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The HeartQuake Dynamic System

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Abstract. The dynamic system presented in this paper is derived from a deterministic game procedure based on a pack of playing cards and two players. We describe the game, give a generalization with its mathematical model, and show some properties of the related dynamic system. A bidimensional geometrical representation is also proposed which exhibits morphological properties, for example, self-similarity.

1. Introduction

HeartQuake is my translation of the name of a traditional Italian game of cards, "*strappacuore*" (pronounced strap'pa'kow:re), which is played by two opponents with the forty cards of the "neapolitan" pack. HeartQuake is a deterministic game: once a shuffling is distributed to the players, the outcome of the game is decided. Moreover, its rules are so naive that it can be played by children. Though the playing of the game is simple, it is quite difficult for an external watcher to foresee the outcome of the game at any moment because of the ease with which the fates of the game rapidly change players. This is probably the main reason for the name and the fortune of this game, and surely is the reason for my decision to start investigating it. Such an investigation is the "heart" of this paper, which follows a description of the original game and a generalization with a formal modelling, some observations on a bidimensional representation of the game, and a conclusion.

2. A simple game of cards

The traditional game HeartQuake is based on the 40 playing cards in the so-called neapolitan pack. Two players are involved, let's call them *Neal* and *Jack*. After a shuffling, 20 cards are given to Neal, 20 to Jack. All cards are unknown to both. A table is available to put cards on. Players are asked to turn the card on top of their pack. Cards belong to two classes: "good" cards (i.e., the four aces, the four twos, and the four threes), and "bad" cards (i.e., the rest).

Neal starts the game by turning the card on top of his pack. The turned card is placed on an initially empty stack face upward. Then it is Jack's turn. This situation goes on while both players turn bad cards and results in a balanced reduction of both players' hoard, increasing the stack on the table. The player who first exhausts their pack and cannot turn loses the game.

As soon as a player, say Neal, turns a good card, say a g, the game enters a second situation in which only Jack reduces his pack: in fact, Jack needs to turn up to g of his cards in search of a good one, if he succeeds the roles swap and it is Neal who needs to exit this dangerous condition. If Jack fails to turn up a good card within g attempts all cards stacked on the table go on the bottom of Neal's pack. Whoever wins the cards, the first situation starts again—they have to turn. If a player exhausts their pack in search of a good card, the game is over and the opponent wins.

3. Generalization and formal definitions

We generalize the game considering a multiset (or pack) of $2 \times n$ cards, $P = \{c_1, c_2, \ldots, c_{2n}\}$. With $C_0, C_1, \ldots, C_m$ such that $\cup_{i=0}^{m} C_i = P$ and $C_m \neq \emptyset$.

Definition 1. *The list of $m + 1$ classes*

$$H = C_0, C_1, \ldots, C_m$$

is called a HeartQuake game.

A card in C_0 is called a bad card, or a zero. A card in C_g, $g \neq 0$ is called a good card, or wherever its value is meaningful, a g.

Definition 2. *Given a HeartQuake game, ν_i is the number of cards in its C_i class, $i = 0, \ldots, m$.*

As a consequence, the number of different configurations for the pack can be expressed as

$$p(\nu_0, \ldots, \nu_m) = \binom{2n}{\nu_0, \nu_1, \ldots, \nu_m} = \frac{(2n)!}{\prod_{i=0}^{m} \nu_i!},$$

that is, as the number of permutations with repetitions.

Let c be any card of a HeartQuake pack. Where this makes sense, we will use $\underline{c}$ to say that c is a covered card, and $\overline{c}$ to say the contrary.

Let S_N, S_J, and S_t be three stacks:

S_N is the stack of covered cards owned by Neal,

S_J is the stack of covered cards owned by Jack, and

S_t is the stack of uncovered cards on the table.

Let S represent any of these stacks; then $\underline{S}$ means that S is a stack of uncovered cards, while $\overline{S}$ means the opposite. An empty stack is indicated as $()$.

Definition 3. *Given a HeartQuake game, the ordered triple*

$$(S_N, S_J, S_t)$$

is called a play configuration. *X is the set of all possible play configurations.*

Definition 4. *Given a HeartQuake game H, the set*

$$M = \{(S_N, S_J, S_t) \in X \mid \nu(S_N) = \nu(S_J) = n\}$$

is called the set of the matches *(or* points*) for H.*

Note that if $(S_N, S_J, S_t) \in M$ then $S_t = ()$.

Definition 5. *Given any HeartQuake game H,* set *Q is the following set of $2m + 3$ symbols:*

$$Q = \{q_N^0, q_J^0, q_N^1, q_J^1, \ldots, q_N^m, q_J^m, \Phi\}.$$

Now let's define the "$\leftarrow$" sign, that is, an infixed "overloaded" operator, or in other words, an operator having different meanings in different contexts.

Definition 6. *Given any card c and any two stacks S_1 and S_2, the "$\leftarrow$" sign is used to indicate the following five instructions.*

$c \leftarrow S_1$ (pick up)*: a "pop" from stack S_1; a player picks up their top card, c; if the stack is empty, a special "empty" card $\sqcup$ is emitted.*

$\overline{c} \leftarrow \underline{c}$ (turn)*: covered card $\underline{c}$ is turned and becomes uncovered card $\overline{c}$.*

$S_1 \leftarrow c$ (put down)*: a "push" of c onto stack S_1.*

$\underline{S_1} \leftarrow \overline{S_1}$ (overturn)*: stack of uncovered cards S_1 is turned over and becomes a stack of opposite ordered, covered cards.*

$S_1 \leftarrow S_2$ (tail after)*: S_2 is tailed after S_1; then S_2 is emptied.*

Of course, context determines which opcode has to be performed.

Such opcodes are clearly Turing machine-computable. We use the instructions in Table 1 to compute the output values $\dot{q} \in Q$ and $\dot{x} \in X$ for any input value $x \in X$ and $q \in Q$. By means of this effective procedure it is therefore possible to define the two functions δ and ω.

Definition 7. *Let $q \in Q$ and $x \in X$. Let $\dot{q}$ and $\dot{x}$ be the output values of procedure hq with input values q and x. Then the functions*

$$\delta : Q \times X \to Q \quad \text{and} \quad \omega : Q \times X \to X$$

are defined so that

$$\delta(q, x) = \dot{q} \quad \text{and} \quad \omega(q, x) = \dot{x}.$$

Note that functions δ and ω are computable by construction.

Table 1: Instruction table for procedure **hq**.

procedure hq: input value $q \in Q, q = \Phi$ or $q = q_p^k, k \in \{0, 1, \ldots, m\}$,
  $p = N$ or $p = J$;
  input value let $x = (\underline{S_N}, \underline{S_J}, \overline{S_t}) \in X$;
  output variables $\dot{q} \in Q$ and $\dot{x} \in X$;
  inner variable $\neg p = \begin{cases} N & \text{if } p = J \\ J & \text{if } p = N \end{cases}$

begin

  if $q = \Phi$ **then** $\dot{q} = \Phi$, $\dot{x} = \sqcup$, **stop**
  $\underline{c} \leftarrow \underline{S_p}$
  if $\underline{c} = \sqcup$ **then** $\dot{q} = \Phi$, $\dot{x} = \sqcup$, **stop**
  $\overline{c} \leftarrow \underline{c}$
  $\overline{S_t} \leftarrow \overline{c}$
  if $c \in C_0$ **then**
    if $k = 0$ **then**
      $\dot{q} = q^0_{\neg p}$, $\dot{x} = (S_N, S_J, S_t)$, **stop**
    if $k = 1$ **then**
      $\dot{q} = q^0_{\neg p}$
      $\underline{S_t} \leftarrow \overline{S_t}$
      $\underline{S_t} \leftarrow \underline{S_{\neg p}}$
      $\underline{S_{\neg p}} \leftarrow \underline{S_t}$
      $\dot{x} = (S_N, S_J, S_t)$
      stop
    else $\dot{q} = q_p^{k-1}$, $\dot{x} = (S_N, S_J, S_t)$, **stop**
  if $c \notin C_0$ **then**
    $\dot{q} = q^c_{\neg p}$, $\dot{x} = (S_N, S_J, S_t)$, **stop**

end

We are now able to introduce the formal model for HeartQuake.

