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Bilinear Cellular Automata

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Abstract. Bilinear cellular automata (CA) are those whose next state may be expressed as a bilinear form (inner product) of the neighboring states. In this paper it is shown that, unlike linear CA, the bilinear CA over $\mathbf{Z}_p^p$ are π-universal, that is, capable of simulating any CA of the same dimension, and hence also capable of simulating any (universal) Turing machine. Evidence is given that the bilinear CA over $\mathbf{Z}_m$, the integers modulo m, may be universal as well. (Although, like Conway's Game of Life, this appears to be difficult to establish, even for a prime number of states.) A fairly complete Wolfram classification of the bilinear CA over $\mathbf{Z}_m$ is also given.

1. Polynomial representations of cellular automaton local rules

A *linear* cellular automaton (CA) is one whose next state is given by a linear [multivariate] polynomial in the neighboring states. As might be expected, the linear (additive) CA have been more amenable to analysis [2, 11, 13, 17, 19]. Accordingly, the authors have applied these results for linear CA to the study of their multiplicative cousins, the monomial CA [4]. Here, we take the next logical step in a progressive algebraic approach to the analysis of CAs, by investigating the bilinear CAs, whose next state is given by a bilinear form, that is, an inner product of the neighboring-state vector with itself. This algebraic approach is motivated by a result in [10], which says that

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every CA over a prime number of states admits a polynomial representation of the local rule.[1]

In this paper we show that the bilinear CA over $\mathbf{Z}_p^p$ are π-universal, that is, capable of simulating any CA of the same dimension. It follows immediately that the bilinear CA over $\mathbf{Z}_p^p$ are also T-universal, that is, capable of simulating any (universal) Turing machine. Hence, there can be no algorithm to predict the dynamics of a bilinear CA over arbitrary commutative rings or modules. This contrasts with recent results in [19], indicating that the linear CA are not T-universal, even over arbitrary commutative rings. However, it appears difficult to establish whether the bilinear CA over $\mathbf{Z}_m$ are T-universal (witness the proof of the T-universality of the Game of Life [5]), though evidence is given herein that they may be. We also give a phenomenological classification of the bilinear CA over the state set $\mathbf{Z}_m$ of integers modulo m along the lines of the Wolfram Classes [20].

1.1 Preliminaries

A *ring* is a set that is closed under two associative binary operations, where one operation (called multiplication) distributes over the other (called addition, assumed commutative, having a neutral element and inverses for all its elements). A ring is commutative if multiplication is commutative. For example, $\langle \mathbf{Z}, +, \times \rangle$, the integers under ordinary addition and multiplication, form a commutative ring. Also, $\langle \mathbf{Z}_m, +, \times \rangle$, the integers under addition and multiplication modulo m, form a commutative ring. Likewise, $\langle \mathbf{Z}[x], +, \times \rangle$ the polynomials over $\mathbf{Z}$, (i.e., with integer coefficients), in one indeterminant, under polynomial addition and multiplication, form a commutative ring. As an example of a noncommutative ring, consider the set of square matrices with integer coefficients under matrix addition and matrix multiplication.

Next we define an one-dimensional euclidean CA. Let Σ denote a finite alphabet. An one-dimensional euclidean *configuration space* $\mathbf{C}$, is given by

$$\mathbf{C} = \{s = \cdots s_{-2}s_{-1}s_0s_1s_2\cdots : s_i \in \Sigma\} = \Sigma^{\mathbf{Z}}.$$

When endowed with the metric

$$\rho\left(s^{(1)}, s^{(2)}\right) = |\Sigma|^{-K} \qquad \text{where } K = \min_i\{i : s_i^{(1)} \neq s_i^{(2)}\}$$

$\mathbf{C}$ becomes a topological space equivalent to the product topology, upon which a dynamical system can be defined. Then a CA is a dynamical system $T : \mathbf{C} \mapsto \mathbf{C}$ that commutes with the shift, $\sigma : \mathbf{C} \mapsto \mathbf{C}$, given by $\sigma(s_i) = s_{i+1}$. That is, T is a CA if it is a continuous map and

$$T \circ \sigma \;=\; \sigma \circ T.$$

This is a fundamental result from [10]. Further background about CA can be found in [7] and [22].

