Complex Systems **9** (1995) 329–348

One-dimensional Deterministic Greenberg–Hastings Models

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Abstract. In this simple model for a one-dimensional array of excitable cells, each site $x \in \mathbf{Z}$ is in one of κ states: 0 (rested state), 1 (excited state), $2, \ldots, \kappa-1$ (refractory states). The states update in discrete time according to a synchronous rule: changes $1 \to 2, \ldots, \kappa-1 \to 0$ happen automatically, while the $0 \to 1$ change is induced by at least a threshold number of 1s in the local neighborhood of x. If indestructible *stable periodic objects* exist, the model evolves into a locally periodic state. In parameter ranges when these structures are impossible, the system approaches the ground state 0: either the dynamics are dominated by annihilating waves, which cause power-law decay, or excitation is unable to propagate and the model experiences exponentially fast relaxation.

1. Introduction

The *Greenberg–Hastings model* (GHM) is probably the simplest model for an *excitable medium* [13, 18]. Imagine points in a lattice either in the *excited* state (coded as 1), in the *rested* state (0), or in one of $\kappa-2$ stages of *recovery* ($2, \ldots, \kappa-1$, also known as *refractory* states). The states then update in discrete time according to the following synchronous rule. An excited state automatically enters into the recovery cycle; in other words, the state changes $1 \to 2$, $2 \to 3, \ldots, \kappa-1 \to 0$ are automatic. When a site is rested, it needs enough excited sites in its neighborhood to become excited.

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In the course of our research on excitable media we carried out extensive computer simulations of the GHM on two-dimensional boxes [9]. The basic message that emerged is that the behavior of the system undergoes several phase transitions as the parameters (number of colors, radius of the neighborhood, and threshold needed for excitation) are varied. By far, the most interesting cases are those in which the system self-organizes into spatial wave structures [3, 6, 9]. These waves are usually driven by stable centers that cannot be disturbed by their surroundings: the *stable periodic objects* (SPOs) [3, 9]. An SPO is a *finite* set colored with κ colors in such a way that every site with color k has at least a threshold number of representatives of the succeeding color $(k+1) \bmod \kappa$ in its neighborhood. If it is possible to construct an SPO for a given set of parameters, then a random initial state on an infinite lattice will contain infinitely many of them. The state of these indestructible structures is periodic with period κ, thus ensuring that excitation never dies out. If parameters are chosen so that SPOs do not exist, the empirical evidence suggests that the evolution either degenerates into a slowly dying annihilating system or even a system experiencing quick global relaxation [9, 10]. It may be also noted that when SPOs are feasible, most are not present at the beginning, but rather they are dynamically generated [10]. Readers interested in interactive explorations of such issues are invited to visit `http://math.wisc.edu/~griffeat/sink.html` and download *WinCA*, a Windows-based program for cellular automata (CA) experimentation written by R. Fisch and D. Griffeath.

Because of the crucial effect that SPOs have on the GHM dynamics, constructing SPOs and devising good arguments for their nonexistence in various parameter ranges are subjects of central theoretical importance [3, 9]. In many cases SPOs are exceedingly rare, so that one cannot rely on computer simulations either to find them or to argue convincingly against their existence. The combinatorial issues arising from two-dimensional GHM rules are extremely complicated and have eluded mathematical analysis. This suggests that one should study the simpler one-dimensional model to gain better understanding of the combinatorial obstacles. On a related note, it seems intriguing to see whether at least part of the complex spatial behavior of the two-dimensional GHM carries into the extremely restrictive environment of the one-dimensional lattice. Our final motivation for investigating the one-dimensional GHM is that the three-parameter rule proposed below appears general enough to demonstrate the types of excitable dynamics possible in one-dimensional CA.

The remainder of the introduction is devoted to basic definitions and notation, as well as an informal presentation of the results of this paper.

GHM dynamics depend on three positive integer parameters: κ, r, and θ. The parameter κ specifies the number of *colors* or *states* in the model; each color will be identified as an element of $\{0, 1, \ldots, \kappa - 1\}$. Parameter r indicates the *range* of the neighborhood of a site. Thus, the *neighborhood* of a point $x \in \mathbf{Z}$ is $\mathcal{N}_x = x + \mathcal{N}$, where $\mathcal{N} = \{-r, -r+1, \ldots, r-1, r\}$. Parameter θ is the excitation *threshold*. The general one-dimensional deterministic GHM

γ_t is then given by the following one-step rule:

$$\gamma_{t+1}(x) = \begin{cases} (\gamma_t(x)+1) \bmod \kappa, & \text{if } \gamma_t(x) \neq 0, \\ 1 & \text{if } \gamma_t(x) = 0 \text{ and} \\ & \quad |\{y \in \mathcal{N}_x : \gamma_t(y) = 1\}| \geq \theta, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Once γ_0 is specified, either as a deterministic or random color configuration, equation (1.1) determines the continued evolution of γ_t.

If A is a finite subset of $\mathbf{Z}$, then the GHM can be defined on A, assuming *free boundary* conditions, so that every $x \in A$ has $\mathcal{N}_x = (x + \mathcal{N}) \cap A$. This will be referred to as a *finite* system, while the GHM on all of $\mathbf{Z}$ will be called the *infinite* system.

We now introduce some terminology which will be used to describe the long-time behavior of the GHM. In the following definitions, we assume that γ_0 is a fixed deterministic initial configuration for the GHM dynamics. (In fact, these concepts make sense for any deterministic sequence of configurations $\gamma_0, \gamma_1, \ldots \in \{0, 1, \ldots\}^{\mathbf{Z}}$.)

We say that γ_t *fixates* if every site changes color only finitely many times; hence, for every site x, $\gamma_\infty(x) = \lim_{t\to\infty} \gamma_t(x)$ exists. If, in addition, $\gamma_\infty(x) = 0$ for every x, then γ_t *dies out strongly*. In part, tractability of GHM dynamics is the result of the fact that a GHM can fixate only by dying out strongly.

If γ_t does not fixate, then different qualitative behaviors are possible. We say that γ_t *dies out weakly* if each site spends an overwhelming proportion of time in the rested state:

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t} 1_{\gamma_s(x)=0} = 1$$

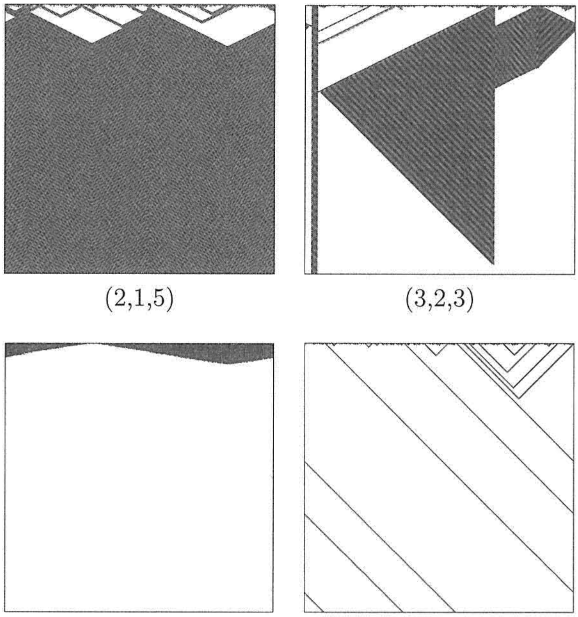
for every $x \in \mathbf{Z}$. Dying out weakly (as we will see) does not necessarily imply strong death, since the evolution may be dominated by rare waves, which can travel over long distances and prevent any site from attaining a final state.