Definition 8. *Given a HeartQuake game H, automaton $\mathcal{H}$ is defined as the 4-tuple*

$$\langle X, Q, \delta, \omega \rangle$$

in which X is both the set of inputs *and that of the* outputs, *Q is the set of* states, *$\delta : Q \times X \to Q$ is the* next-state function, *and $\omega : Q \times X \to X$ is the* output function *for $\mathcal{H}$.*

Table 2 shows the transitions for $\mathcal{H}$. Note that finiteness of $\mathcal{H}$ depends on finiteness of Q.

Table 2: Transition table of the HeartQuake automata. The Is represent values *picked up* from the stack of the current player, $\sqcup$ is the symbol coming from an empty stack, and "-" is an impossible combination.

$Q \backslash I$	0	1	...	m	$\sqcup$
q_N^0	q_J^0	q_J^1	...	q_J^m	Φ
q_J^0	q_N^0	q_N^1	...	q_N^m	Φ
q_N^1	q_J^0	q_J^1	...	q_J^m	Φ
q_J^1	q_N^0	q_N^1	...	q_N^m	Φ
q_N^2	q_N^1	q_J^1	...	q_J^m	Φ
q_J^2	q_J^1	q_N^1	...	q_N^m	Φ
$\vdots$			$\vdots$		
q_N^i	q_N^{i-1}	q_J^1	...	q_J^m	Φ
q_J^i	q_J^{i-1}	q_N^1	...	q_N^m	Φ
$\vdots$			$\vdots$		
q_N^m	q_N^{m-1}	q_J^1	...	q_J^m	Φ
q_J^m	q_J^{m-1}	q_N^1	...	q_N^m	Φ
Φ	-	-	-	-	Φ

Definition 9. *A HeartQuake automaton $\mathcal{H} = \langle X, Q, \delta, \omega\rangle$ is given. For any $x \in M$, given a fixed initial state, say q_N^0, the series*

$$(\delta_i(x))_{i\in\mathbf{N}} \quad \text{and} \quad (\omega_i(x))_{i\in\mathbf{N}}$$

are uniquely defined and determined as follows

$$\begin{array}{llll}
\delta_0(x) & = q_N^0 & \omega_0(x) & = x \\
\delta_1(x) & = \delta(q_N^0, x) & \omega_1(x) & = \omega(q_N^0, x) \\
\delta_2(x) & = \delta(\delta(q_N^0, x), \omega(q_N^0, x)) & \omega_2(x) & = \omega(\delta(q_N^0, x), \omega(q_N^0, x)) \\
 & = \delta(\delta_1(x), \omega_1(x)) & & = \omega(\delta_1(x), \omega_1(x)) \\
 & \vdots & & \vdots \\
\delta_{k+1}(x) & = \delta(\delta_k(x), \omega_k(x)) & \omega_{k+1}(x) & = \omega(\delta_k(x), \omega_k(x)) \\
 & \vdots & & \vdots
\end{array}$$

Definition 10. *Let $\mathcal{H}$ be a HeartQuake automaton and $x \in M$ one of its matches; then the recursive procedure*

$$\begin{cases} \delta_{k+1}(x) = & \delta(\delta_k(x), \omega_k(x)) \\ \omega_{k+1}(x) = & \omega(\delta_k(x), \omega_k(x)) \end{cases}$$

is called a HeartQuake system.

The recursive procedure in Definition 10 is the mathematical model of a dynamical system. In particular, the series $(\omega_i(x))_{i\in\mathbf{N}}$ may be seen by an external watcher as a triple of meters oscillating in time in one of two modes: "regular" and "critical." Regular mode corresponds to a series of bad cards: one meter goes up while the others go down in a steady, linear way. Critical mode corresponds to a mixture of good and bad cards: the meters may stop, go up and down, or reach a higher value instantaneously. Due to these sudden *earthquakes*, predicting the outcome of the match at any moment is often not an easy job. At a certain instant of the run, one of the meters may be quite close to its 0 value, but a few turns later that same meter may unexpectedly recover and dominate the other. So though simple in its definition, the HeartQuake dynamical system is a candidate for depicting complex behavior. Section 4 discusses an investigation of this behavior.

4. Some properties of HeartQuake

What we have defined as a HeartQuake game in Definition 1 is the parameter of a dynamic system. A mathematical object playing the same role as the k in, for example, $S_{n+1} = kS_n(1 - S_n)$ that is, the logistic procedure. The only difference is that this time the parameter is not a number but a list of sets—in both cases, a choice of H or k is the choice of a member in a family of functions of one variable. Likewise the role of a match $x \in M$ (an initial play configuration) is the same as that of a number S_0 (an initial point). Both systems are decided given a parameter and an initial configuration, in the sense that one can observe their state at any time. So an investigation of the properties of HeartQuake is conducted considering different values for H and x.

This dynamical system admits "by construction" a fixed point in $(\Phi, \sqcup)$.

Claim 1. *For any game H and point x: the only fixed point for $\mathcal{H}$ is $(\Phi, \sqcup)$.*

Proof. (ab absurdo) Suppose that there is another fixed point for $\mathcal{H}$; let (Δ, Ω) be such a point. Then $\exists x \in X$ and $\exists k \in \mathbf{N}$ such that $\delta_k(x) = \Delta$ and $\omega_k(x) = \Omega$, and $(\Delta, \Omega) \neq (\Phi, \sqcup)$. Then there are two distinct cases: $\Delta = \Phi$ and $\Delta \neq \Phi$.

1. $\Delta = \Phi$. In this case $\exists l < k \ni' \delta_l(x) \neq \Phi$ and $\delta_{l+1}(x) = \Phi$, which results in the two following cases ($\star$ means "any value").

   1.1 $\delta_l(x) = q_N^\star$. Then

$$\exists z \in X \ni' \begin{cases} \delta_{l+1}(x) = \Phi = \delta(q_N^\star, z) \\ \omega_{l+1}(x) = \omega(q_N^\star, z) \end{cases} \tag{1}$$

   which is only satisfied by $((), S_J, S_t)$. But in this case $\omega_{l+1}(x) = \sqcup$, which is a contradiction.

   1.2 $\delta_l(x) = q_J^\star$. With a similar reasoning we reach the same conclusion.

2. $\Delta \neq \Phi$. This results in the following two cases.

2.1 $\Delta = \delta_k(x) = q_N^\star$. Then, by definition of fixed point,

$$\begin{cases} \delta_{k+1}(x) = \delta(q_N^\star, \Omega) = q_N^\star \\ \omega_{k+1}(x) = \omega(q_N^\star, \Omega) = \Omega. \end{cases} \tag{2}$$

This is a contradiction, because $\Omega = \sqcup$ contradicts the first equation, while $\Omega \neq \sqcup$ contradicts the second one.

2.2 $\Delta = \delta_k(x) = q_J^\star$. With a similar reasoning we reach the same conclusion. ■

For any match attracted by the fixed point we define its *length* in Definition 11.