[1] The study in [10] was mainly concerned with symbolic dynamical systems in one dimension, the result is easily seen to hold for CA in higher dimensions, since any neighborhood may be ordered in such a way as to produce a neighborhood vector, and hence a polynomial representation of the local CA rule.

Definition 1. A CA is defined as T-universal or π-universal as follows.

1. T-*universal* if it is capable of simulating an arbitrary (universal) Turing machine.
2. π-*universal* if it is capable of simulating an arbitrary CA on the same underlying lattice π.

Now let $\delta : \Sigma^n \to \Sigma$ denote the *local rule* of a CA over a prime number of states (ostensibly, $\Sigma = \mathbf{Z}_p$). Let $\vec{x}_i = (x_i, x_{i+1}, ..., x_{i+d-1})$ be a vector of indeterminates denoting the corresponding states, (herein called the *neighborhood state vector*). Then there is a *unique* polynomial $P(\vec{x})$, in n variables $\vec{x} := x_0, \ldots, x_{n-1}$, such that $P(\vec{x}) = \delta(\vec{x})$. P may also can be expressed as a sum of monomials:

$$P(\vec{x}) \;=\; \sum_k a_k \left(\prod_j x_j^{p_j} \right) \qquad (\text{mod } p).$$

The $a_k \in \mathbf{Z}_p$ are then the coefficients of the monomial terms. This is another result found in [10].

For CA over a composite number of states, there may be no polynomial representation of the local rule, or there may be more than one polynomial representation of the local rule. However, we may augment the original state set to obtain a prime number of states, and use a projection of the local rule from the larger state set onto the original state set. We shall make use of this technique in Example 1.

Example 1. The general polynomial modulo 2 for an elementary CA (with $m := 2$ states and radius $r := 1$ (3 neighbors) in dimension one is given by

$$\begin{aligned} P_{\text{elem}}(x_{-1}, x_0, x_1) \;&=\; c_0 + c_1 x_{-1} + c_2 x_0 + c_3 x_1 + c_4 x_{-1} x_0 \\ &\quad + c_5 x_{-1} x_1 + c_6 x_0 x_1 + c_7 x_{-1} x_0 x_1. \end{aligned}$$

Since the number of states is a prime $p = 2$, each of the $2^8 = 256$ distinct binary assignments for the coefficients c_i, corresponds to a distinct elementary CA rule. We note that the table given in [22] provides *boolean expressions* for the elementary CA, which is not the same as the polynomials representation modulo 2, given here (the difference lies in the XOR operation used here instead of the OR used in standard boolean forms).

Definition 2. A bilinear CA is one whose local rule $\delta : \Sigma^{2r+1} \to \Sigma$ is of the form

$$\delta(\vec{x}) = \vec{x} B \, \vec{x}^T = \sum_i \sum_j b_{ij} x_i x_j$$

where Σ is the finite set of states with an addition and multiplication by a set of scalar coefficients, $\vec{x}^T$ is the transpose of $\vec{x}$, and $B = (b_{ij})$ is the matrix of coefficients with entries $b_{ij} \in \Sigma \equiv \mathbf{Z}_k$.

The Elementary Bilinear CA (in Wolfram numbers)							
0	6	10	12	18	20	24	30
34	36	40	46	48	54	58	60
66	68	72	78	80	86	90	92
96	102	106	108	114	116	120	126
130	132	136	142	144	150	154	156
160	166	170	172	178	180	184	190
192	198	202	204	210	212	216	222
226	228	232	238	240	246	250	252

Table 1: Wolfram numbers of the elementary bilinear CA.