We say that γ_t is *locally periodic* if the following three conditions are satisfied.

1. For each x there is a finite time T_x so that $\eta_t(x)$ is periodic for $t \geq T_x$.
2. γ_t does not fixate.
3. There is no global spatial periodicity, that is, for each $x \in \mathbf{Z}$ there is a positive integer n and a time t so that $\eta_s(x) \neq \eta_s(nx)$ for $s \geq t$.

In addition, γ_t is *uniformly locally periodic* if it is locally periodic and the final period is the same for all sites.

The third condition specifies the locality of local periodicity. The periodicity should not manifest itself spatially throughout $\mathbf{Z}$ but should emanate from infinitely many different finite collections of sites. In this case, different regions of $\mathbf{Z}$ will be out of phase with each other. For example, we do not want to use the term "locally periodic" to describe the evolution of



(2,1,5) (3,2,3)

(13,10,2) (1,1,4)

Figure 1: Characteristic behaviors of one-dimensional GHM.

the two-color GHM with $r = \theta = 1$, starting from $\ldots 0101010101 \ldots$ (or $\ldots 000010000 \ldots$ or any other configuration with only a finite number of 1s).

If the initial state is random, then we say that the system has one of the above properties if the appropriate condition holds for almost all initial states. For example, "locally periodic" will mean the same as "locally periodic a.s." Throughout the remainder of this paper, unless explicitly stated otherwise, the initial state of the GHM γ_0 is random with each $\gamma_0(x)$ uniform over the κ colors and independent of $\gamma_0(y)$ for $y \neq x$. Figure 1 displays simulations of typical examples (determined by their (r, θ, κ) coordinates) of (clockwise from top left) uniform local periodicity, nonuniform local periodicity, nonstrong weak death, and strong death. Each system was run on an interval of 500 sites with periodic boundary conditions until time 499. (As mentioned earlier, such simulations can in general be deceptive due to rare nucleation.)

Analysis of one-dimensional systems related to GHM started in [1, 4, 11, 12]. The earliest references study the related *cyclic cellular automaton* (CCA). In the CCA ζ_t *every* color advances by contact, that is, x changes color from k to $(k+1) \bmod \kappa$ only if the neighborhood of x contains more than θ sites with color $(k+1) \bmod \kappa$. More formally,

$$\zeta_{t+1}(x) = \begin{cases} (\zeta_t(x) + 1) \bmod \kappa, & \text{if } |\{y \in \mathcal{N}_x : \zeta_t(y) = (\zeta_t(x) + 1) \bmod \kappa\}| \geq \theta, \\ \zeta_t(x), & \text{otherwise.} \end{cases} \tag{1.2}$$

Table 1: Conjectured threshold-range phase diagram for the one-dimensional GHM.

$\theta/r \in$	$(0, 1/\kappa)$	$(1/\kappa, 1/3)$	$(1/3, 1/2)$	$(1/2, \infty)$
phase	uniformly locally periodic	locally periodic	weak death	strong death

All of the four references cited study the *basic* case $r = \theta = 1$. Using ideas from [1], it was shown in [11] that the basic CCA fixates if and only if $\kappa \geq 5$. The precise nature of fluctuations in the basic three-color models is determined in [4, 12]. For basic GHM with $\kappa = 3$, $P(\gamma_t(0) \neq 0) \sim C_1/\sqrt{t}$ for a constant C_1 ([4]), while the analogous CCA result has the form $P(\zeta_t(0) \neq \zeta_t(1)) \sim C_2/\sqrt{t}$ for a constant C_2 ([12]). General one-dimensional two-color CCA are studied in [5, 16], where large deviation techniques are used to compute a "critical threshold" for large r. For rigorous results about the two-dimensional GHM and CCA, as well as conjectures based on computer experiments, we refer the reader to [3, 6, 8, 9, 10, 17].

The major aim in studying the one-dimensional GHM is to determine the long-term behavior of γ_t, for any given r, θ, and κ. The results of this paper suggest the following conclusions.

1. If $r > \kappa(\theta - 1)$, then γ_t is uniformly locally periodic.

2. If $3\theta - 1 \leq r \leq \kappa(\theta - 1)$, then the system is locally periodic, but not uniformly locally periodic.

3. If $2\theta - 1 \leq r < 3\theta - 1$, and $\kappa \geq \kappa_0 = \kappa_0(\theta)$, then γ_t dies out weakly, but not strongly.

4. If $r < 2\theta - 1$, and $\kappa \geq 3$, then the GHM dies out strongly.

By far, the most elusive of these is 3, for which we have only partial rigorous results. Assuming the correctness of 3, one can summarize the four conclusions as shown in the phase diagram of Table 1, which can be assumed valid only for large r, θ, and κ.

The rest of this paper is organized as follows. Section 2 formally introduces the crucial combinatorial concept of an SPO, provides some examples of SPOs, and discusses their influence on the environment. Section 3 gives sufficient conditions for strong death. Section 4 proves a theorem on nonexistence of SPOs. This theorem is sharp for a large number of colors and gives some insight into conditions under which the GHM dies out weakly. The only case for which we can rigorously establish weak death is the basic case $r = \theta = 1$ with an arbitrary number of colors. Section 5 is somewhat technical and includes a fairly detailed asymptotic analysis of the basic case, explaining such issues as nucleation and survival of traveling waves. The paper concludes with section 6 presenting some special combinatorial properties of two-color models.

2. Stable periodic objects

We call a set $A \subset \mathbf{Z}$ together with a coloring $\eta : A \to \{0, 1, \ldots, \kappa\}$ an SPO if every $x \in A$ has at least θ sites of color $(\eta(x) + 1) \bmod \kappa$ in $\mathcal{N}_x$.

Note that, under the dynamics of both GHM and CCA, an SPO repeats its initial color pattern every κ times independently of colors of other sites. It follows that if an SPO exists in the initial state, or is created at some other time, then the system cannot die out weakly. Existence of SPOs for a particular model is therefore crucial for its ergodic behavior but is in general a difficult combinatorial problem. Two-dimensional SPOs are introduced in [9], together with some remarkable examples due to Dan Pritikin.

The following proposition is the basic observation about the existence of SPOs. Throughout this paper we will use the following notation, introduced by an example:

$$\underset{k}{[\,013\,]} \qquad \text{stands for an interval of } 3k \text{ sites with colors} \qquad 013013\ldots013.$$

Below we give several examples of SPOs for various ranges of parameters. We leave out the somewhat tedious checking of the SPO hypotheses.