Definition 11. *Given a HeartQuake automaton $\mathcal{H}$ and a point $x \in M$, if the set*

$$\{k \in \mathbf{N} \mid (\delta_k(x), \omega_k(x)) = (\Phi, \sqcup)\}$$

is not empty, then the length of the game in x *is defined as the minimum of that set.*

So we have found the first remarkable dynamic behavior of HeartQuake: attraction towards its sole fixed point $(\Phi, \sqcup)$. Such behavior can be observed in matches of any game because it is always possible to make up a biased pack such that the game ends.

Are there dynamic behaviors other than this? For example, are there matches which bring the players into "runaway" conditions or endless cycling through a loop? In other words, we ask whether the function

$$F_\mathcal{H} : M \to \{\texttt{'N'}, \texttt{'J'}\}$$

defined so that

$$\forall x \in M : F_\mathcal{H}(x) = \begin{cases} \texttt{'N'} & \text{if } x \text{ is won by Neal} \\ \texttt{'J'} & \text{if } x \text{ is won by Jack} \end{cases}$$

is a total function or not.

Looking for the answer to this question we take into consideration a subfamily of games, that is, those with $\nu_0 = 1$ (i.e., games with one zero card). It is possible to show that in this case *all matches* reach the fixed point, regardless of the number of state transitions needed and so it is always possible for $F_\mathcal{H}$ to issue an output value.

Claim 2. $\forall H = (1, C_1, \ldots, C_m) : \forall x \in M\, \exists k \in \mathbf{N} \ni' (\delta_k(x), \omega_k(x)) = (\Phi, \sqcup)$.

Proof. A HeartQuake automaton $\mathcal{H}_{1,\nu_1,\nu_2,\ldots,\nu_m}$ is given. Let k be an integer such that $\sum_{i=1}^m \nu_i = 2k+1$, and r is an integer such that $0 < r \leq k$. Moreover, $\forall w \in \mathbf{N}$. Let y^w mean "w occurrences of whatsoever good cards," and z mean a certain good card. Then the following two game scenarios exist.

1. First case: given any $r \le k$,

step	S_N	S_J	S_t	state
1)	y^r0y^{k-r}	$y^{r-1}zy^{k-r+1}$	$()$	q_N^0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2r+1)$	$0y^{k-r}$	y^{k-r+1}	$y^{2r-1}z$	q_N^z
		Case 1: $z = 1$		
$2r+2)$	y^{k-r}	$y^{k-r+1}y^{2r-1}z0$	$()$	q_J^0
		Jack wins.		
		Case 2: $z > 1$		
$2r+2)$	y^{k-r}	y^{k-r+1}	$y^{2r-1}z0$	q_N^{z-1}
		Jack wins.		

for any such r and k, $(\Phi, \sqcup)$ is reached.

2. Second case: given any $r \le k$,

step	S_N	S_J	S_t	state
1)	y^rzy^{k-r}	y^r0y^{k-r}	$()$	q_N^0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2r+1)$	zy^{k-r}	$0y^{k-r}$	y^{2r}	q_N^y
$2r+2)$	y^{k-r}	$0y^{k-r}$	$y^{2r}z$	q_J^z
		Case 1: $z = 1$		
$2r+3)$	y^{k-r}	y^{k-r}	$y^{2r}z0$	q_N^0
	$y^{k+r}z0$	y^{k-r}	$()$	q_N^0
		Neal wins.		
		Case 2: $z > 1$		
$2r+3)$	y^{k-r}	y^{k-r}	$y^{2r}z0$	q_J^{z-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k+3)$	$()$	$()$	$y^{2r}z0y^{2k-2r}$	q_J^y
$2k+4)$	$()$	$()$	$y^{2r}z0y^{2k-2r}$	Φ
		Neal wins.		

for any such r and k, $(\Phi, \sqcup)$ is reached. ■

That is, a game with one bad card is intrinsically unbalanced and it is decided by that card regardless of both the cardinality of the pack and the number of its classes. The player with the bad card loses.

Note that Claim 2 shows both the existence of an unlimited series of games for which all matches end, and the existence of games whose length is unlimited.

The tool used for demonstrating this is a table tracing the orbits of the two series in Definition 10; we call such a tool a "trace table."

A number of other results about HeartQuake have been reached considering the games with $m = 1$ (i.e., games with only zeroes and ones). Such games can be arranged into a matrix of games like the following one, in which

$$M_1 = \begin{pmatrix} \mathcal{H}_{1,1} & & \mathcal{H}_{1,3} & \dots \\ & \mathcal{H}_{2,2} & & \dots \\ \mathcal{H}_{3,1} & & \mathcal{H}_{3,3} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \xrightarrow{(1)} \begin{pmatrix} \mathcal{H}_{1,2k+1} \\ \mathcal{H}_{2,2k+2} \\ \mathcal{H}_{3,2k+1} \\ \vdots \end{pmatrix}$$

$$\downarrow(2) \qquad\qquad \searrow(3)$$

$$(\mathcal{H}_{2k+1,1}, \mathcal{H}_{2k+2,2}, \mathcal{H}_{2k+1,3}, \dots) \qquad \begin{pmatrix} \mathcal{H}_{1,1} \\ (\mathcal{H}_{4-i,i})_{i=1,2,3} \\ \vdots \\ (\mathcal{H}_{2k-i,i})_{i=1,\dots,2k-1} \\ \vdots \end{pmatrix}$$

Figure 1: Matrix M_1 may be scanned in three ways: (1) row by row, (2) column by column, and (3) diagonal by diagonal. Each scanning has a different dynamical interpretation.

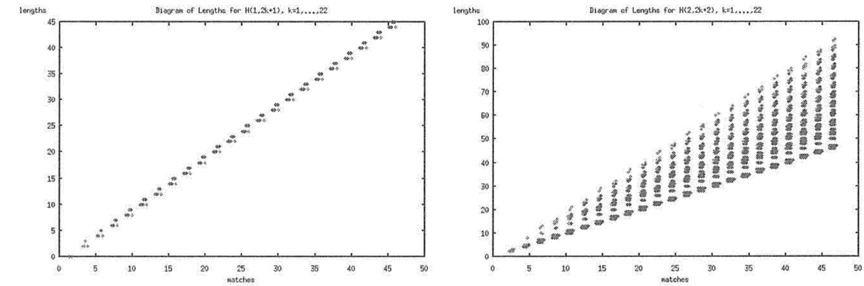
only elements with an even number as the sum of their indices are present. We call "M_1 games" those in M_1:

$$M_1 = \begin{pmatrix} \mathcal{H}_{1,1} & & \mathcal{H}_{1,3} & & \mathcal{H}_{1,5} & \dots & \\ & \mathcal{H}_{2,2} & & \mathcal{H}_{2,4} & & \mathcal{H}_{2,6} & \dots \\ \mathcal{H}_{3,1} & & \mathcal{H}_{3,3} & & \mathcal{H}_{3,5} & \dots & \\ & \vdots & & & \vdots & & \end{pmatrix}. \tag{3}$$

Such a matrix may of course be scanned in many ways: row by row, column by column, and through the diagonals. These paths correspond to a number of experiments that may be carried out (see Figure 1. For example, scanning (3) row by row, left to right means considering the influence of the arrival of more and more aces in a fixed population of zero cards. Proceeding column by column, up to down, means considering the effect of the arrival of more and more zero cards in a fixed population of aces. Considering the secondary diagonals, left to right, means investigating the influence of that ratio in packs with a fixed size. These experiments have been carried out, and the seemingly most fruitful, row by row, is discussed in section 4.1.