We distinguish bilinear CA from quadratic CA, which in addition to a sum involving products of pairs of neighboring states, also have a linear component and a constant term:

$$\begin{aligned} \delta(\vec{x}_i) &= \vec{x}_i B\, \vec{x}_i^T + L(\vec{x}_i) + c \\ &= \sum_j \sum_k b_{j,k} x_{i+j} x_{i+k} + \sum_j a_j x_{i+j} + c. \end{aligned}$$

Since $x_i^2 = x_i \pmod 2$, we can present the following example.

Example 2. A general polynomial for the *elementary* bilinear CA is

$$\begin{aligned} P(x_{-1}, x_0, x_1) &= c_6 x_0 x_1 + c_5 x_{-1} x_1 + c_4 x_{-1} x_0 \\ &\quad + c_3 x_1 + c_2 x_0 + c_1 x_{-1} \pmod 2, \end{aligned}$$

which is the polynomial of Example 1, with $c_0 = c_7 = 0$.

Hence, there are $2^6 = 64$ elementary bilinear CA. Table 1 lists the elementary bilinear CA by their Wolfram numbers.

While in [22] it is indicated that the elementary CA seem to be too simple to be T-universal, [16] indicates that the elementary rule 54 might be T-universal. Hence, the presence of rule 54 in Table 1 suggests that the bilinear CA over $\mathbf{Z}_m$ might be T-universal. However, we have been unable to find a bilinear polynomial representation for a known T-universal CA. For example, we have established that John Conway's Game of Life [5] *cannot* be expressed as a bilinear polynomial over $\mathbf{Z}_m$ for any modulus m. (A proof is available from the authors.)

2. π-universality of bilinear cellular automata over $\mathbf{Z}_p^p$

In [19], previously known results about linear CA [2, 11, 13, 17] have been extended to linear CA with state sets over arbitrary commutative rings. Hence, it is reasonable to explore the T-universality of bilinear CA over commutative rings other than $\mathbf{Z}_m$. We show that, in contrast, the bilinear CA over $\mathbf{Z}_p^p$ are π-universal, and hence T-universal. Our result relies upon the original construction in [1] of an one-dimensional π-universal CA (UCA). In the same paper, the following was also established.

> For every CA A with m states, there exists an one-way CA A' which simulates A twice slower and A' needs at most $m^2 + m$ states.

Hence, there exists an one-way π-UCA U, with $m = 14^2 + 14 = 210$ states and $n = 2$ neighbors. If we add one more state, we obtain a prime number of states, $p = 211$. This also adds $211^2 - 210^2$ new neighborhoods on which U are not defined. However, we may obtain a new one-way π-UCA U', over $p = 211$ states simply by assigning a random next state, (say 0), to the new neighborhoods. One is then assured by Theorem 19.1 in [10] that the local rule $\delta(x_i, x_{i+1}) = U'(x)_i$ has a polynomial representation $P(x_0, x_1)$ over $\mathbf{Z}_{211}$, such that $P(x_i, x_{i+1}) = \delta(x_i, x_{i+1})$.

Now $P(x_0, x_1)$ can be expressed as a bilinear form in the powers of x_0 and x_1, with coefficient matrix $B = (b_{u,v})$ as follows:

$$P(x_0, x_1) = \sum_{0 \le u,v < p} b_{u,v} x_0^u x_1^v \pmod{p}. \tag{1}$$

We therefore expand each cell x_i, to a p-tuple $\vec{x}_i$, consisting of the powers of x_i over $\mathbf{Z}_p$, (with the convention $x_i^0 = 1$, even when $x_i = 0$), that is,

$$\vec{x}_i = (x_i^0, x_i^1, x_i^2, \ldots, x_i^{p-1}).$$

The expansion of x_i to $\vec{x}_i$ can be illustrated for one-dimensional CA, by writing $\vec{x}_i$ vertically under x_i:

$$x = \cdots [\ x_i\ ]\ [\ x_{i+1}\ ] \cdots$$

$$\vec{x} = \cdots \begin{bmatrix} x_i^0 \\ x_i^1 \\ x_i^2 \\ \vdots \\ x_i^{p-1} \end{bmatrix} \begin{bmatrix} x_{i+1}^0 \\ x_{i+1}^1 \\ x_{i+1}^2 \\ \vdots \\ x_{i+1}^{p-1} \end{bmatrix} \cdots.$$