Proposition 2.1. *The following are SPOs.*

$\kappa = 2,\ r + 1 \geq \frac{3}{2}\theta$:

$$\underset{\lceil\frac{\theta}{2}\rceil}{[\,0\,]}\ \underset{\theta}{[\,1\,]}\ \underset{\lceil\frac{\theta}{2}\rceil}{[\,0\,]},$$

$\kappa = 3, r + 1 \geq 2\theta, r \geq 2$:

$$0\ \underset{\theta}{[\,1\,]}\ \underset{\theta}{[\,2\,]}\ \underset{\theta-1}{[\,0\,]}\ \underset{\theta-1}{[\,1\,]}\ \underset{\theta-1}{[\,0\,]}\ \underset{\theta}{[\,2\,]}\ \underset{\theta}{[\,1\,]}\ 0,$$

$\kappa = 4, r \geq 2\theta$:

$$\underset{\theta}{[\,0\,]}\ \underset{\theta}{[\,1\,]}\ 3\ \underset{\theta}{[\,2\,]}\ \underset{2}{[\,1\,]}\ \underset{\theta-1}{[\,3\,]}\ \underset{\theta}{[\,0\,]}\ \underset{\theta-2}{[\,1\,]}\ \underset{3}{[\,3\,]}\ \underset{\theta}{[\,2\,]}\ \underset{4}{[\,1\,]}\ \underset{\theta-3}{[\,3\,]}$$
$$\ldots\ \underset{\theta}{[\,0\,]}\ 1\ \underset{\theta}{[\,3\,]}\ \underset{\theta}{[\,2\,]},$$

$\kappa = 5, r \geq 2\theta + 1$:

$$\underset{\theta-1}{[\,0\,]}\ \underset{\theta}{[\,1\,]}\ 4\ \underset{\theta}{[\,2\,]}\ 0\ \underset{\theta}{[\,3\,]}\ 1\ \underset{\theta-1}{[\,4\,]}\ \underset{2}{[\,2\,]}\ \underset{\theta-1}{[\,0\,]}\ \underset{2}{[\,3\,]}\ \underset{\theta-1}{[\,1\,]}$$
$$\underset{2}{[\,4\,]}\ \underset{\theta-2}{[\,2\,]}\ \underset{3}{[\,0\,]}\ \underset{\theta-2}{[\,3\,]}\ \underset{3}{[\,1\,]}\ \underset{\theta-2}{[\,4\,]}\ \underset{3}{[\,2\,]}\ \underset{\theta-3}{[\,0\,]}\ \underset{4}{[\,3\,]}\ \ldots,$$

κ *arbitrary,* $r + 1 \geq 3\theta$:

$$\underset{\theta}{[\,0\,]}\ \underset{\theta}{[\,\kappa-1\,]}\ \underset{\theta}{[\,1\,]}\ \underset{\theta}{[\,\kappa-2\,]}\ \underset{\theta}{[\,2\,]}\ \ldots\ \underset{\theta}{[\,\kappa-2\,]}\ \underset{\theta}{[\,1\,]}\ \underset{\theta}{[\,\kappa-1\,]}\ \underset{\theta}{[\,0\,]}.$$

The "monster" SPOs in the five-color cases are very hard to write down for general θ, so just the left end is indicated above. To show what complicated objects these are, we write them down explicitly in the two smallest cases, when $r = 5$ and $\theta = 2$ (28 sites):

$$0\,11\,4\,22\,0\,33\,1\,4\,22\,0\,33\,1\,4\,0\,22\,4\,11\,3\,00\,4 \tag{2.1}$$

and when $r = 7$ and $\theta = 3$ (109 sites):

$$\begin{gathered}00\,111\,4\,222\,0\,333\,1\,44\,22\,00\,33\,11\,44\,2\,000\,3\,111\,4\,222\,00\,33\,11\,44\,22\,00\,333\,1\\ 444\,2\,00\,11\,33\,00\,22\,44\,11\,3\,000\,2\,444\,1\,333\,00\,22\,44\,11\,33\,00\,222\,4\,111\,3\,000\,44.\end{gathered} \tag{2.2}$$

Of course, both objects (2.1) and (2.2) are astronomically rare in the initial state, but they are readily able to generate waves of excitation and thus influence distant sites in the surrounding environment. There are, however "holes," consisting of 0s, which remain unfilled by either dynamics. However, if we increase the radius r to 6 (while keeping $\theta = 2$ and $\kappa = 5$), then the SPO (2.1) becomes "unstoppable;" it will eventually control every site that is not in the domain of another SPO. These effects are formulated more precisely (and more generally) in the following theorem. Many of the arguments in its proof are similar to the ones in [3, 8].

Theorem 2.1.

1. *If an SPO exists, then γ_t is locally periodic.*
2. *If an SPO exists and $r > \kappa(\theta - 1)$, then γ_t is uniformly locally periodic.*
3. *Assume that there is an SPO with θ contiguous 0s at its left end. If, in addition, $r \leq \kappa(\theta - 1)$, then the system is not uniformly locally periodic.*

Proof. Regions between SPOs are finite, hence periodic. Part 3 of the locally periodic definition is clear because there are always sites that are part of SPOs and "out of phase." This proves statement 1.

To prove statement 2, first note that somewhere in the initial state there is an SPO interval. Let I be the largest interval containing it with the property that every site in I is eventually periodic with period κ. If it is not $\mathbf{Z}$, then there is a site x on its boundary, that is, $x \notin I$ and, say, $x + 1 \in I$.

Let us assume first that $r > (\theta - 1)\kappa$. This implies that there is at least one color that has θ representatives in $\mathcal{N}_x \cap I$. Thus there exists a t_0 such that $\gamma_{t_0+n\kappa}$ has θ representatives of color 1 in $\mathcal{N}_x \cap I$ for $n = 0, 1, \ldots$. Define

$$D(n) = \begin{cases} 0, & \text{if } \gamma_{t_0+n\kappa}(x) = 0, \\ \kappa - \gamma_{t_0+n\kappa}(x), & \text{otherwise.} \end{cases}$$

Assume first that $D(n) > 0$ for every n. Then $D(n)$ is nonincreasing; hence there exists some n_0 so that $D(n)$ is a nonzero constant for $n \geq n_0$. Hence x must change color every time for $t \geq t_0 + n_0\kappa$. If $D(n) = 0$ for some n_0, then $\gamma_{t_0+n\kappa+1}(x) = 1$ for all $n \geq n_0$, hence again x changes every time. It follows that $x \in I$, a contradiction.

It remains to prove statement 3. Let $k = \lfloor r/(\theta - 1) \rfloor$ and $a = r - k(\theta - 1)$. Then $k \leq \kappa$. If $a \neq 0$, form the following object

$$\begin{array}{cccccccccccc} [\,0\,] & [\,1\,] & \cdots & [\,k-1\,] & [\,k\,] & 1 & 2 & & k-1 & [\,k\,] & [\,k+1\,] & \\ \theta - 1 & \theta - 1 & & \theta - 1 & a & & & \cdots & & \theta - a & \theta & \ldots, \end{array} \tag{2.3}$$

where the block of θ sites with color $(k+1) \bmod \kappa$ is the left end of an SPO. If $a = 0$, then form the same object, except start the SPO with the block of θ sites with color k. This object is clearly an SPO if $a = 0$ and otherwise under the additional requirement that $r + 1 \geq 2\theta + k - 1$, which is for $\theta \geq 3$ equivalent to $r \geq 2\theta + 1_{\theta=3}$. Luckily, the exceptional case $\theta = 3$, $r = 2\theta = 6$ has $a = 0$; therefore object (2.3) is always an SPO. A sufficiently long interval of 0s, bordered by object (2.3) on the right and its mirror image on the left, will never change. ■

We should note that we do not address the important statistical question that arises in the nonuniform local periodic cases: Which sites are more prevalent—those that are eventually periodic or those that are eventually 0 (assuming that there is no third possibility)? (See [5, 16] for such analysis of the two-color CCA.)

3. Strong death

An argument based on spatial restrictions (see Theorem 1 of [10]) can be used to establish *global* relaxation in a finite number of steps if $\theta > r$ and $\kappa \geq 3$. Here we use a one-dimensional setting to prove much better sufficient conditions for strong death.

Theorem 3.1.