4.1 Scanning M_1 row by row

Let us consider the rows in M_1. Claim 2 implies that games $\mathcal{H}_{1,2k+1}, k \in \mathbf{N}$, that is, games in its first row, are fully determined. It can be formally proven that this holds also for its second row, that is, for $\mathcal{H}_{2,2k+2}, k \in \mathbf{N}$. This seems less intuitive than with $\nu_0 = 1$ because with two zero cards there is always a number of balanced matches in which the players have one zero card each. For these two families it is possible to draw pictures like those in Figure 2, which are diagrams with the lengths of all their matches for some values of k.



$\mathcal{H}_{1,2k+1}, k \in \mathbf{N}$ $\mathcal{H}_{2,2k+2}, k \in \mathbf{N}$

Figure 2: Diagram of lengths of games $\mathcal{H}_{1,2k+1}$ and $\mathcal{H}_{2,2k+2}$, for $k = 0, \ldots, 22$. The y-axis represents *lengths*, that is, the number of transitions from initial state q_N^0 to the first occurrence of Φ. Abscissas represent all possible matches in the family, arranged so that a game corresponding to a certain k is placed in interval $[k, k+1[$, with its matches lexicographically ordered.

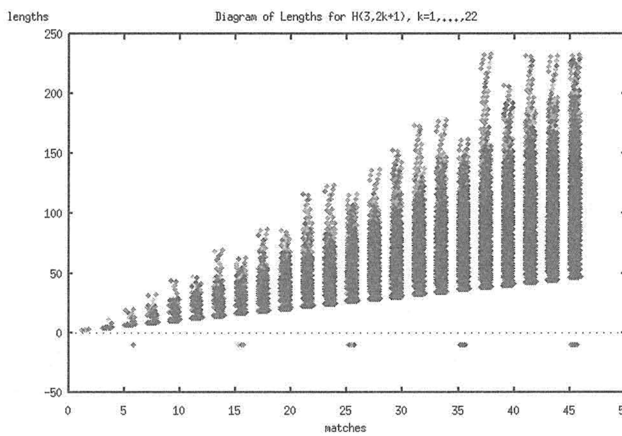


Figure 3: Diagram of lengths of games $H_k = \mathcal{H}_{3,2k+1}$, $k = 0, \ldots, 22$. Again, all matches in H_j lie in interval $[j, j+1[$, lexicographically ordered. Along line $y = -10$, all matches that don't end within a certain threshold value are plotted. Such matches may last more, or may have entered a *cycle*.

Considering the third row in matrix (3), $\mathcal{H}_{3,2k+1}, k \in \mathbf{N}$, and trying to draw a diagram of lengths, we find that for some values of k the length of the game seems to be out of reach for any number of iterations. This is shown in Figure 3, wherein games which appear to be longer than a threshold value are plotted with $y = -10$. It seems that the degree of uncertainty related with the intrinsically unbalanced case of a pack with three zero cards drives the system into a novel kind of behavior. In order to investigate the phenomenon, first we sieve the "suspicious" points with a diagram of length; one such point

Table 3: Trace table for $((1101), (0110), ())$, that is, a match in $\mathcal{H}_{3,5}$. Note that this point enters (at the third iteration) a period-6 cycle.

step	S_N	S_J	S_t	**state**
1)	1101	0110	()	q_N^0
2)	101	0110	1	q_J^1
3)	101	110	10	q_N^0
3)	10110	110	()	q_N^0
4)	0110	110	1	q_J^1
5)	0110	10	11	q_N^1
6)	110	10	110	q_J^0
6)	110	10110	()	q_J^0
7)	110	0110	1	q_N^1
8)	10	0110	11	q_J^1
9)	10	110	110	q_N^0
9)	10110	110	()	q_N^0
⋮	⋮	⋮	⋮	⋮

is for example $((1101), (0110), ())$, a match for game $\mathcal{H}_{3,5}$. We can easily show that this point falls into a *cycle* by means of trace table showing the first nine orbits of that point as in Table 3.

As a consequence, the HeartQuake family of dynamical systems has at least *two* remarkable dynamic behaviors: attraction towards its fixed point and attraction into cycles. Cycles appear in the third row of (3) and seem to be related with the uncertainty in the process of predicting the outcome of a match of a given game.

Having shown the existence of cycles, we can now define their period.

Definition 12. *Given a HeartQuake automaton $\mathcal{H}$ and a point $x \in M$: if $\exists k \in \mathbf{N}$ such that the set*

$$P_k(x) = \{l \in \mathbf{N} \mid (\delta_{k+l}(x), \omega_{k+l}(x)) = (\delta_k(x), \omega_k(x))\}$$

is not empty, then it can be shown that these sets do not depend on k and so collapse to one subset of $\mathbf{N}$; then the period of the game *in x is defined as the minimum of that set and indicated as $|\mathcal{H}(x)|$.*

Let's consider sub-family $\mathcal{H}_{3,2k+1}, k \in \mathbf{N}$. As suggested by Figure 3, it seems that in this case suspicious points are regularly distributed. It can be proven that all such points are cycles. Figure 4 shows their distribution and the length of their period. The regularity found in Figure 4 brings us to the two results in Claims 3 and 4.

Claim 3. $\forall k \in \mathbf{N}, \mathcal{H}_{3,5+10k} = \langle X, Q, \delta, \omega \rangle : \exists x \in X \ni' |\mathcal{H}_{3,5+10k}(x)| = 8k + 6.$

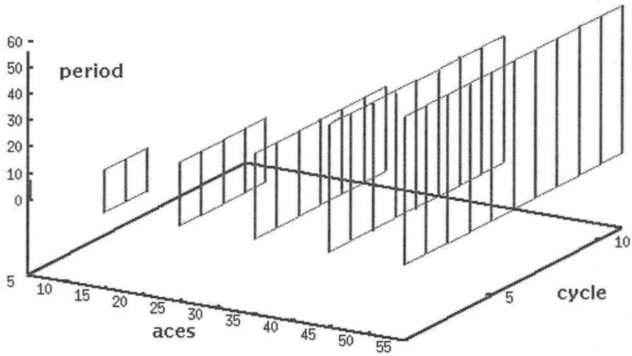


Figure 4: The cycles of games $\mathcal{H}_{3,2k+1}, k \in \mathbf{N}$ and their period.

Proof. Let $k \in \mathbf{N}$. Consider the following match (1^k means $\overbrace{11\ldots1}^{k \text{ times}}$):

$$((101^{3k+3}01^{2k-2}), (1^{k+3}01^{4k}), ()),$$

that is, a match of game $\mathcal{H}_{3,5+10k}$. It is easy to show that this configuration enters a cycle whose period is $8k+6$:

step	S_N	S_J	S_t	state
1)	$101^{3k+3}01^{2k-2}$	$1^{k+3}01^{4k}$	()	q_N^0
2)	$01^{3k+3}01^{2k-2}$	$1^{k+3}01^{4k}$	1	q_J^1
3)	$01^{3k+3}01^{2k-2}$	$1^{k+2}01^{4k}$	11	q_N^1
4)	$1^{3k+3}01^{2k-2}$	$1^{k+2}01^{4k}$	110	q_J^0
4)	$1^{3k+3}01^{2k-2}$	$1^{k+2}01^{4k+2}0$	()	q_J^0
$\vdots$		$\vdots$		$\vdots$
$2k+8)$	$1^{2k+1}01^{2k-2}$	$01^{4k+2}0$	1^{2k+4}	q_J^1
$2k+9)$	$1^{2k+1}01^{2k-2}1^{2k+4}0$	$1^{4k+2}0$	()	q_N^0
$\mathbf{2k+9)}$	$1^{2k+1}01^{4k+2}0$	$1^{4k+2}0$	()	q_N^0
$\vdots$		$\vdots$		$\vdots$
$6k+11)$	$01^{4k+2}0$	$1^{2k+1}0$	1^{4k+2}	q_N^1
$6k+12)$	$1^{4k+2}0$	$1^{2k+1}0$	$1^{4k+2}0$	q_J^0
$6k+12)$	$1^{4k+2}0$	$1^{2k+1}01^{4k+2}0$	()	q_J^0
$\vdots$		$\vdots$		$\vdots$
$10k+14)$	$1^{2k+1}0$	$01^{4k+2}0$	1^{4k+2}	q_J^1
$\mathbf{10k+15)}$	$1^{2k+1}01^{4k+2}0$	$1^{4k+2}0$	()	q_N^0

■

Claim 3 shows that, for any integer k, it is always possible to find a cycle whose period is greater than k, that is, there exist cycles whose period is as big as one likes.

