixed states are those in which all nonsink vertices are passive.*

Having located the forever-passive vertices from the structure of D alone, what can be said about forever-active vertices?

Proposition 2.4. *Let N be finite.*

(a) *No vertex with $d^+(x) > d^-(x)$ is forever active.*

(b) *A vertex x with $d^+(x) = d^-(x)$ is forever active if and only if there is some n such that all its in-neighbors are forever active in $\Phi^n f$.*

Proof. Let x be forever active and $d^+(x) \geq d^-(x)$. We get $f(x) \geq \Phi f(x) \geq \Phi^2 f(x) \geq \cdots$, and since $f(x) \leq N$ is finite, there is some n with $\Phi^n f(x) = \Phi^{n+1}(x) = \cdots$. This implies that for every $j \geq n$, x has $d^+(x)$ active in-neighbors; that is, $d^+(x) = d^-(x)$ and all in-neighbors of x are forever active at state $\Phi^n f$. ∎

Lemma 2.5. *Let the vertex x be not totally active in the periodic state f. If $d^+(x) \geq d^-(x)$, then $f(x) \leq 2d^+(x) - 1$.*

Proof. If x is passive at $\Phi^{per(f)-1} f$, then

$$f(x) = \Phi^{per(f)} f(x) \leq \Phi^{per(f)-1} f(x) + d^-(x) \leq d^+(x) - 1 + d^-(x)$$
$$= 2d^+(x) - 1.$$

If x is active at $\Phi^{per(f)-t+1} f, \ldots, \Phi^{per(f)-1} f$, but active at $\Phi^{per(f)-t} f$, then

$$f(x) \leq \Phi^{per(f)-1} f(x) \leq \cdots \leq \Phi^{per(f)-t+1} f(x),$$

but $\Phi^{per(f)-t+1} f(x) \leq 2d^+(x) - 1$ as above. ∎

2.2 Activity sequences

We next look at the sequence $a_f(x), a_{\Phi f}(x), \ldots$ of activities for a certain vertex x. Surely the sequences for all vertices are not independent from each other. The following two lemmas, extensions of lemmas in [2], describe some connection.

Lemma 2.6. *Let $d^+(x) \geq d^-(x)$, and assume that $a_f(x) = 0$ but $a_{\Phi f}(x) = \cdots = a_{\Phi^t f}(x) = 1$. Then there is some $y \in N^-(x)$ with $a_f(y) = \cdots = a_{\Phi^{t-1} f}(y) = 1$.*

Proof. Since x is passive at state f, $f(x) < d^+(x)$. Now assume there is no such in-neighbor y of x as described above. Then until $\Phi^t f$, from every in-neighbor it receives at most $t-1$ chips. On the other hand it loses $(t-1)d^+(x)$ chips, so

$$\Phi^t f(x) \leq f(x) + (t-1)d^-(x) - (t-1)d^+(x)$$
$$= f(x) + (t-1)(d^-(x) - d^+(x))$$
$$\leq f(x) < d^+(x),$$

a contradiction to $a_{\Phi^t f}(x) = 1$. ∎

Lemma 2.7. *Let $d^+(x) \leq d^-(x)$, and assume $a_{\phi f}(x) = \cdots = a_{\Phi^t f}(x) = 0$. Then there is some $y \in N^-(x)$ with $a_f(y) = \cdots = a_{\Phi^{t-1} f}(y) = 0$.*

Proof. Assume there is no such in-neighbor y. Then

$$\begin{aligned}\Phi^t f(x) &\geq f(x) + d^-(x) - a_f(x)d^+(x) \\ &= (f(x) - a_f(x)d^+(x)) + d^-(x) \geq d^+(x),\end{aligned}$$

a contradiction to x being passive in state $\Phi^t f$. ∎

The activity vector $(a_f(x)/x \in V)$ can be viewed as a rough description of the state. It is rough since Φf, and even its activity vector, does not depend on $(a_f(x)/x \in V)$ alone. In particular, if the activity vectors of f and $\Phi^r f$ are identical, those of Φf and Φ^{r+1} may differ. On the other hand, we shall show in the next lemma that if the activity vector of a periodic state is *strictly* periodic with period r, then r equals the period of the state.

Lemma 2.8. *Let f be some periodic state, and let r denote the smallest positive integer for which $a_{\Phi^i f}(x) = a_{\Phi^{r+i} f}(x)$ for every $i \geq 0$ and every vertex x. Then $per(f) = r$.*

Proof. Surely such an integer r exists and is smaller than or equal to $per(f)$. Now let x be any vertex. The difference $\Phi^r f(x) - f(x)$ depends only on the activities of x and all its in-neighbors through the states $f, \Phi f, \ldots, \Phi^{r-1} f$. By the assumption

$$0 = \Phi^{per(f)r} f(x) - f(x) = per(f)(\Phi^r f(x) - f(x)),$$

hence $\Phi^r f(x) = f(x)$. Since this works for every vertex x, $r \leq per(f)$ implies $r = per(f)$. ∎

2.3 Fixed states

We call a state *totally active* or *totally passive* if all vertices are active or passive, respectively. Surely every totally passive state is fixed. Totally active states are fixed if and only if the multidigraph is Eulerian. Moreover, there are no other fixed states.