1. *If $r + 1 < \frac{3}{2}\theta$, and $\kappa = 2$, then γ_t dies out strongly.*
2. *If $r + 1 < 2\theta$ and $\kappa \geq 3$, then γ_t dies out strongly.*

Proof. In both cases we show that a sufficiently large block of 0s has to expand. Specifically, we show that any block of $\ell > 4r$ sites with color 0 at time t will produce a block of at least size $\ell + 1$ at some time $s \in \{t+1, \ldots, t + 2\kappa - 1\}$. Without loss of generality we can assume that γ_t is 0 on $\{x < 0\}$, $\gamma_t(0) = \kappa - k$ for some $k \in [1, \kappa - 1]$, and prove that $\gamma_s = 0$ on $\{x \leq 0\}$ for some s specified above.

We now prove statement 1. Let a be such that the site $r - a$ is the position of the θth 1 on the nonnegative axis at time t. If $a \leq 0$, then we may take $s = t + 1$. If $a > 0$, then we claim that there are at most $\theta - 1$ sites with color 1 in $\mathcal{N}_0$ at time $t + 1$. There are exactly a 1s on the negative axis and at most $r - a - \theta + 1$ of them in $[1, r - a]$. Even assuming that all sites $r - a + 1, \ldots, r$ have color 1, we cannot have more than $r + a + 1 - \theta$ 1s in $\mathcal{N}_0$. But $a \leq r - \theta + 1$, and therefore $r + a + 1 - \theta \leq 2r - 2\theta + 2 < \theta$. Hence we can take $s = t + 2$ in this case.

We now prove statement 2. We can assume that at some time $v \in [t + 1, t + \kappa]$ a site on the negative axis becomes 1. Then there exists an integer $a \in [1, r]$ so that $\gamma_v(x)$ is 0 for $x < -a$, is 1 for $x \in [-a, -1]$, has exactly $\theta - 1$ 2s in $[0, r - a - 1]$, and is 2 for $x = r - a$.

As a first step, we claim that $\gamma_u(x) = 0$ for $x \leq -a - 1$ and $v \leq u \leq v + \kappa - 1$. At times from v to $v + \kappa - 2$, the number of sites with color different from 1 in $\mathcal{N}_{-a-1} \cap \{x \geq -a\}$ is at least $\theta - 1$; since $r < 2\theta - 1$, this does not leave enough space for κ sites with color 1.

Now the argument splits into two cases. One case is that $\gamma_v(0) \neq 0$. We claim that then $\gamma_{v+\kappa-1}(0) = 0$. This is enough, since $\gamma_{v+\kappa-1}(x) = 0$ for $x \leq 0$. The claim is a triviality if $\gamma_v(0) = 1$ and is otherwise implied by the fact that for times between $v + 1$ and $v + \kappa - 2$ there are at most $r - a - (\theta - 1) + a = r - \theta + 1 < \theta$ sites with color 1 in $\mathcal{N}_0$.

The other case is that $\gamma_v(0) = 0$. If $\gamma_{v+1}(0) = 0$, then the argument proceeds as above. However, if $\gamma_{v+1}(0) = 1$, then there must exist at least $\theta - a$ 1s in $[1, r]$ at time v. At time $v + \kappa - 2$ we have the following situation: $(\kappa - 1)$ on $[-a, -1]$, $\kappa - 2$ at the origin, at least $\theta - 1$ colors 0 in $[1, r - a - 1]$, 0 at $r - a$, and at least $\theta - a$ sites with color $\kappa - 1$ in $[1, r]$. We claim that there are less than θ 1s in $[1, r - 1]$ at time $v + \kappa - 1$. Assuming this claim, it is clear that $\gamma_{v+\kappa} = 0$ on $\{x \leq 0\}$.

To prove the claim, assume that there are b 0s in $[r - a + 1, r]$ at time $v + \kappa - 2$. Since there are at least $2\theta + b + 1 - a$ sites with color different from 1 in $[0, r]$ (this is the only place where we use that $\kappa \geq 3$; i.e., $\kappa - 1 \neq 1$), every 0 that turns to 1 at the next time needs to reach at least $3\theta + b - a - r$ 1s outside $[0, r]$. To have θ 1s in $[0, r]$ at time $v + \kappa - 1$, we have to have $\theta - b$ 0s in $[0, r - a]$, which at time $v + \kappa - 1$ turn into 1s. However, this implies that the site $r - a - (\theta - b) + 1$ reaches θ 1s at time $v + \kappa - 2$. Therefore $3\theta + b - a - r + a + (\theta - b) \leq r + 1$, or $4\theta \leq 2r + 1$, an impossibility. This ends the proof of statement 2. ∎

We are now in position to characterize the behavior of the general three-color GHM.

Corollary 3.1. *Assume that $\kappa = 3$. Then γ_t*

1. *dies out strongly if and only if $r + 1 < 2\theta$,*
2. *dies out weakly, but not strongly, if and only if $r = \theta = 1$,*
3. *is locally periodic if and only if $r + 1 \geq 2\theta$, $r \geq 2$, or it*
4. *is uniformly locally periodic if and only if $r + 1 \geq 3\theta - 1$, $r \geq 2$.*

Proof. This follows immediately from Proposition 2.1, Theorem 2.1, and Theorem 1 in [4] (see also Theorem 5.1 below). ∎

4. Nonexistence of stable periodic objects

Assume that r, θ, and $\kappa \geq 3$ are such that an SPO does not exist. In such a case there is still the possibility of traveling waves. Indeed, consider the configuration

$$\begin{matrix} [\,1\,] & [\,2\,] & [\,3\,] & & [\,\kappa-1\,] \\ \lfloor\frac{r+1}{2}\rfloor & \lceil\frac{r+1}{2}\rceil & \lfloor\frac{r+1}{2}\rfloor & \cdots & a \end{matrix},$$

where a is either $\lfloor\frac{r+1}{2}\rfloor$ or $\lceil\frac{r+1}{2}\rceil$, depending on the parity of κ. This configuration is able to move across 0s, as soon as $r+1 \geq 2\theta$, suggesting that γ_t cannot die out strongly. On the other hand, nonexistence of SPOs seems to indicate that no "stationary" finite configurations are stable; hence the following conjecture.

Conjecture 4.1. Assume that r, θ, and $\kappa \geq 3$ are such that $r+1 \geq 2\theta$, and an SPO does not exist. Then γ_t dies out weakly, but not strongly.

Taking Conjecture 4.1 for granted still leaves one intriguing question open: Does an SPO exist if $r+1 \geq 2\theta$? In view of exotic SPO architectures, exemplified by the example of (2.2), this question is not easy to answer for small numbers of colors. On the other hand, it turns out, perhaps surprisingly, that the large κ design in Proposition 2.1 is *optimal*; there are no SPOs for thresholds larger than $(r+1)/3$. This is the subject of our next result.

Theorem 4.1. *Assume that $r+1 < 3\theta$. Moreover, if $\theta \leq 5$, then assume that $\kappa > 12\theta + 3$, and if $\theta > 5$ assume that $\kappa > 14\theta - 7$. Then an SPO does not exist.*

Proof. We assume that an SPO interval exists and is positioned on $\mathbf{Z}$ so that the site $z > 0$ is its rightmost site, that the origin has color 0, and that there are at least θ representatives of each of the colors $1, \ldots, \kappa-1$ in $[1, z]$, but exactly $\theta - 1$ representatives of color 0 in $[1, z]$. We can also assume that this SPO is the shortest possible. This, in particular, implies that every color must eat; in other words, if the color of an x is k, then there is at least one site with color $(k-1) \bmod \kappa$ in $\mathcal{N}_x$.