Note that the jth component of $\vec{x}_i$ is $x_i^{(j)} = x_i^j$, so that P may be written as a polynomial in $2p$ unknowns, given by

$$P(x_i^0, x_i^1, \ldots, x_i^{p-1}, x_{i+1}^0, x_{i+1}^1, \ldots, x_{i+1}^{p-1}) = P(\vec{x}_i, \vec{x}_{i+1}) = \vec{x}_i^T B \vec{x}_{i+1}.$$

Let $\mathbf{C} = \mathbf{Z}_p^{\mathbf{Z}}$ denote the *configuration space* of U', that is, the set of bi-infinite strings over $\mathbf{Z}_p$. Then $P : \mathbf{C} \to \mathbf{C}$ is the bilinear form of U' given in equation (1). Similarly, let $\mathbf{C}^p$ denote the configuration space consisting

$$\begin{tikzcd} \mathbf{C}^p \arrow[r, "Q"] & \mathbf{C}^p \\ \mathbf{C} \arrow[u, "E"] \arrow[r, "P"'] & \mathbf{C} \arrow[u, "E"'] \end{tikzcd}$$

Figure 1: Lift to a bilinear CA.

of bi-infinite strings of expanded cells $\vec{x}_i$ over $\mathbf{Z}_p^p$. And let $E : \mathbf{C} \to \mathbf{C}^p$ given by $E(x)_i = \vec{x}_i$ be the map that *lifts* $\mathbf{C}$ to $\mathbf{C}^p$. We shall construct a bilinear polynomial $Q : \mathbf{C}^p \to \mathbf{C}^p$ for the CA operating on the expanded neighborhoods.

First, we define a polynomial $P^{(j)}(\vec{x}_i, \vec{x}_{i+1})$, the jth component of $\vec{y}_i = E(P(x_i, x_{i+1}))$, as the jth power of P:

$$P^{(j)}(\vec{x}_i, \vec{x}_{i+1}) = (P(\vec{x}_i, \vec{x}_{i+1}))^j = P^j(\vec{x}_i, \vec{x}_{i+1}).$$

Thus, each $P^{(j)}$ is bilinear in $\vec{x}_i, \vec{x}_{i+1}$:

$$P^{(j)}(\vec{x}_i, \vec{x}_{i+1}) \quad = \quad \vec{x}_i^T B^{(j)} \vec{x}_{i+1}.$$

We may therefore define a bilinear polynomial $Q([\vec{x}_i], [\vec{x}_{i+1}])$ over $\mathbf{Z}_p^p$, (p direct products of $\mathbf{Z}_p$), by

$$Q([\vec{x}_i], [\vec{x}_{i+1}]) \quad := \quad \sum_{j=0}^{p-1} P^{(j)}(\vec{x}_i, \vec{x}_{i+1}) \tag{2}$$

so that the diagram in Figure 1 commutes.

We have thus established the following theorem.

Theorem 1. *There exists an euclidean one-dimensional, π-universal, bilinear CA over $\mathbf{Z}_p^p$, that is, it is capable of simulating any other one-dimensional CA.*

Moreover, we may generalize this result to obtain the following theorem.