Theorem 2.9. *Let D be some finite, strongly connected multidigraph.*

(a) If $d^+(x) = d^-(x)$ for every vertex x, then the fixed states are exactly all totally active or totally passive states.

(b) Otherwise the fixed states are exactly the totally passive states.

Proof. Let f be a fixed state. Every passive vertex is forever passive and every active vertex is forever active. If there is some (forever) passive vertex at f, then by Remark 2.1 all vertices must be forever passive, since D is strongly connected. So f is totally passive in that case. Otherwise, if there is no passive vertex, then all vertices are active and f is totally active. However, by Proposition 2.4 this is impossible if not all vertices have equal in- and out-degree—at least one vertex must have out-degree greater than in-degree in that case. ∎

3. Average activity and the Laplacian

We assume throughout this section that the vertex set of D is finite and ordered as $\{v_1, v_2, \ldots, v_n\}$. The *Laplacian* L of D is the $n \times n$ matrix (L_{ij}) with

$$L_{ij} := \begin{cases} \mu(v_j v_i) & \text{if } v_j v_i \in A \text{ and } i \neq j \\ 0 & \text{if } v_j v_i \notin A \text{ and } i \neq j \\ -d^+(v_i) + \mu(v_i v_i) & \text{if } i = j. \end{cases}$$

Thus the Laplacian is constructed from the adjacency matrix by subtracting the out-degrees on the diagonal and transposing the matrix. Having been very useful in the investigation of the chip firing game in [3] and [4], the Laplacian is also applicable in the parallel process.

Let f be some p-periodic state. We define the *average activity* $aa_f(x)$ of a vertex x as $\sum_{t=0}^{p-1}(1/p)a_{\Phi^t f}(x)$.

Proposition 3.1. *Let f be some periodic state for finite, strongly connected D. Then the average activity vector $(aa_f(v_i))_{i=1,\ldots,n}$ lies in the null space (kernel) of the Laplacian of D.*

Proof. Let v_i be any vertex. During the whole period $d^+(v_i)aa_f(v_i)per(f)$ chips start at v_i. From every in-neighbor v_j, $\mu(v_j v_i)aa_f(v_j)per(f)$ chips arrive at v_i. Then $d^+(v_i)aa_f(v_i) = \sum_{j=1,\ldots,n/v_j v_i \in A} aa_f(v_j)\mu(v_j v_i)$, or in other words, $(aa_f(v_1), \ldots, aa_f(v_n)) \times L^T = \mathbf{0}$. ∎

In [3] it was shown that this null space is one-dimensional in the strongly connected case, and has the vector $(1, 1, \ldots, 1)$ as a basis provided that D is strongly connected and Eulerian.

Corollary 3.2. *For every strongly connected, Eulerian multidigraph, and for every periodic state f, all average activities of the vertices are equal.*

For non-Eulerian multidigraphs, there is some vector $(\ell_1, \ell_2, \ldots, \ell_n)$ in the null space of L with all ℓ_i integers, and $\gcd(\ell_1, \ldots, \ell_n) = 1$. This unique vector is called the *primitive* vector for D.

Corollary 3.3. *Let $(\ell_1, \ldots, \ell_n)$ denote the primitive vector for the strongly connected finite multidigraph D. Then no period smaller than $\max_{i=1,\ldots,n} \ell_i$ occurs except 1. If this period $\max_{i=1,\ldots,n} \ell_i$ occurs for N chips, then it also occurs for $N+1$ chips.*

A typical example for the behavior of non-Eulerian multidigraphs is presented in the left part of Figure 1, and has $(2, 3, 1, 1, 1, 4, 2, 1)$ as its primitive vector. Thus no period 2 or 3 occurs there. Figure 2 indicates which periods occur for the various values of N, based on thousands of computations on the multidigraph. If period p has been found for N chips, then '♣' stands in the corresponding entry (N, p) of the table.

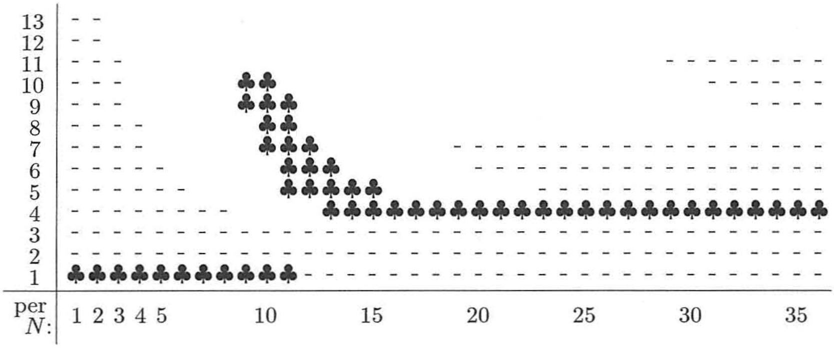

Figure 2: Possible periods versus number of chips N, for the multidigraph in the left part of Figure 1.