Step 1. For each $k = 1, 2, \ldots, \kappa-1$, every site in $[0, z]$ with color $\kappa - k$ is in the interval $[0, r + (k-1)(r-\theta+1)]$.

Each of $(\kappa - 1)$ must lie in $[0, r]$, because otherwise some $\kappa - 1$ would only see colors in $[1, z]$, which contain only $\theta - 1$ 0s. For each $k < \kappa$, determine $x_k \geq 0$ so that $\kappa - k$ is the color of x_k, and there are exactly $\theta - 1$ colors $\kappa - k$ in $[x_k + 1, z]$. By the same argument, all colors $\kappa - k - 1$ must lie in $[0, x_k + r]$.

We now prove the claim in this step by induction. If it is true for some $k < \kappa - 1$, then $x_k \leq r + (k-1)(r-\theta+1) - \theta + 1$, and, consequently, all colors $\kappa - (k+1)$ lie in $[0, r + (k-1)(r-\theta+1) - \theta + 1 + r]$.

Step 2. For each $k = 0, 1, \ldots, \kappa - 1$, every site with color k lies in $[0, (k+2)r]$.

If a 0 lies inside $[2r+1, z]$, then (since, by Step 1, every $\kappa - 1$ in $[0, z]$ lies in $[0, r]$) we can discard it, thus reducing the size of the SPO. Hence all 0s lie in $[0, 2r]$. Similarly, if y_k is the position of the rightmost representative of the color $k < \kappa - 1$, then all colors $k+1$ must lie in $[0, y_k + r]$. An induction argument completes the proof.

Step 3. Fix two colors $0 \leq c_1 < c_2 \leq \kappa - 1$. Assume that there exists an $x \in [0, z]$, such that each color $k \in [c_1, c_2]$ has at least θ representatives in $[x, z]$, but does not appear in $[x - r, x - 1]$. Then each color $k \in [c_1, c_2]$ has at least θ representatives in $[0, (c_1 + 2)r + (k - c_1)(r - \theta + 1)]$.

By Step 2, we know this holds for $k = c_1$. Fix a $y \in [x, z]$, and assume that a color $k \in [c_1, c_2)$ has at least θ representatives in $[x, y]$. Then there is at least one k in $[0, y - \theta + 1]$ and, consequently, at least θ colors $k + 1$ in $[0, y - \theta + 1 + r]$ (nothing to the left of 0 can influence these representatives with color k). Again, a simple induction argument establishes Step 3.

Step 4. Assume that $\kappa \geq 21$. Then none of the colors $7, 8, \ldots, \kappa - 14$ appears in $[3r + 1, 4r]$.

By Step 1, all sites in $[0, z]$ with colors $\kappa - 1, \kappa - 2, \ldots, \kappa - 10$ lie in $[0, 10(r - \theta + 1) + \theta - 1] \subset [0, 7r + 3]$. By Step 2, all sites in $[0, z]$ with colors 0,1,2,3,4,5 are in $[0, 7r]$. Together, these account for at least 16θ of the sites in $[0, 7r + 3]$. Since $7r + 4 \leq 21\theta - 10$, these leave at most $5\theta - 10$ sites in $[0, 7r + 3]$ unaccounted for.

Assume now that the claim of this step is not true and let k be one of the colors $7, 8, \ldots, \kappa - 14$ which is represented in $[3r + 1, 4r]$. Since this site must eat something, there has to be at least $\theta - 1$ other ks in $[r + 1, 6r]$, and at least one $k - 1$ in $[2r + 1, 5r]$. In turn, there must be at least θ colors $k - 1$ in $[1, 7r]$. Also, the original color k must be eaten by θ colors $k + 1$, all of which have to be $[2r + 1, 5r]$. These force at least θ colors $k + 2$ in $[r + 1, 6r]$, and, in turn, at least θ colors $k + 3$ in $[1, 7r]$. We thus have 5θ colors among $6, 7, \ldots, \kappa - 11$, a contradiction.

Step 5. Assume $\theta \leq 5$, and $\kappa > 12\theta + 3$. Then the theorem is true.

The colors $0, \ldots, 6\theta + 2$ and $\kappa - 6\theta - 1, \ldots, \kappa - 1$ are in this case all distinct. We will count the number of colors in the interval $I = [0, 12\theta^2 - 3\theta - 2]$.

If $1 \leq k \leq 6\theta + 1$, then $k(r - \theta + 1) + \theta - 1 \leq 12\theta^2 - 3\theta - 2$, hence, by Step 1, there are at least θ of each of the colors $\kappa - 6\theta - 1, \ldots, \kappa - 1$ in I. On the other hand, if $7 \leq k \leq 6\theta + 2$, then $10r + (k - 8)(r - \theta + 1) \leq 12\theta^2 - 2\theta - 7 \leq 12\theta^2 - 3\theta - 2$. By Steps 3 and 4 (note that $6\theta + 2 \leq \kappa - 14$), there are at least θ of colors $7, \ldots, 6\theta + 2$ in I. Hence we have produced $\theta(12\theta - 3)$ sites in I, an impossibility.

Step 6. Assume that $\theta \geq 5$, and $\kappa > 14\theta - 7$. Then the theorem is true.

Then the colors $0, 1, \ldots, 7\theta - 3$ and $\kappa - 7\theta + 4, \ldots, \kappa - 1$ are all different. This time, let $I = [0, 14\theta^2 - 13\theta - 2]$. By Step 1, there are at least θ of each of $\kappa - 1, \kappa - 2, \ldots, \kappa - 7\theta + 4$ in I. By Steps 3 and 4, there are also at least θ of the colors $7, \ldots, 7\theta - 3$ in I, producing $14\theta^2 - 13\theta$ sites in I, a contradiction. ■

In addition to the method used in the above proof, which works well for a large number of colors, we present one that is more appropriate for a smaller number of colors.

Proposition 4.1. *Assume that $\kappa \geq 4$ and $r+1=2\theta$. Then an SPO does not exist.*

Proof. Let $\tilde{\gamma}_t \in \{0,1,\dots,\kappa-1\}^{\mathbf{Z}}$ be the following modification of the GHM:

$$\tilde{\gamma}_{t+1}(x) = \begin{cases} (\tilde{\gamma}_t(x)+1) \bmod \kappa, & \text{if } \tilde{\gamma}_t(x) > 0 \\ & \text{or } (|\{y \in \mathcal{N}_x : \tilde{\gamma}_t(y) = 1\}| \geq \theta \text{ and} \\ & \qquad \kappa - 1 \in \tilde{\gamma}_t(\mathcal{N}_x)), \\ 0, & \text{otherwise.} \end{cases}$$

We conjecture that $\tilde{\gamma}_t$ dies out strongly if and only if there is an SPO for γ_t. We do not know how to prove this, but at least one direction is clear, if $\tilde{\gamma}_t$ dies out strongly, then there can be no SPOs of minimal length, and hence no SPOs at all, for γ_t. The proof that $\tilde{\gamma}_t$ dies out strongly in this case is very similar to the proof of statement 2 of Theorem 3.1 and hence omitted. ∎

5. The basic case $r = \theta = 1$

The method of [4], which assumes that $\kappa = 3$, does not carry immediately over to arbitrary κ, because, vaguely speaking, it is not straightforward to define an edge so that the only way it can be annihilated is by collision with another edge. We will make this clear below; we begin with the statement of the main result.