Theorem 2. *The multilinear CA over $\mathbf{Z}_p^p$ are π-universal, capable of simulating any other CA of the same dimension.*

Proof. As a sketch of the proof we first expand the neighborhoods N_i of each cell i with size $n = |N_i|$ into n vectors, as follows:

$$\vec{N}_i \quad = \quad (\vec{x}_i, \vec{x}_{i+1}, ..., \vec{x}_{i+n-1}).$$

Now the polynomial representation of a CA defined over $\mathbf{Z}_p$, for some prime p, can be expressed as a sum of monomial terms, that is,

$$P(x_i, x_{i+1}, \ldots, x_{i+n-1}) \quad = \quad \sum_{k=1}^{p^{p^n}} c_k \prod_{j=0}^{n-1} x_{i+j}^{p_j}.$$

Three Extensions of the Logical AND					
x_i	x_{i+1}	T	T_0	T_1	T_2
0	0	0	0	0	0
0	1	0	0	0	0
0	2	und	0	0	2
1	0	0	0	0	0
1	1	1	1	1	1
1	2	und	0	0	2
2	0	und	0	0	2
2	1	und	0	0	2
2	2	und	0	2	2

Table 2: Three possible extensions of the logical AND to $\mathbf{Z}_3$.

We interpret each power $x_{i+j}^{p_j}$ as a component of $\vec{x}_{i+j}$, so that P may be expressed as a *multilinear* polynomial over $\mathbf{Z}_p^p$, (as a sum of the component polynomials P^j). The rest of the argument is the same as that for bilinear CAs, but independent of neighborhood size and dimension. ■

An example

As an illustration of the preceding sketch, consider the following one-way one-dimensional CA, the logical AND of x_i with x_{i+1}, given by

$$T(x)_i \quad = \quad x_i x_{i+1} \pmod 2.$$

For the sake of our illustration, we shall extend T to include a rule with state set $\mathbf{Z}_3$. There are several possibilities, three of which; T_0, T_1, and T_2, are given in Table 2. For rule T_0, the next state of each neighborhood containing the new state 2, is assigned state 0. For rule T_2, the next state of each neighborhood containing state 2, is assigned state 2. Rule T_1 is the *match* rule, which returns $x_i = x_{i+1}$ if $x_i = x_{i+1}$ and 0 otherwise.

The corresponding polynomials P_0, P_1, and P_2 over $\mathbf{Z}_3$ are:

$$\begin{aligned} P_0(x_i, x_{i+1}) &= x_i x_{i+1} + x_i x_{i+1}^2 + x_i^2 x_{i+1} + x_i^2 x_{i+1}^2 && \pmod 3, \\ P_1(x_i, x_{i+1}) &= 2x_i x_{i+1}(x_i + x_{i+1}) && \pmod 3, \\ P_2(x_i, x_{i+1}) &= 2x_i + 2x_{i+1} + (x_i + x_{i+1})^2 + x_i^2 x_{i+1}^2 && \pmod 3. \end{aligned}$$

Naturally, we choose to employ the simplest polynomial, P_1, for the purposes of illustrating our methods. First, we give the polynomials for each component of the next state:

$$\begin{aligned} P_1^0(x_i, x_{i+1}) &\equiv 1 \pmod 3, \\ P_1^1(x_i, x_{i+1}) &= 2x_i x_{i+1}(x_i + x_{i+1}) \equiv 2x_i^2 x_{i+1} + 2x_i x_{i+1}^2 \pmod 3, \\ P_1^2(x_i, x_{i+1}) &= x_i^2 x_{i+1}^2 (x_i^2 + 2x_i x_{i+1} + x_{i+1}^2) \\ &\equiv 2x_i^2 x_{i+1}^2 + 2x_i x_{i+1} \pmod 3. \end{aligned}$$

Combining the $P_1^j(x_i, x_{i+1})$, $j = 0, 1, 2$ into $Q_1([\vec{x}_i], [\vec{x}_{i+1}])$ over $\mathbf{Z}_3^3$, we get