A few more remarks about these computations are appropriate. I chose about 100 multidigraphs, all with fewer than 15 vertices. For every fixed multidigraph D and every total chip number N in the range considered, I chose about 1000 (from 100 up to 100,000, depending on the size of the digraph and the first results) initial states at random. Thereby I chose two different models, since the natural one ("For every chip, put it on a vertex at random with all probabilities equal") resulted in very equal distributed distributions. Each such initial state was transformed, until we arrived at a periodic state.

4. Bounds for N

Let f be a periodic state. We want to derive upper and lower bounds for $\sum_{i=1,\ldots,per(f)} \Phi^i f(x)$ by local observations, for every vertex x. Since $per(f)N = \sum_{x \in V} \sum_{i=1,\ldots,per(f)} \Phi^i f(x)$, such bounds would yield upper and lower bounds for the total number of chips N in terms of $per(f)$ (or conversely).

Which sequences $f = f_0, f_1, \ldots, f_{per(f)-1}, f_{per(f)}$ are possible with $f_{i+1} = \Phi f_i$? The only restrictions we consider are local at x, that is,

1. $f_i \leq f_{i+1} \leq f_i + d^-$ for $f_i < d^+$,

2. $f_i - d^+ \leq f_{i+1} \leq f_i - d^+ + d^-$ for $f_i \geq d^+$, and

3. exactly pa of the numbers are greater than or equal to d^+,

where all indices are taken modulo p, and where pa, p, d^+, and d^- are abbreviations for $per(f)aa_f(x)$, $per(f)$, $d^+(x)$, and $d^-(x)$, respectively. Let us investigate how large or small the sum of p integers $\sum_{i=0,\ldots,p-1} f_i$ can be under these restrictions.

(I) Let us begin by searching for the minimum. If $d^+ \leq d^-$, then it is easy to see that this minimum is pad^+: we choose $f_0 = f_1 = \cdots = f_pa = d^+$, and $f_{pa+1} = \cdots = f_{p-1} = 0$. If $d^+ > d^-$, then we assume $2pa \leq p$. In this case it is not difficult to show that the minimum is

$$h(p, pa, d^+, d^-) := pa(d^+ + (d^+ - d^-)) + \sum_{i=1}^{p-2pa} \max\{d^+ - (1+i)d^-, 0\} \tag{1}$$

Since $per(f)aa_f(x)$ must be equal to or a multiple of the corresponding value in the primitive vector, we get the following.

Proposition 4.1. *Let f be some periodic state, and let $(\ell_1, \ldots, \ell_n)$ be the primitive vector of the finite strongly connected multidigraph D. Then*

$$per(f)N \geq \sum_{i=1,\ldots,n} h(per(f), \ell_i, d^+(v_i), d^-(v_i)),$$

where $h(\ldots)$ is defined by equation (1) if $d^+(x) > d^-(x)$, and equals $per(f)\ell_i d^+(v_i)$ otherwise.

(II) On forever-active vertices there may lie arbitrarily many chips in periodic states. Thus, if $aa_f(x) = per(f)$, then there is no maximum value for the problem. This stands in sharp contrast to the behavior of vertices x that are *not* forever active. If the state f being considered is periodic, then $f(x)$ must be bounded by some constant that depends on $per(f)$, $d^+(x)$, $d^-(x)$, and $aa_f(x)$; thus there must be some maximum value of $\sum_{i=0,\ldots,p-1} f_i$.

It is not difficult to show that for $d^+ \leq d^-$ the maximum is achieved by clustering the positive and negative activities. We start with $f_0 = d^+ - 1$, going as steep as possible upwards, but keeping in mind that we must be under d^+ for $i \geq pa + 1$. Surely we should not go below $d^+ - 1$. Thus we set $f_0 = f_{pa+1} = \cdots = f_{p-1} = d^+ - 1$. The first restriction yields

$$\begin{aligned} f_i &\leq (d^+ - 1) + d^- + (i-1)(d^- - d^+) \\ &= id^- + (2-i)d^+ - 1 \qquad \text{for } i \geq 1, \end{aligned}$$

and the second restriction yields

$$f_i \leq (d^+ - 1) + (pa + 1 - i)d^+ = (pa + 2 - i)d^+ - 1 \qquad \text{for } i \leq pa.$$

The rational point s, where these two restrictions intersect, is

$$s = pa\frac{d^+}{d^-}\,. \tag{2}$$

If we choose all f_i on these two restriction lines, we find after some arithmetic

$$g(p, pa, d^+, d^-) := p(d^+ - 1) + d^+\left(\frac{(pa)^2}{2} + pa\left(\frac{1}{2} - \lfloor s \rfloor\right)\right) + d^-\binom{\lfloor s \rfloor + 1}{2} \tag{3}$$

as the maximum value for $\sum_{i=0,\dots,p-1} f_i$ (using the convention $\binom{1}{2} = 0$).