Theorem 5.1. *Assume that $r = \theta = 1$ and $\kappa \geq 3$.*

1. *There exists a constant $A = A(\kappa)$ so that $P(\gamma_t(0) = 1) \sim A/\sqrt{t}$ as $t \to \infty$.*
2. *The system dies out weakly, but not strongly.*
3. *As $\kappa \to \infty$, $A(\kappa) \sim \left(\dfrac{8\kappa}{\pi}\right)^{1/4} e^{-\kappa/2}$.*

It is perhaps possible to give further terms in the asymptotic expansion of $A(\kappa)$, but explicit computation may be hard even for $\kappa = 4$.

Before starting with the proof, we introduce some useful language. For a fixed $z \in \mathbf{Z}$, and an integer $k \in [2,\kappa]$, we say that $z, z+1, \dots, z+k-1$ is a *right percolating path of length k*, if $\gamma_0(z) = 1$, and $\gamma_0(z+i) \in \{0\} \cup [\kappa-i+1, \kappa-1]$ for $i = 1, \dots, k-1$. A *left* percolating path is defined analogously.

Fix an $x \in \mathbf{Z}$, assume that $\gamma_0(x) = 1$, and fix $t \geq 1$. If it is *not* the case that $\gamma_t(x-t) = 1$ and $\gamma_t(x-t+1) = 2$, then the 1 at x *dies out on its left by time t*. If it dies out by time $t+1$, but not by time t, we say that it dies out *at time t*. The definition of a 1 that *dies out on its right* is analogous.

For a $x \in \mathbf{Z}$, define by x^+ the bond $\{x, x+1\}$ and x^- by $(x-1)^+$. A bond x^+ is *open* at time t if $\gamma_t(x) - \gamma_t(x+1) \in \{-1, 0, 1\} (\bmod \kappa)$, and *closed* otherwise. It is easy to check that an open bond stays open forever.

Assume that $\gamma_0(x) = 1$ and that this 1 dies out on its left at time $t-1$. Then $(x-t)^+$ is either open at time $t-1$, in which case the 1 "collides with another 1," or else it is closed. The second possibility does not arise when $\kappa = 3$, this being the only case when all bonds are open at time 0.

Next in line is a key lemma that enables us to use the techniques from [4, 12]. For $x, y \in \mathbf{Z}$ we define the interval $[x, y]$ to be *harmless* if $\gamma_0(x) \neq 1$, $\gamma_0(y) \neq 1$, and there are no 10 or 01 pairs in $[x, y]$ at time 0. We define a site x to be a *left edge* if the following two conditions hold.

- $\gamma_0(x) = 1$.
- If $[y, y+2\kappa-5]$ is the first harmless interval of length $2\kappa-4$ completely on the left of x, and the colors of sites z, $z < y$, are changed to 0, while preserving other colors, the 1 at x never dies out on its left.

Again, a *right edge* is defined analogously. What the next lemma says, in short, is that a left edge can die only by annihilation against a right edge.

Lemma 5.1. *Assume there is a left edge at x which dies out on its left at time $t-1$. Then there is a right edge either at $x-2t+1$ or at $x-2t+2$, which dies out on its right at time $t-1$.*

Proof. We start by figuring out how a bond can stay closed for a long time. Fix a $y \in \mathbf{Z}$ and a time t and define $F_t^l(y)$ to be the event at which the following happens.

- y^+ is closed at time 0.
- There is a $y_1 \in [y-\kappa+3, y]$ such that $\gamma_0(y_1) = 1$, and this 1 dies out on its right at time $y - y_1$.
- There is a $y_2 \in [y-y_1+2, y-y_1+\kappa-1]$ with $\gamma_0(y_2) = 1$, and this 1 dies out on its left at time $y_2 - y - 1$.
- There is a $y_3 \in [y-y_2-\kappa+3, y-y_2]$ with $\gamma_0(y_3) = 1$, and this 1 dies out on its right at time $y - y_3$.

And so on, alternating between left and right of y, until a y_i is found so that $|y - y_i| \geq t - 2\kappa$. Note that 1s that arrive to 0 from the left are no more that $2\kappa - 4$ units apart. A harmless interval of length at least $2\kappa-4$ in $[y-t+2\kappa, 0]$ hence prevents $H_t^l(y)$ from happening.

We define $F_t^r(y)$ by reflection of $F_t^l(y)$ over the bond y^+. The claim is that

$$\{y^+ \text{ is closed at time } t\} \subset F_t^l(y) \cup F_t^r(y). \tag{5.1}$$

To prove this, observe that a closed bond $0k$ at time t will become open at the time $t+\kappa-1-k$ unless the 0 changes to 1 at some time $s \in [t+1, t+\kappa-1-k]$.

Now, (5.1) implies that the edge $(x-t)^+$ is open at time $t-1$, since the length of the harmless interval implies that neither $H_t^l(x-t)$ nor $H_t^r(x-t)$ can happen. Hence there is a 1, either at $x-2t+1$ or at $x-2t+2$, that dies on its right at time $t-1$. We have to prove that this 1 is a right edge. Let $[y-\kappa+5, y]$ be the first harmless interval of length $2\kappa-4$ on the right of this 1. Since $y<x$, changing all colors in $[y+1,\infty)$ to 0 also eliminates the 1 at x. With this initial state, the 1 at $x-2t+1$ or $x-2t+2$ hence lives forever on its right and is therefore a right edge. ■

For a site x we define the probability measures $P_x^r = P(\cdot\,|x$ is a right edge) and $P_x^l = P(\cdot\,|x$ is a left edge). Let E_x^l, E_x^r, Var_x^l, and Var_x^r be the respective expectation and variance operators. Moreover, we define random variables L_x and R_x. Let $y>x$ be the position of the first right (resp. left) edge on the right of x. Then L_x (resp. R_x) is the number of left (resp. right) edges in (x,y). Define

$$\begin{aligned} p &= p_\kappa = P(0 \text{ is a right edge}), \\ q &= q_\kappa = P_x^r(L_x = 0). \end{aligned}$$

Lemma 5.2.

1. *$q = P_x^l(R_x = 0)$.*
2. *Define the random variables $R_k(x)$ by $R_0(x) = x$ and $R_{k+1}(x) = \inf\{y > R_k(x) : y \text{ is a right edge}\}$. Under P_x^r, $R_{k+1}(x) - R_k(x)$, $k \geq 0$ are i.i.d. and so are $L_{R_k(x)}$, $k \geq 0$.*
3. *$E_x^r(L_x) = E_x^l(R_x) = 1$.*
4. *$\mathrm{Var}_x^r(L_x) = \mathrm{Var}_x^l(R_x) = \dfrac{2q}{1-q}$.*

Proof. We argue by symmetry that statement 1 holds:

$$\begin{aligned} P_x^r(L_x = 0) &= \sum_{y>x} P_x^r(y \text{ a right edge, no edges in } (x,y)) \\ &= p^{-1} \sum_{y>x} P(x \text{ a right edge, } y \text{ a right edge, no edges in } (x,y)) \\ &= p^{-1} \sum_{y>x} P(x \text{ a left edge, } y \text{ a left edge, no edges in } (x,y)) \\ &= P_x^l(R_x = 0). \end{aligned}$$