Claim 4. $\forall k \in \mathbf{N}, \mathcal{H}_{3,5+10k} = \langle X, Q, \delta, \omega \rangle : \exists (x_0, x_1, \ldots, x_{k-1}) \in M^k \ni' \forall i \in \{0, 1, \ldots, k-1\} : |\mathcal{H}_{3,5+10k}(x_i)| = 8k + 6$.

Proof. Let $k \in \mathbf{N}$. Consider game $\mathcal{H}_{3,5+10k}$ again. Then $\forall i \in \mathbf{N}, i < k$, consider the following initial play configurations:

$$x_i = ((1^{i+1}01^{3k+3+i}01^{2k-2-2i}), (1^{k+3+2i}01^{4k-2i}), ()). \tag{4}$$

All such points enter a cycle whose period is $8k + 6$:

step	S_N	S_J	S_t	**Q**
1)	$1^{i+1}01^{3k+3+i}01^{2k-2-2i}$	$1^{k+3+2i}01^{4k-2i}$	-	q_N^0
$\vdots$	$\vdots$			$\vdots$
$2i+3$)	$01^{3k+3+i}01^{2k-2-2i}$	$1^{k+2+i}01^{4k-2i}$	1^{2i+2}	q_N^0
$2i+4$)	$1^{3k+3+i}01^{2k-2-2i}$	$1^{k+2+i}01^{4k-2i}1^{2i+2}0$	-	q_J^0
	$1^{3k+3+i}01^{2k-2-2i}$	$1^{k+2+i}01^{4k+2}0$	-	q_J^0
$\vdots$	$\vdots$			$\vdots$
$2k+4i+8$)	$1^{2k+1}01^{2k-2-2i}$	$01^{4k+2}0$	$1^{2k+4+2i}$	q_J^1
$2k+4i+9$)	$1^{2k+1}01^{2k-2-2i}1^{2k+4+2i}0$	$1^{4k+2}0$	-	q_N^0
$\mathbf{2k+4i+9}$)	$1^{2k+1}01^{4k+2}0$	$1^{4k+2}0$	-	q_N^0
$\vdots$	$\vdots$			$\vdots$
$6k+4i+11$)	$01^{4k+2}0$	$1^{2k+1}0$	1^{4k+2}	q_N^1
$6k+4i+12$)	$1^{4k+2}0$	$1^{2k+1}0$	$1^{4k+2}0$	q_J^0
$6k+4i+12$)	$1^{4k+2}0$	$1^{2k+1}01^{4k+2}0$	-	q_J^0
$\vdots$	$\vdots$			$\vdots$
$10k+4i+14$)	$1^{2k+1}0$	$01^{4k+2}0$	1^{4k+2}	q_J^1
$10k+4i+15$)	$1^{2k+1}0$	$1^{4k+2}0$	$1^{4k+2}0$	q_N^0
$\mathbf{10k+4i+15}$)	$1^{2k+1}01^{4k+2}0$	$1^{4k+2}0$	-	q_N^0

Note how, at row $2k + 4i + 9$, i disappears from the stacks and all points enter the same cycle, exactly that of Claim 3. ■

Claim 4 shows that for any integer k there exists a HeartQuake game in which at least k matches enter a cycle whose period is greater than k.

Note that matches in (4) are such that Neal has two zeroes in his pack. It can be shown that there exists a dual series of k matches in which Neal has only one zero. Experimentally it seems that no other matches fall into cycles; if this is true, then also this game is fully decided—one only has to inspect the match rather than running the game procedure on it in order to predict its outcome.

What happens for game $\mathcal{H}_{4,2k+2}, k \in \mathbf{N}$, that is, for games in the fourth row of (3)? A graph of lengths shows that in this case cycles also appear (see Figure 5). But this time their distribution seems to be completely random and so is unpredictable. The degree of uncertainty of a game with at least four zeroes is such that prediction of its long term behavior or eventual

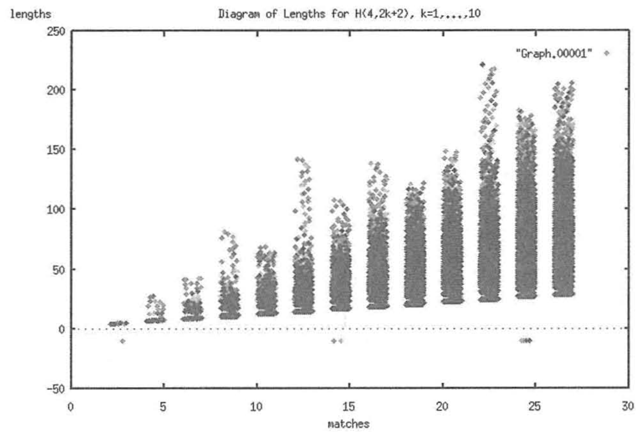


Figure 5: Length of games $\mathcal{H}_{4,2k+2}, k = 0, \ldots, 12$.

outcome is no longer possible. In other words, HeartQuake is another simple, deterministic system which, under certain conditions, behaves unpredictably or randomly.

4.2 Other paths

A number of other sub-families of $\mathcal{H}_{\nu_0,\nu_1}$ have been considered, that is, a number of other paths into matrix (3) have been explored. All these paths exhibit cycles whose number and period seem completely random. In no other case except scanning row by row have we found the equilibrium point between decidability and chaos that can be observed at the third row of M_1. For example, considering games in

$$(\mathcal{H}_{2k-i,i})_{k=1,2,\ldots} \quad i = 1, 2, \ldots, 2k-1$$

we move the attention from games in the rows of matrix (3) to those in the secondary diagonals. Note that this time the kth element of the series is not an infinite series itself, but is made up of $2k-1$ games. An interpretation for this family is: consider increasing populations of cards, for every fixed population we change the ratio of good over bad cards and see what happens. For $n = 1$ and 2 all matches end, for $k = 3$ there is only one point which enters a cycle, and for $k > 3$ the number of cycles and their period seem to be completely unpredictable. Figure 6 shows an excerpt from the "one-page dictionary" for cycles in this family.