For $d^- < d^+$ we abbreviate

$$t := \left\lfloor \frac{pa}{p - pa} \right\rfloor, \tag{4}$$

thus obtaining

$$\begin{aligned} g(p, pa, d^+, d^-) &:= p(d^+ - 1) + pad^- \\ &\quad + (d^+ - d^-)t\left(pa - \frac{1}{2}(p - pa)(1 + t)\right) \end{aligned} \tag{5}$$

Proposition 4.2. *For every periodic state f without forever-active vertices, we get*

$$per(f)N \leq \sum_{x \in V} g(per(f), per(f)aa_f(x), d^+(x), d^-(x)),$$

where g is defined by equations (2) and (3) if $d^+(x) \leq d^-(x)$, and by equations (4) and (5) otherwise.

What about upper bounds for $aa_f(x)$? Let $(\ell_1, \dots, \ell_n)$ be the primitive vector of a strongly connected finite multidigraph. There must be some integer k such that $per(f)aa_f(v_i) = k\ell_i$ for every $i = 1, \dots, n$. Surely $k \leq per(f)/\max_{i=1,\dots,n} \ell_i$. If the period is not a multiple of $\max_{i=1,\dots,n} \ell_i$, then there is no forever-active vertex.

Corollary 4.3. *Let f be periodic in the finite strongly connected multidigraph D. Let $(\ell_1, \dots, \ell_n)$ be the primitive vector. If $per(f)$ is not divisible by $\max_{i=1,\dots,n} \ell_i$, then*

$$per(f)N \leq \sum_{i=1,\dots,n} g(per(f), k\ell_i, d^+(v_i), d^-(v_i)),$$

with $k = \lfloor per(f)/\max_{i=1,\dots,n} \ell_i \rfloor$ and g as above.

There is another very useful bound, which has been given in a "note added in proof" in [3]. Let the *feedback number* $\varphi(D)$ of a strongly connected multidigraph $D = (V, A, \mu)$ be the minimum $\sum_{a \in A'} \mu(a)$ for every set A' of arcs whose removal destroys all directed cycles. For instance, the left multidigraph of Figure 1 has feedback number 3, while the right one has feedback number 4.

Theorem 4.4 ([3]) *For $N < \varphi(D)$ chips, only period 1 occurs.*

We are now able to explain the minus signs in Figure 2. They indicate the situation in which, for a given chip number N, period p is impossible according to Theorem 2.9, Corollary 3.3, the bounds above, and Theorem 4.4. If nothing stands at the entry, then by computational experiences on random instances almost surely no period p occurs for N chips, but I do not know why.

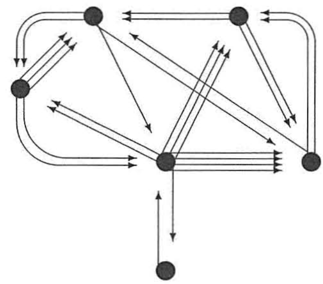

Figure 3: Some digraph that allows relatively large periods.

5. Large periods

Given a finite multidigraph $D = (V, A, \mu)$ and a finite number N, there are exactly $\binom{N+|V|-1}{N}$ states. Consequently, as a first very rough result, the period of any periodic state is bounded by this number. But, with the results of section 4 we are in a position to give also lower bounds for periods for non-Eulerian multidigraphs. For instance, the multidigraph in Figure 3 has $(328, 235, 160, 72, 362, 72)$ as its primitive vector (with the usual row-wise labeling of its vertices). For $N > \mu(A) - |V| = 21$ vertices, there is no fixed (totally passive) state, hence every periodic state has period at least 362. For instance, for $N = 22$ the state $(3, 4, 6, 4, 5, 0)$ is 464-periodic. Compared with the total number $\binom{27}{5} = 80{,}730$ of states, this seems large enough.

We shall show that the period is not bounded by some polynomial in $\mu(A)$. For this purpose we construct a family of digraphs, even without multiple arcs, that is, with the constant function $\mathbf{1}$ as μ. For an integer $k \geq 2$ let the digraph D_k be constructed from the directed 3-cycle by replacing one of its vertices by k copies. For integers $k, b_1, b_2, \ldots, b_k \geq 2$ we construct the digraph $D(b_1, b_2, \ldots, b_k)$ from the disjoint union of $D_{b_1}, \ldots, D_{b_k}$ by identifying all vertices with in-degree greater than 1. Figure 4 shows an example.