Furthermore, statement 2 holds because both $R_{k+1}(x) - R_k(x)$ and $L_{R_k(x)}$ depend only on colors to the right of $R_k(x)$. Finally, to prove statements 3 and 4, we note that, by statement 1 and standard markovian arguments,

$$P_x^r(L_x = k) = q^{k-1}(1-q)^2,$$

for $k \geq 1$. ■

Proof of statement 1 in Theorem 5.1. Define the events

$$\begin{aligned} G_t^l(x) &= \{\gamma_0(x) = 1, \text{this 1 survives on its left up to time } t\}, \\ G_t^r(x) &= \{\gamma_0(x) = 1, \text{this 1 survives on its right up to time } t\}, \\ H_t^l(x) &= G_t^l(x) \cap \{x \text{ is a left edge}\}, \text{ and} \\ H_t^r(x) &= G_t^r(x) \cap \{x \text{ is a right edge}\}. \end{aligned}$$

Note that $P(G_t^l(x) \setminus H_t^l(x)) \leq P(\text{there is no harmless interval in } [x, x+t]) \leq e^{-Ct}$, where $C = C(\kappa) > 0$ is a constant. It follows that

$$0 \leq P(\gamma_t(0) = 1) - P(H_t^l(t) \cup H_t^r(-t)) \leq 2e^{-Ct}. \tag{5.2}$$

Assume for the moment that we have proved that, as $t \to \infty$,

$$P(H_t^r(0)) \sim \frac{B}{\sqrt{t}}, \tag{5.3}$$

for some constant $B = B(\kappa)$. Then it follows by symmetry and translation invariance that $P(H_t^l(t)) = P(H_t^r(-t)) \sim B/\sqrt{t}$, while

$$\begin{aligned} P(H_t^l(t) \cap H_t^r(-t)) &\leq P(G_{t/2}^l(t) \cap G_{t/2}^r(-t)) \\ &= P(G_{t/2}^l(t))P(G_{t/2}^r(-t)) \sim \frac{2B^2}{t} \end{aligned}$$

as $t \to \infty$. Hence (5.2) and (5.3) together imply that $P(\gamma_t(0) = 1) \sim 2pB/\sqrt{t}$.

It remains to prove (5.3); as this consists of checking that the technique in [4] applies to our case, we merely sketch the argument. Let M_n be the number of right edges in $[0, n]$. It follows from the result in [15] that $P(|M_n - pn| > \epsilon n) < e^{-Cn}$ for a sufficiently large n, where $C = C(\kappa, \epsilon)$ is a positive constant. For a fixed m, define the event

$$F_m = \left\{ \sum_{i=0}^{k} L_{R_i(x)} < k \text{ for } k = 1, \ldots, m \right\}.$$

It follows from theorems about random walks ([4, 7]) that

$$P_x^r(F_m) \sim \sqrt{\frac{q}{\pi(1-q)}} \cdot \frac{1}{\sqrt{m}},$$

as $m \to \infty$. Now assume that 0 is a right edge. Then

$$F_{M_{2t-1}+1} \subset H_t^r(0) \subset F_{M_{2t-1}}.$$

It is now straightforward to see that this implies that for each $\epsilon > 0$ there exist constants $C_1, C_2 \in (0, \infty)$ (depending on ϵ and κ) so that for large t

$$|\sqrt{t}P(H_t^r(0)) - B| < C_1\epsilon\sqrt{t} + C_1\sqrt{t} \cdot e^{-C_2 t},$$

where

$$B = \sqrt{\frac{pq}{2\pi(1-q)}}. \tag{5.4}$$

Sending first $t \to \infty$ and then $\epsilon \to 0$ completes the proof. ■

Note, for later use, that $A = 2B$, where B is given by (5.4).

Lemma 5.3. *γ_t does not die out strongly.*

Proof. Let $F_x = \{\gamma_t(x) \text{ changes i.o.}\}$. Since $F_0 = \cap_{x\in\mathbf{Z}} F_x$, $P(F_0)$ is either 0 or 1. Therefore, we need only show that $P(F_0) > 0$.

Abbreviate $H_t = H_t^r(-t)$ (the event from the foregoing proof), and denote

$$\alpha = \liminf_{t\to\infty} \frac{\sum_{0\le i\le j\le t} P(H_i \cap H_j)}{\left(\sum_{0\le i\le t} P(H_i)\right)^2}. \tag{5.5}$$

If we prove that $\alpha < 1$, then it will follow by the Rényi–Lamperti lemma ([2], page 87), that $P(H_t \text{ i.o.}) > 0$, but $\{H_t \text{ i.o.}\} \subset F_0$.

The denominator in (5.5) is clearly $\sim 4B^2 t$ as $t \to \infty$. Call the numerator in (5.5) s_t. We will use the estimate $\limsup_{t\to\infty} s_t/t \le \limsup_{t\to\infty}(s_t - s_{t-1})$. Fix an $\epsilon > 0$ and let t be so large that for $i > t^{1/4}$, $P(H_i) \le (B+\epsilon)/\sqrt{i}$. Then, by the previous proof,

$$\begin{aligned}
s_t - s_{t-1} &= \sum_{i=0}^{t} P(H_i \cap H_t) \le \sum_{i=0}^{t} P(H_i \cap H_{t-i}) \\
&\le \sum_{t^{1/4}\le i\le t-t^{1/4}} \frac{(B+\epsilon)^2}{\sqrt{t}\sqrt{t-i}} + \frac{2t^{1/4}(B+\epsilon)}{\sqrt{t}} \\
&\le (B+\epsilon)^2 \int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx + 2(B+\epsilon)t^{-1/4} \\
&= (B+\epsilon)^2\pi + 2(B+\epsilon)t^{-1/4}.
\end{aligned}$$

Hence, $\alpha \le \pi/4 < 1$, and the lemma is proved. ■

Lemma 5.4. *γ_t dies out weakly.*

Proof. The argument is a standard application of the Borel–Cantelli lemma, so we leave out the details. ■

Before we prove statement 3 in Theorem 5.1, let us define an *immediate left* (resp. *right*) *edge* to be a right (resp. left) percolating path of length κ followed immediately by a harmless interval of $\kappa - 3$ sites.

Lemma 5.5. *As $\kappa \to \infty$, $p_\kappa \sim \sqrt{2\pi\kappa}\cdot e^{-\kappa}$.*

Proof. A necessary condition for 0 to be a right edge is that a right percolating path starts at 0. A sufficient condition is that an immediate right edge begins at 0. Both comparison events are $\sim \kappa!/\kappa^\kappa$. ■

Lemma 5.6. $\limsup_{\kappa\to\infty} q_\kappa \le \frac{1}{2}$.

Proof. Recall that $q_\kappa = P_0^l(R_x = 0)$. Let F be the event that, moving from 0 to the right, we see a right percolating path before we see a left one (a percolating path starting at 0 counts here). Clearly $1 - q_k \ge P(F)\left(1 - \frac{4}{\kappa}\right)$.

Let $0 = X_0, X_1, X_2, \ldots$ be the successive positions of 1s on the nonnegative axis. Let $\tau = \inf\{k \ge 0$: there is exactly one percolating path in $[X_k, X_{k+1}]\}$, and let G be the event that the percolating path in $[X_\tau, X_{\tau+1}]$ is going to the right. Clearly, $G \subset F$ and $P(G) = \frac{1}{2}$. ∎

Lemma 5.7. $\liminf_{\kappa\to\infty} q_\kappa \ge \frac{1}{2}$.

Proof. Now let F be the event that, going from 0 to the right, we see an immediate left edge before we see a right percolating path. Clearly $q_\kappa \ge P(F)$. Let F' be the event that we see an immediate left edge before an immediate right edge. Then $P(F^c)(1 - \frac{4}{\kappa}) \le P(F'^c)$.