5. Images of HeartQuake

Given a HeartQuake automaton $\mathcal{H} = \langle X, Q, \delta, \omega\rangle$ and a point $x \in M$, the recursive procedure in Definition 10 is fully decided. In some sense, all the information of a match resides in its initial play configuration. Every such

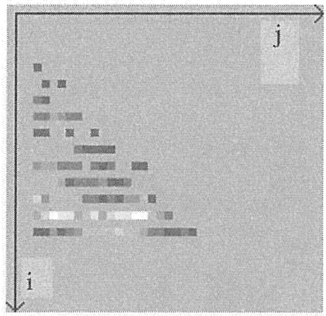


Figure 6: A "one-page dictionary" for cycles in games $(\mathcal{H}_{i-j,j})_{i=2,4\ldots,26}$, $j = 1, \ldots, i-1$. The i axis represents the number of cards in the packs, the j axis is the number of aces in the pack, and gray levels express periods—the whiter the gray level, the longer the cycle. No regularity appears to be detectable.

point is made of *two* meaningful fields, that is, the two stacks S_N and S_J. These two data structures may be simply regarded as strings of digits in a base-$m+1$ numeral system. As a consequence, a mapping can be set up between the stacks in a match and two integer numbers $b(S_N)$ and $b(S_J)$. Such numbers can be in turn interpreted as a row and a column index into a matrix of pixels—an image. It is therefore possible to set up an injective mapping between initial play configurations and pixels of a matrix of $p \times p$ pixels, p being

$$p = \max\{y \in \mathbf{N} \mid y = b(S_N),\ (S_N, S_J, S_t) \in M\}.$$

Pixel (x, y) will be turned on if there exists a match $(S_N, S_J, ())$ so that $y = b(S_N)$ and $x = b(S_J)$. Moreover, it is possible to map some information regarding a match with the natural attribute of pixels, that is, *color*. For example, the length of a game won by Jack may be represented with different tones of green, red may be used for Neal, and gray levels may represent cycles. Figure 7 shows the image corresponding to game $\mathcal{H}_{4,2}$.

5.1 Images of games with two classes of cards

First we focus our attention on M_1-games, that is, games with $m = 1$ or consisting of zeroes and ones. Consider a matrix of graphs arranged in a way similar to matrix (3), for example:

$$G_1 = \begin{pmatrix} & & G(0,2) & & G(0,4) & & \ldots \\ & G(1,1) & & G(1,3) & & G(1,5) & \ldots \\ G(2,0) & & G(2,2) & & G(2,4) & & \ldots \\ & G(3,1) & & G(3,3) & & G(3,5) & \ldots \\ G(4,0) & & G(4,2) & & G(4,4) & & \ldots \\ & \vdots & & \vdots & & \vdots & \end{pmatrix}. \tag{5}$$

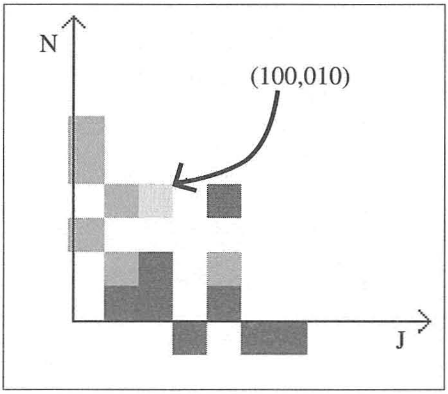


Figure 7: An image with all matches in $\mathcal{H}_{4,2}$. Match (x, y) is mapped onto $[x, x+1[\times[y-1, y[$ in order to indicate the outcome. The light gray pixel $(2, 4)$ pointed to by the arrow corresponds to match $((100), (010), ())$, which falls into a period-6 cycle.

Figure 8 depicts 15 images of G_1 which suggest a number of observations. Let r and c be any two integers such that $r + c$ is even.

Observation 1. *For any $r > 0$ and $c > 0$: $G(r, c)$ can be partitioned into four equal blocks.*

This is clearly visible, for example, in the two last rows of Figure 8.

Observation 2. *For any r and c: the pattern represented in $G(r, c)$ is one of those contained in $G(r, c+2)$, that is, every graph is fully contained in its right neighbor. (This is shown as an example in Figures 8 and 9.)*

Observation 3. *The four patterns in $G(r, c)$ are the same as those in the following images: $G(r-2, c)$, $G(r-1, c-1)$, and $G(r, c-2)$. Patterns are arranged according to the following scheme:*

$$\begin{matrix} G(r-1, c-1) & G(r, c-2) \\ G(r-2, c) & G(r-1, c-1) \end{matrix},$$

that is, a 2×2 matrix in which the diagonal contains the one repeated pattern.

Another way to represent Observation 3 is

$$G(r, c) = G(r-2, c) + 2G(r-1, c-1) + G(r, c-2). \tag{6}$$

The validity of this property seems to be strenghtened by the fact that if you consider the number of matches in a HeartQuake game with two classes, $p(r, c) = \frac{(r+c)!}{r!c!}$, then

$$p(r, c) = p(r-2, c) + p(c, r-2) + 2 \times p(r-1, r-1),$$

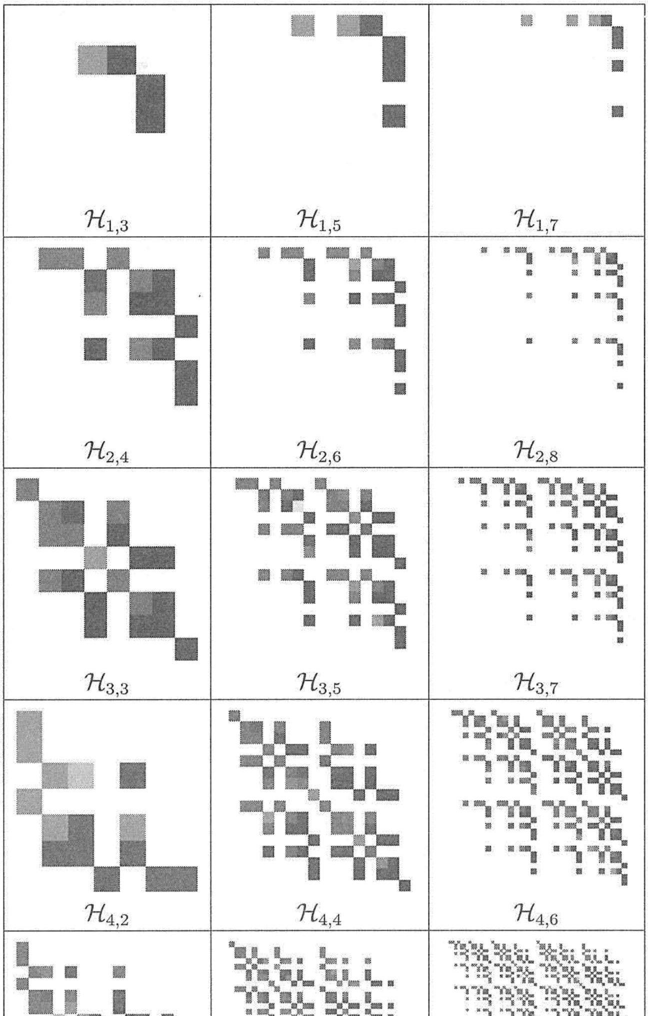


Figure 8: Images of 15 HeartQuake games plotted with the same size. Light gray pixels fall into cycles.

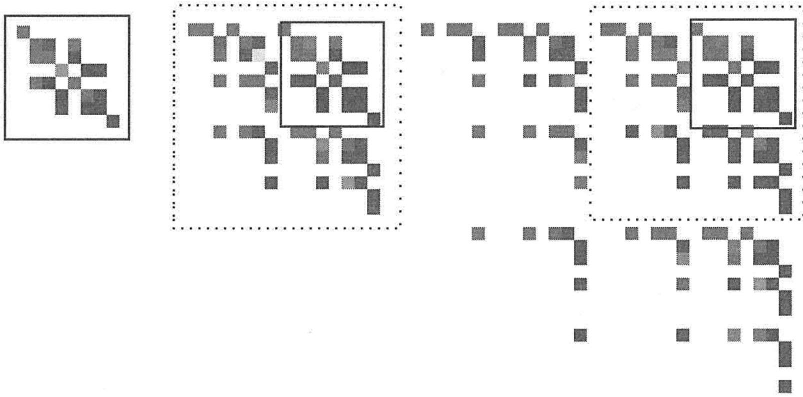


Figure 9: Patterns of games $\mathcal{H}_{3,3}$, $\mathcal{H}_{3,5}$, and $\mathcal{H}_{3,7}$.

that is, the number of matches of the four games equals that of $G(r,c)$. This is true only for the patterns depicted, outcomes are different in general.