After some suitable relabeling as in Figure 4, it is easy to see that a vector $\mathbf{v} = (v_1, v_2, \ldots, v_{1+\sum b_i})$ that satisfies $L \times \mathbf{v} = \mathbf{0}$ is defined by $v_1 := \prod_{j=1}^{k} b_j$ and $v_2, \ldots, v_{1+b_1} := c_1, v_{2+b_1}, \ldots, v_{1+b_2} := c_2, \ldots,$ where $c_i := \prod_{j=1,\ldots,k, j\neq i} b_j$. There is no common divisor if the b_i are different prime numbers.

Theorem 5.1. *There is no polynomial $h(n)$ such that $per(f) \leq h(n)$ for every periodic state f in a digraph $D = (V, A)$ with n arcs.*

Proof. Assume on the contrary there is such a polynomial. Then we also find such a polynomial of the form $h(n) = Mn^k$. We choose $k+1$ "large" prime numbers $p_1 < p_2 < \cdots < p_{k+1}$ that are "close together." "Large" means $M2^{k(k+3)} < p_1$, and "close together" means $p_{i+1} \leq 2p_i$ for every $i = 1, \ldots, k$—this is possible by Bertrand's postulate ([7], Theorem 418).

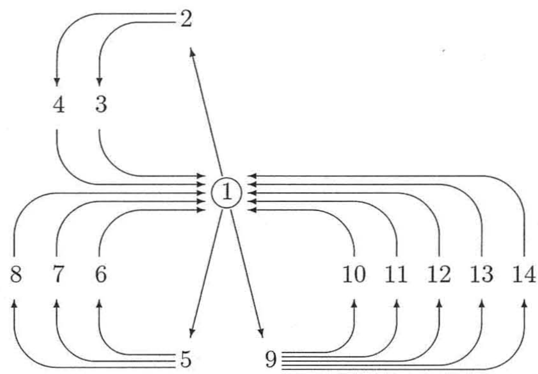

Figure 4: The digraph $D(2,3,5)$.

Then the digraph $D(p_1, p_2, \ldots, p_{k+1})$ will serve as a counterexample. When taking $N > |A| - |V|$ chips, any periodic state f has period

$$per(f) \geq \prod_{i=1}^{k+1} p_i \geq p_1^{k+1}.$$

On the other hand, $|A| = k+1+2\sum_{i=1}^{k+1} p_i$. Applying the "closeness" property several times, we obtain

$$|A| \leq k+1+2^{k+2}p_1 \leq 2^{k+3}p_1$$

and

$$h(|A|) \leq M(2^{k+3}p_1)^k = M2^{k(k+3)}p_1^k < p_1^{k+1} \leq per(f),$$

since p_1 is large enough, yielding a contradiction to our assumption in this way. ∎

Recently Kiwi, Ndoundam, Tchuente, and Goles [8] have given examples of superpolynomial periods for the chip firing process for undirected graphs, so even for a primitive vector $(1,1,\ldots,1)$ this is possible for certain N.

Problem 5.2. *Given integers n and m, what is the largest entry in some primitive vector of some strongly connected (multi)digraph with n vertices and m arcs? What is the largest period occurring in such a strongly connected (multi)digraph?*

As an aside, is it true that the multidigraph of Figure 3 may serve as a calendar (after some initial time), when using 28 chips? Is it true that only period 365 occurs there?

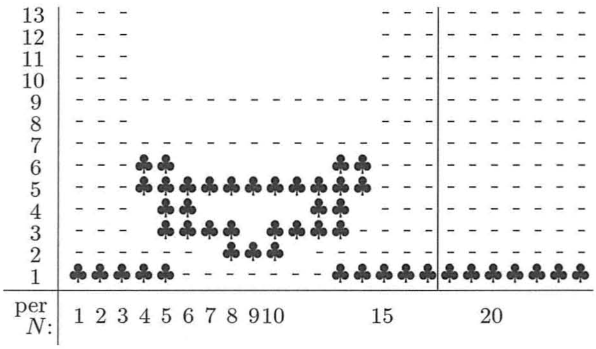

Figure 5: Possible periods versus number of chips N, for the right digraph in Figure 1.

6. Eulerian multidigraphs

It has been emphasized several times so far that the behavior of Eulerian multidigraphs is quite different from arbitrary multidigraphs. A typical example of the relation between period and N in the Eulerian case is given in Figure 5. The most striking difference appears to be that the pattern is symmetric along the axis $N = \mu(A) - |V|/2$. Moreover, it turns out that the periods are relatively small, and the period-2 line is connected.

6.1 Symmetry

Chip firing on Eulerian multidigraphs is only interesting with $N < 2\mu(A) - |V|$ chips. The more precise bound depends on its *minimum in-degree* $\delta^-(D)$.