First, define $Y_1 = \inf\{x > 0 : \gamma_0(x) = 1, \gamma_0(x+1) = 0\}$, and let $Y_k = \inf\{x > Y_{k-1} : \gamma_0(x) = 1, \gamma_0(x+1) = 0\}$ for $k \ge 2$ be the successive 10s on the positive axis. Define σ to be the first k for which Y_k is an immediate right edge (note that Y_σ is the first immediate right edge on the nonnegative axis, unless there is one exactly at the origin). Then $P(\gamma_0(Y_\sigma - 1) = 0) \le 2/\kappa$.

Now define $0 = X_0, X_1, X_2, \ldots$ recursively by $X_k = \inf\{x > X_{k-1} + 1 :$ $(\gamma_0(x), \gamma_0(x+1))$ is either (1,0) or (0,1)$\}$. Define τ to be the first index $k \ge 1$ such that there is exactly one immediate (left or right) edge in $[X_k, X_{k+1}+1]$. Moreover, let the stopping time τ' be the first k for which there are two immediate edges in $[X_k, X_{k+1}+1]$. (Note that there cannot be more than 2, one left and one right.) Let G be the event that the immediate edge in $[X_\tau, X_{\tau+1}+1]$ is the left one. By a symmetry argument, $P(G) = 1/2$. The last step is to show that $P(G \setminus F') \to 0$ as $\kappa \to \infty$. To this end, observe that

$$G \setminus F' \subset \{\text{a right percolating path starts at } 0\} \cup \{\tau' < \tau\} \cup \{\gamma_0(Y_\sigma - 1) = 0\}.$$

The only thing left to show is that $P(\tau' < \tau)$ is small. Actually, $P(\tau' < \tau) \le P(\text{a right percolating path at } 0 | \gamma_0(0) = 1, \gamma_0(1) = 0)$, which is exponentially small in κ. ∎

6. The two-color models

Certain "energy" techniques seem to apply only to two-color models [14]. In this section we show how they imply a fairly complete description of the behavior of the two-color GHM. We include some discussion about the two-color CCA mainly to illustrate how much trickier cyclic models are.

Proposition 6.1. *Assume that $\kappa = 2$. Then, for every x, $\gamma_t(x) = \gamma_{t+2}(x)$ for all but finitely many t. Moreover, γ_t*

1. *dies out strongly if and only if $r + 1 < \frac{3}{2}\theta$,*
2. *is locally periodic if and only if $r + 1 \ge \frac{3}{2}\theta$, or*
3. *is uniformly locally periodic if and only if $r + 1 \ge 2\theta$.*

Hence, every point eventually becomes either 0 or a part of an SPO. Of course, this says nothing new if γ_t either is uniformly locally periodic or dies out strongly.

Proof. The three assertions 1, 2, and 3 follow from Proposition 2.1, Theorem 2.1, and Theorem 3.1. To prove that $\gamma_{t+2}(x) = \gamma_t(x)$ eventually for every site x, we first notice that it is enough to prove this when an SPO exists; therefore we can restrict our attention to γ_t on a finite interval. More precisely, we fix an $n > 0$, investigate γ_t on $[0, n-1]$ with free boundary, and prove that $\gamma_t = \gamma_{t+2}$ eventually, for arbitrary γ_0. For this, we use an idea from [14]. Let us interpret γ_t as an n-dimensional column vector, and consider the following "energy" functional

$$F(t) = \langle A\gamma_t, \gamma_{t-1}\rangle - \langle b, \gamma_t + \gamma_{t-1}\rangle. \tag{6.1}$$

Here, A is an $n \times n$ matrix with entries $a_{i,j} = 0$ if $|i-j| > r$, $a_{i,j} = 1$ if $0 < |i-j| \le r$, and $a_{i,i} = -2r - 1$, whereas b is an n-dimensional vector with all entries $\theta - \frac{1}{2}$. These are chosen so that the ith coordinate of $A\gamma_t - b$ is never 0 and positive if and only if $\gamma_{t+1}(i) = 1$. Since A is symmetric,

$$F(t+1) - F(t) = \langle A\gamma_t - b, \gamma_{t+1} - \gamma_{t-1}\rangle,$$

hence $F(t+1) > F(t)$ unless $\gamma_{t+1} = \gamma_{t-1}$. However, $F(t) \le (4r + 2\theta)n$ for each t, hence F must be eventually constant. ■

Proposition 6.2. *Assume that $\kappa = 2$. Again, for each x, $\zeta_t(x) = \zeta_{t+2}(x)$, except for finitely many t. Moreover, ζ_t*

1. *fixates if $r + 1 \le \frac{4}{5}\theta + 1$ or $r + 1 = \theta$,*
2. *is locally periodic if $r + 1 \ge \theta + 1$, or*
3. *is uniformly locally periodic if $r + 1 \ge 2\theta$.*

We conjecture that ζ_t fixates if and only if $\theta > r$.

Proof. Assume first that $r + 1 < 2\theta$. Assume that $\zeta_0(x)$ is 0 for $x \ge 0$ and arbitrary for $x < 0$. Then we claim that $\zeta_t(x) = 0$ for all t and $x \ge \max\{r - \theta + 1, 0\}$. By monotonicity, it is enough to prove the claim under the assumption that $\zeta_0(x) = 1$ for $x < 0$. Furthermore, we assume that $r \ge \theta$, for otherwise this claim is a triviality. Let $a = r - \theta$. Then ζ_1 is 1 on $[0, a]$, 0 on $[-(a+1), -1]$, and equal to ζ_0 elsewhere. Since $a + 1 - r \le -(a+1)$ and $r - (a+1) < \theta$, $\zeta_2 = \zeta_0$. This proves the claim.

The preceding paragraph has several consequences. First, ζ_t in any case dissolves into countably many finite systems. Second, ζ_t is locally periodic for $\theta + 1 \le r + 1$, but not uniformly so if $r + 1 < 2\theta$. This proves statements 2 and 3.

Let us now assume that $r \le \frac{4}{5}\theta$. Assume that the sites in $[-r, -1]$ have fixated, with the number of 1s there being a. If the origin changes at time t from 0 to 1, then there must be at least $\theta - a$ 1s in $[1, r]$ at time t. Let $T \ge t$

be the first time at which one of the 1s in $[1, r]$ changes to 0. The time T must be finite; otherwise the origin is unable to change back to 0 after t. At time T, there is a site $x \in [1, r]$ that sees $\theta - a$ 1s and θ 0s; hence $2r + 1 \geq 2\theta - a$. Since we can assume that $a \leq \frac{1}{2}r$ (otherwise we can exchange roles of 0s and 1s), we get $r > \frac{4}{5}\theta$, a contradiction. Hence, the origin must fixate, and the procedure can be iterated.

If $\theta = r + 1$, then a very easy argument works. Take an interval of 0s, at least $r+1$ sites long, and let x be the first site to the right of it (the color of x is 1). Either this site becomes 0 at some time (and is stuck at 0 afterwards), or else it never becomes 0 (hence it is stuck at 1). In the second case, all sites in $[x, x+r]$ must be 1 at all times, and the procedure can be continued.

Finally, the first claim is proved with the functional (6.1) with the same b, and A is given by $a_{i,i} = 2\theta - 2r - 1$, $a_{i,j} = 1$ if $0 < |i - j| \leq r$, and $a_{i,j} = 0$ otherwise. ∎

Acknowledgements

We wish to thank David Griffeath and Dan Pritikin for valuable input during the development of this paper.

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