Note that (6) represents a relationship between games made of $r+c$ cards and games with 2 cards less. Iterating the process, we "decompose" any $G(r,c)$ into a number of atomic patterns, or patterns that cannot be further decomposed. Such patterns are all arranged in a frame made of the first two rows and the first two columns of matrix (5). Moreover, given any r and c greater than 2, the decomposition of $G(r,c)$ is a linear combination of games $G(0,i)$, $G(1,i)$, $G(j,0)$, and $G(j,1)$ with $i \le c$, and $j \le r$. For example, $G(8,6)$ can be "factored" into the following atomic patterns:

$$\begin{aligned} G(8,6) = {} & G(0,6) + 28 \times G(0,4) + 210 \times G(0,2) + 210 \times G(2,0) \\ & +420 \times G(1,1) + 112 \times G(1,3) + 8 \times G(1,5) + 70 \times G(4,0) \\ & +196 \times G(3,1) + 15 \times G(6,0) + 50 \times G(5,1) + G(8,0) + 6 \times G(7,1) \end{aligned}$$

Figure 10 shows the distribution of the basic bricks of $G(8,6)$ within matrix (5), while Figure 11 shows its decomposition tree.

$$\begin{pmatrix} \cdot & & G(0,2)^{210} & & G(0,4)^{28} & & G(0,6)^{1} \\ & G(1,1)^{420} & & G(1,3)^{112} & & G(1,5)^{8} & \uparrow \\ G(2,0)^{210} & & \cdot & & \cdot & & \uparrow \\ & G(3,1)^{196} & & \cdot & & \cdot & \uparrow \\ G(4,0)^{70} & & \cdot & & \cdot & & \uparrow \\ & G(5,1)^{50} & & \cdot & & \cdot & \uparrow \\ G(6,0)^{15} & & \cdot & & \cdot & & \uparrow \\ & G(7,1)^{6} & & \cdot & & \cdot & \uparrow \\ G(8,0)^{1} & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow & G(8,6) \end{pmatrix}$$

Figure 10: The basic blocks of image $G(8,6)$. G^k means k occurrences of pattern G. A $\cdot$ means "pattern not involved."

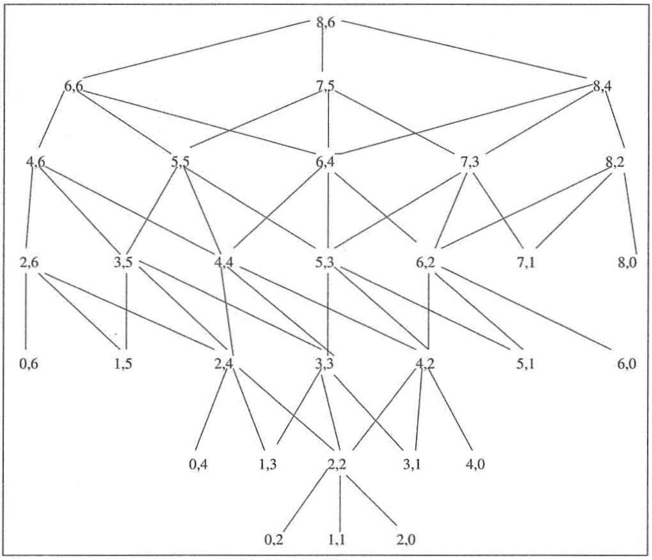


Figure 11: Decomposition tree for image $G(8,6)$.

Observation 4. *$\forall r, c : G(r,c)$ is specular to $G(c,r)$, that is, G and its transpose depict equal patterns up to a resizing and a rototranslation.*

This is clearly visible comparing couples of games in Figure 8. Figure 12 shows three such couples.

Observation 5. *Some images depict a certain degree of self-similarity.*

As an example see Figure 13, which depicts $G(8,6)$.

5.2 Images of games with more than two classes of cards

In this section we take into consideration games in M_k, with $k > 1$, that is, games in the k-dimensional extension of matrix (3), or games consisting of k classes of cards.

A number of observations made for games with two classes of cards can be extended to those with more classes. For example, images depicting M_2-games seem to consist of nine blocks. Such blocks represent patterns of other games according to the following rule of decomposition:

$$\begin{aligned} G(i,j,k) = {} & 2G(i-1,j,k-1) + 2G(i,j-1,k-1) + 2G(i-1,j-1,k) \\ & + G(i,j-2,k) + G(i-2,j,k) + G(i,j,k-2). \end{aligned} \tag{7}$$

As an example, Figure 14 shows image $G(6,4,2)$ indicating its first-level blocks. Moreover, note $G(6,3,1)$ and the third block in the first row of

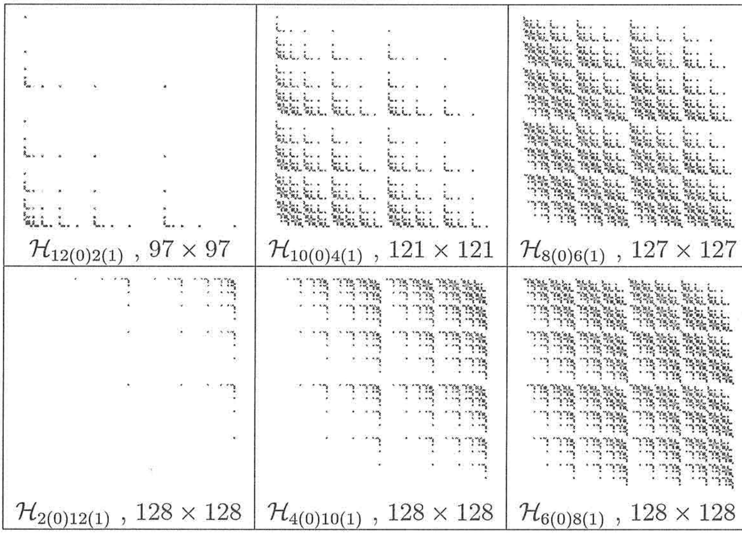


Figure 12: A pack of fourteen cards is considered. Each image is labelled with its game and its original dimension. Ignoring dimension and rotation, images $G(i, j)$ and $G(j, i)$ seem to depict the same pattern.

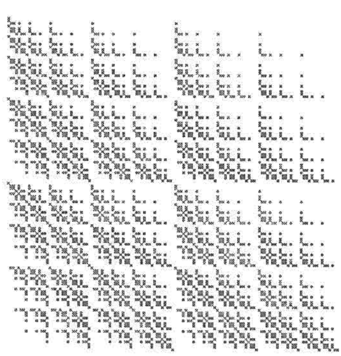


Figure 13: Image $G(8, 6)$.

$G(5, 3, 2)$, that is, $G(5, 2, 1)$. They both seem to draw a "gingerbreadman" like pattern [3] (see Figures 15 and 16, respectively).