Theorem 6.1. *The only periodic states for Eulerian multidigraphs $D = (V, A, \mu)$ with $N > 2\mu(A) - |V| - \delta^-(D)$ are the totally active states. Only period 1 occurs.*

Proof. Assume there is some periodic state f that is not totally active. Then there is at least one passive vertex y, and

$$N = f(y) + \sum_{x \in V, x \neq y} f(x) \leq d^+(y) - 1 + \sum_{x \in V, x \neq y} 2d^+(x) - 1 \leq 2\mu(A) - |V| - \delta^-(D)$$

by Lemma 2.5. ∎

Let f be some state with $f(x) \leq 2d^+(x) - 1$ for every vertex x. We define the *reflection* $\overline{f}$ by $\overline{f}(x) := 2d^+(x) - 1 - f(x)$. Then $\overline{f}$ is again a state.

Proposition 6.2. *Let f be some periodic but not totally active state in some Eulerian multidigraph. Then the reflection $\overline{f}$ is also periodic (with $per(f) = per(\overline{f})$ and $aa_{\overline{f}}(x) = 1 - aa_f(x)$).*

Proof. The results follow from the fact that $\overline{\Phi f} = \Phi \overline{f}$, which can be seen as follows. Note that a vertex is active in f if and only if it is passive in $\overline{f}$. Let x be any vertex. Then

$$\begin{aligned}\Phi f(x) + \Phi \overline{f}(x) &= f(x) - a_f(x)d^+(x) + \sum_{yx\in A} a_f(y)\mu(yx) \\ &+ \quad \overline{f}(x) - a_{\overline{f}}(x)d^+(x) + \sum_{yx\in A} a_{\overline{f}}(y)\mu(yx) \\ &= 2d^+(x) - 1 - d^+(x) + \sum_{yx\in A} \mu(yx) \\ &= d^+(x) - 1 + d^-(x) \\ &= 2d^+(x) - 1. \blacksquare\end{aligned}$$

6.2 Periods and dicycle lengths

This subsection relies on the fact that Lemmas 2.6 and 2.7 apply for all vertices in Eulerian multidigraphs. Throughout let f be some periodic state in such a digraph. We use the abbreviations $a_i(x) := a_{\Phi^i f}(x)$ for $i = 0, 1, \ldots, per(f) - 1$, and all indices are taken modulo $per(f)$. For integers z, let $\overline{z}$ denote that integer $i \in \{0, 1, \ldots, per(f) - 1\}$ congruent to z modulo $per(f)$. The *cyclic distance* of $i, j \in \{0, 1, \ldots, per(f) - 1\}$ is defined by $d(i,j) := \min\{\overline{i-j}, \overline{j-i}\}$.

Note that the lemmas in section 2 do *not* imply that every vertex x has some in-neighbor y having all $a_i(y) = a_{i+1}(x)$. This would be convenient, but unfortunately it applies only in special cases.

We call a maximal subsequence of consecutive 1s or consecutive 0s in the activity pattern of a vertex a *1-block* or *0-block*, respectively. Let r and s denote the length of a longest 1-block and 0-block, respectively, appearing in all these activity sequences of the vertices.

For $i = 1, \ldots, per(f) - 1$, let V_i (respectively W_i) denote the set of those vertices x where a 1-block of length r (respectively 0-block of length s) begins at $\Phi^i f$. That is, $x \in V_i$ if $a_i(x) = \cdots = a_{i+r-1}(x) = 1$, and $y \in W_i$ if $a_i(y) = \cdots = a_{i+r-1}(y) = 0$. Since the 1-blocks being considered have maximal length, $V_i \cap V_j$ is empty if $d(i,j) \le r$. In the same manner, $d(i,j) \le s$ implies $W_i \cap W_j = \emptyset$. By Lemma 2.6 every $x \in V_i$ has some in-neighbor in V_{i-1}, and every $y \in W_i$ has some in-neighbor in W_{i-1}. Consequently all these sets $V_0, \ldots, V_{per(f)-1}, W_0, \ldots, W_{per(f)-1}$ are nonempty.

Now we choose any $x \in V_0$. We find some in-neighbor x_1 in $V_{per(f)-1}$, some in-neighbor x_2 of x_1 in $V_{per(f)-2}$, and so on. Let t be the smallest index for which $x_t \in \{x_0, x_1, \ldots x_{t-1}\}$, say $x_t = x_j$ with $0 \le j < t$. Then $x_t \to x_{t-1} \to \cdots \to x_j = x_t$ is a dicycle of length $t - j$ in D. Since x_t lies in both $V_{per(f)-t}$ and $V_{per(f)-j}$ (with all indices again modulo $per(f)$), we get $d(\overline{t}, \overline{j}) = d(\overline{per(f) - t}, \overline{per(f) - j}) > r$.

Let $c(D)$ denote the length of the longest dicycle in D, if there is one, and ∞ if D is acyclic (note that acyclic digraphs are treated sufficiently in

Corollary 2.3). In the argument above, $c(D) \geq t - j \geq d(\bar{t}, \bar{j}) > r$, and when treating the 0-block case in a similiar way, we get the following.