In games with a class consisting of exactly *two* cards a collapse phenomemon occurs. One of the blocks of the decomposition is the pattern of a game with one class less than in the original, that is, if the original game was in M_k, that block belongs to M_{k-1}. This phenomenon is clearly observ-

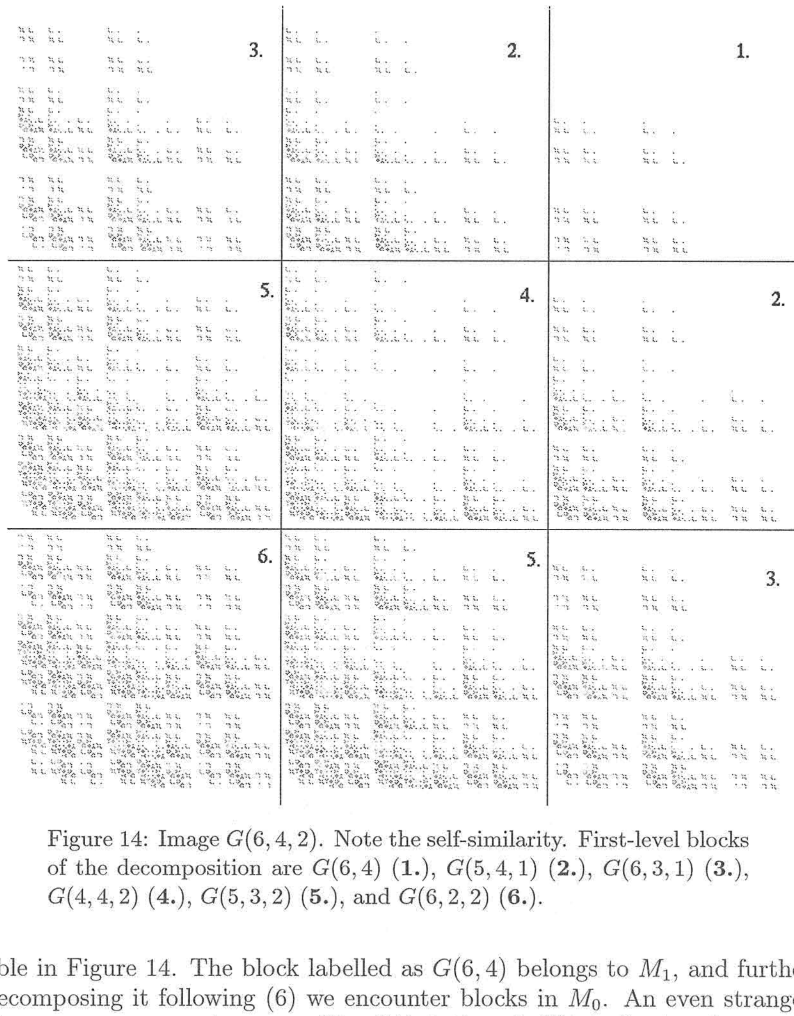


Figure 14: Image $G(6,4,2)$. Note the self-similarity. First-level blocks of the decomposition are $G(6,4)$ (**1.**), $G(5,4,1)$ (**2.**), $G(6,3,1)$ (**3.**), $G(4,4,2)$ (**4.**), $G(5,3,2)$ (**5.**), and $G(6,2,2)$ (**6.**).

able in Figure 14. The block labelled as $G(6,4)$ belongs to M_1, and further decomposing it following (6) we encounter blocks in M_0. An even stranger phenomenon occurs in games like $G(6,3,1)$ and $G(5,2,1)$, that is, games with $\nu_m = 1$ (see Figure 15). If we apply (7) to $G(6,3,1)$ we get

$$G(6,3,1) = \underbrace{2G(5,3) + 2G(6,2)}_{\text{collapse to } M_1} + 2G(5,2,1) + G(6,1,1) + G(4,3,1) + \underbrace{G(6,3,-1)}_{\text{impossible!}}.$$

In other words, the decomposition rules still apply, and impossible "imaginary" blocks like $G(6,3,-1)$ simply disappear.

Can we generalize the decomposition rule to M_k games for any k? Experimental results suggest that, for any integer $k > 1$, a k-class, p-card game

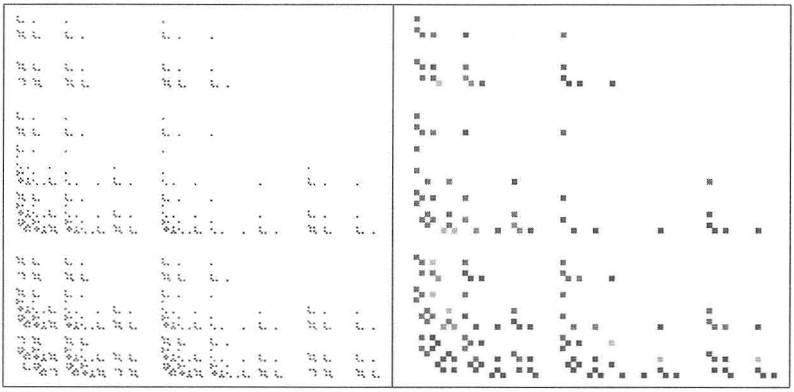


Figure 15: Images $G(6,3,1)$ and $G(5,2,1)$. Note the similarity and the empty ("imaginary") block on the right top of both images.

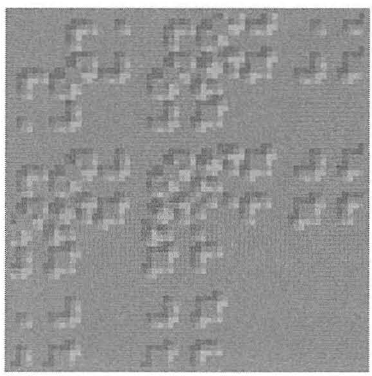


Figure 16: A "gingerbreadman" like pattern [3] found in image $G(6,4,2)$.

can be decomposed into a matrix of $k \times k$ blocks. This decomposition can be made as follows: k single blocks of $p-2$ cards each are disposed through the main diagonal and $\frac{k\times(k-1)}{2}$ double blocks of $p-2$ cards each are symmetrically disposed with respect to the main diagonal. More precisely: given $I_k = \{0,1,\ldots,k-1\}$ a set of indices, we can denote with $\{i\}$ the generic singleton and with $\{i,j\}$ the generic subset of two elements of I_k. Experimental results show that the general decomposition rule is

$$G(\nu_0, \nu_1, \ldots, \nu_{k-1}) = \sum_{\{i\} \subset I_k} G(\nu_{a<i}, \nu_i - 2, \nu_{i<b<k}) \\ +2 \sum_{\{i,j\} \subset I_k, i<j} G(\nu_{a<i}, \nu_i - 1, \nu_{i<b<j}, \nu_j - 1, \nu_{j<c<k}). \quad (8)$$

Relation (8) is consistent with previously described relations (6) and (7). Likewise, it again represents a relationship between games made of $r+c$ cards and games with 2 cards less. We can iterate the process producing patterns that are "atomic" for M_k. Such patterns inherit the same distribution of those in M_1: they lie in the first two superficial strata of the hypercube. Moreover, given any game in M_k, its basic patterns are localizable with the same method shown in Figure 10.

6. Conclusions and future directions

We have just described a dynamic system whose simple formulation hides a complex behavior which often makes it very hard to predict the outcome of a run ahead of time. A number of results have been described including the existence of one fixed point and of cycles. The existence of runs whose dynamic behavior is detectable only after a number of iteration steps as big as one likes has also been shown. A straightforward mapping of run procedures into bidimensional graphics has been introduced. A number of morphological properties have been observed, including self-similarity and a sort of factorizability. Future directions may include the following.

- Further generalizations, for example, considering a ring of r players, with a control token flowing clockwise among the players. The player that has the token picks up a card from their pack: if it is a bad card, they simply pass the token to the next player. If it is a good card, then that player and the next one fight for the hoard on the table with the rules of standard HeartQuake. This may result in multidimensional images.

- Further investigations through other paths in $M_k, k > 1$ in search of unknown phenomena and behavior. Such studies should necessarily take advantage of sharp algorithms and of parallel computing.

- Applications in other fields, including data encryption and computer music.

- Analysis of the meaning of the properties observed in HeartQuake's images.

- Investigation concerning relationships with cellular automata.

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