Lemma 6.3. *For every periodic state in an Eulerian multidigraph D, $r, s < c(D)$ with the notations above.*

This lemma has been proven in [2] for $c(D) = 2$. Together with Lemma 2.8 there follows their result that only period 1 or 2 occurs for undirected trees. On the other hand, if all V_i are disjoint, then $t \equiv j \pmod{per(f)}$ for the dicycle constructed above.

Lemma 6.4. *Let f be a periodic state in the Eulerian multidigraph D. If the activity sequence of every vertex contains at most one 1-block of maximal length r, or if the activity sequence for every vertex contains at most one 0-block of length s, then the length of some dicycle in D must be divisible by $per(f)$.*

Corollary 6.5. *Let f be a periodic state in the strongly connected Eulerian multidigraph D. If there is some block of length at least $\lfloor per(f)/2 \rfloor$, then there must be some dicycle whose length is divisible by $per(f)$.*

Corollary 6.6. *If f is periodic with $per(f) \in \{2, 3, 4, 5, 6, 7, 9\}$ in the Eulerian multidigraph D, then D contains some dicycle whose length is divisible by $per(f)$.*

Proof. Assume D contains no such dicycle. Then $r, s < \lfloor per(f)/2 \rfloor$ by Corollary 6.5. This is impossible for $per(f) = 2$, 3, or 5. In the following we may assume without loss of generality that $s \leq r$.

For $per(f) = 4$ we get $r = s = 1$; consequently the activity sequences of every vertex read either $0, 1, 0, 1$ or $1, 0, 1, 0$, beginning with f, which is a contradiction to Lemma 2.8.

By the same reason $r = s = 1$ is impossible in the case $per(f) = 6$, hence $r = 2$. Then either $s = 1$ and again we have a contradiction to Lemma 2.8, or $s = 2$. The result then follows from Lemma 6.4.

If $per(f) = 7$, then $r = 2$ and $aa_f = 3/7$ or $= 4/7$. In both cases we have contradictions with Lemma 6.4, in the first case for the 1-blocks, in the other for the 0-blocks.

The case $per(f) = 9$ can be treated in the same way. ∎

For instance, in Figure 6 we see the pattern for the cube digraph—the undirected graph on 8 vertices whose edges are the edges of the cube. Since there are no cycles of length 5, 7, 9, and multiples, period 5, 7, and 9 do not occur. Note that period 3 occurs (since the graph contains cycles of length 6). It also follows that no period 7 or 9 occurs or the right digraph in Figure 1.

Note that by the result in [8], the preceding corollary cannot be true for arbitrary periods.

We shall construct periodic states in the following way.

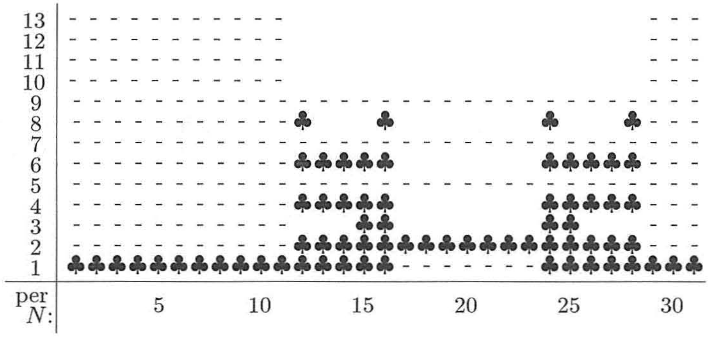

Figure 6: Possible periods versus number of chips N, for the symmetrical cube digraph.

Lemma 6.7. *If there is some partition $V = V_0 \cup V_1 \cup \cdots \cup V_{p-1}$ of the vertex set of an Eulerian multidigraph such that for every $i \in \{0, 1, \ldots, p-1\}$, every vertex in V_i has some in-neighbor in V_{i-1}, then period p occurs (for some appropriate N).*

Proof. For every $i \in \{0, 1, \ldots, p-1\}$ and every $x \in V_i$, let $g(x) := \sum_{y \in V_{i-1}} \mu(yx)$. By the assumption above, all these values are greater than 0. For every $i \in \{0, 1, \ldots, p-2\}$ and every $x_{i+1} \in V_{i+1}$ we define

$$j(x_{i+1}) := \sum_{k=0}^{i-1} \sum_{y \in V_k} \mu(y, x_{i+1}).$$

We simply define states f of period p. For every $i \in \{0, 1, \ldots, p-2\}$ and every $x_{i+1} \in V_{i+1}$ we choose f such that

$$d^+(x_{i+1} - j(x_{i+1}) - g_{i+1}) \leq f(x_{i+1}) < d^+(x_{i+1}) - j(x_{i+1}).$$

For $x_0 \in V_0$ we choose f such that

$$d^+(x_0) \leq f(x_0) < d^+(x_0) + g(x_0).$$

It is not difficult to show that the resulting state is periodic with period p, and the average activity equals $1/p$. ■

In fact, all periodic states with average activity $1/per(f)$ occur with this construction.

Proposition 6.8. *In every finite, strongly connected, Eulerian multidigraph, every divisor of every dicycle length occurs as a period.*

Proof. Assume $x_0 \to x_1 \to \cdots \to x_{pk-1} \to x_0$ is a dicycle of length kp. First we partition the vertices of the dicycle by $V_0^1 := \{x_0, x_p, x_{2p}, \ldots\}, \ldots, V_i^1 := \{x_i, x_{i+p}, x_{i+2p}, \ldots\}, \ldots$. The sets are disjoint, and very $x \in V_i^1$ has some in-neighbor in V_{i-1}^1. Next we define V_0^2 as the union of V_0^1 and all out-neighbors of vertices of V_0^1 outside $V_0^1 \cup \cdots \cup V_{p-1}^1$. If $V_0^2, \ldots, V_i^2$ are defined, we define V_{i+1}^2 as the union of V_{i+1}^1 and all out-neighbors of vertices of V_{i+1}^2 outside $V_0^2 \cup \cdots \cup V_i^2 \cup V_{i+1}^1 \cup \cdots \cup V_{p-1}^1$. By the construction, all these resulting sets $V_0^2, \ldots, V_{p-1}^2$ are disjoint, and every vertex in V_i^2 again has some in-neighbor in V_{i-1}^2. Furthermore $\bigcup_{i=0}^{p-1} V_i^1$ equals the vertex set V of the multidigraph, or is strictly contained in the set $\bigcup_{i=0}^{p-1} V_i^2$, since our multidigraph is strongly connected. Proceeding in this way, we finally arrive at a partition of V that obeys the property stated in Lemma 6.7, and we apply this lemma. ∎

6.3 Period 2

We call *convex* the sets of integers of the form $\{i, i+1, \ldots, j\}$. The following property is unique for period 2.

Proposition 6.9. *Let $D = (V, A, \mu)$ be some finite, strongly connected, Eulerian multidigraph. Then the set of integers N for which period 2 occurs forms a convex subset of the set $\{\lceil \frac{1}{2}\mu(A) \rceil, \ldots, \lfloor \frac{3}{2}\mu(A) - |V| \rfloor\}$.*

Proof. In every periodic state of period 2, all vertices have average activity $\frac{1}{2}$. Then let $V = V_0 \cup V_1$ be any fixed partition of V with the property as in Lemma 6.7.

The total number of chips for which period 2 occurs, given this bipartition, is the convex set

$$\left\{ \mu(A) - \sum_{x \in V_1} g(x), \ldots, \mu(A) - |V| + \sum_{x \in V_0} g(x) \right\}$$

by Lemma 6.7 and the equality $\mu(A) = \sum_{x \in V} d^+(x)$. By the property of the bipartition mentioned above, $\sum_{x \in V_0} g(x) \geq |V_0|$ and $\sum_{x \in V_1} g(x) \geq |V_1|$. Since both sums count disjoint subsets of the arc set, we obtain

$$\sum_{x \in V_0} g(x) + \sum_{x \in V_1} g(x) \leq \mu(A).$$

Since D is Eulerian, $\sum_{x \in V_0} g(x) = \sum_{x \in V_1} g(x)$. Hence

$$\begin{aligned}\frac{1}{2}\mu(A) &\leq \mu(A) - \sum_{x \in V_1} g(x) \leq \mu(A) - \frac{1}{2}|V| \leq \mu(A) - |V| + \sum_{x \in V_0} g(x) \\ &\leq \frac{3}{2}\mu(A) - |V|.\end{aligned}$$

Thus, for every such bipartition, the set of possible N's forms a convex subset of integers in this range, but containing $\mu(A) - \frac{1}{2}|V|$, hence the result. ∎

6.4 Sharper bounds

At least we can give sharper bounds for N for (possible) larger periods. Note that the bounds of section 5 read as

$$aa_f(x)\mu(A) \leq N \leq (1 + aa_f(x))\mu(A) - |V|$$

for Eulerian multidigraphs. (Note also that all average activities $aa_f(x)$ of the distinct vertices are equal here.) So in general we get

$$\frac{1}{per(f)}\mu(A) \leq N \leq \left(2 - \frac{1}{per(f)}\right)\mu(A) - |V|.$$

Lemma 6.10. *Let f be some periodic state in the strongly connected Eulerian multidigraph D, with $per(f)$ longer than the longest dicycle of D. Then $per(f)aa_f(x) \geq 2$. If $per(f)$ is odd, then $per(f)aa_f(x) \geq 3$.*

There is still another bound for the average activities, which may be better than the previous one if $c(D)$ is small and the period large, as follows.

Lemma 6.11. *In every periodic state in any Eulerian multidigraph, $aa_f(x) \geq 1/c(D)$.*

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