$$\begin{aligned} Q_1([\vec{x}_i], [\vec{x}_{i+1}]) &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} x_i^0 \\ x_i^0 \\ x_i^0 \end{bmatrix} \begin{bmatrix} x_{i+1}^0 \\ x_{i+1}^0 \\ x_{i+1}^0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} x_i^2 \\ x_i^2 \\ x_i^2 \end{bmatrix} \begin{bmatrix} x_{i+1}^1 \\ x_{i+1}^1 \\ x_{i+1}^0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} x_i^1 \\ x_i^1 \\ x_i^1 \end{bmatrix} \begin{bmatrix} x_{i+1}^2 \\ x_{i+1}^2 \\ x_{i+1}^2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} x_i^1 \\ x_i^1 \\ x_i^1 \end{bmatrix} \begin{bmatrix} x_{i+1}^1 \\ x_{i+1}^1 \\ x_{i+1}^1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} x_i^2 \\ x_i^2 \\ x_i^2 \end{bmatrix} \begin{bmatrix} x_{i+1}^2 \\ x_{i+1}^2 \\ x_{i+1}^2 \end{bmatrix} \end{aligned}$$

where $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbf{Z}_p^p$, and multiplication is component-wise.

The bilinear form of Q_1 over $\mathbf{Z}_3^3$ is:

$$Q_1[\vec{x}_i], [\vec{x}_{i+1}]) =$$

$$\left(\begin{matrix} \begin{bmatrix} x_i^0 \\ x_i^0 \\ x_i^0 \end{bmatrix} \\ \begin{bmatrix} x_i^1 \\ x_i^1 \\ x_i^1 \end{bmatrix} \\ \begin{bmatrix} x_i^2 \\ x_i^2 \\ x_i^2 \end{bmatrix} \end{matrix} \right)^T \left(\begin{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \end{matrix} \right) \left(\begin{matrix} \begin{bmatrix} x_{i+1}^0 \\ x_{i+1}^0 \\ x_{i+1}^0 \end{bmatrix} \\ \begin{bmatrix} x_{i+1}^1 \\ x_{i+1}^1 \\ x_{i+1}^1 \end{bmatrix} \\ \begin{bmatrix} x_{i+1}^2 \\ x_{i+1}^2 \\ x_{i+1}^2 \end{bmatrix} \end{matrix} \right).$$

Theorem 1 raises the question of how simple the state set can be made for bilinear CA to remain π-universal. It implies the existence of a π-universal automaton over $\mathbf{Z}_p^p$ for small p, but we have been unable to settle the matter for a cyclic, or even prime, number of states.

3. Classification of bilinear cellular automata over $\mathbf{Z}_m$

If we assume that the bilinear CAs over $\mathbf{Z}_m$ are T-universal, then as an alternative to a complete analysis, we might settle for a classification along the lines of the Wolfram Classes [20]. In this section, we present a classification of the bilinear CA over $\mathbf{Z}_m$ according to properties of the coefficient matrix.

Definition 3. We define the following bilinear CA rule types.

1. A *column rule* is a bilinear CA where all the nonzero coefficients appear in one column, that is, $b_{ij} = 0$ for $j \neq k$ (fixed), and $b_{ik} \neq 0$ for more than one i.

2. A *transverse-diagonal rule* is a bilinear CA where all the nonzero coefficients appear in the transverse diagonal, that is, $b_{ij} = 0$ for $j \neq n-1-i$, and $b_{i,n-1-i} \neq 0$ for more than one i. Also we require that the central entry $b_{rr} \neq 0$.
3. A *main-diagonal rule* is a bilinear CA where all the nonzero coefficients appear in the main diagonal, that is, $b_{ij} = 0$ for $i \neq j$, and $b_{ii} \neq 0$ for more than one i.
4. A *randomly-distributed rule* is a bilinear CA where the nonzero coefficients may appear anywhere in the coefficient matrix b_{ij}.

Observation 1. For bilinear CAs over $\mathbf{Z}_m$.

1. The column rules fall into (an extended version of) Wolfram's Class I.
2. The transverse-diagonal rules fall into (a restricted version of) Wolfram's Class II.
3. The main-diagonal rules fall into (a restricted version of) Wolfram's Class III.
4. The rules with random coefficients appear to exhibit Wolfram's Class IV-like behavior.

Note that our classification is not consistent with the equivalence classes formed by diagonalizing B in the manner of classical [bi]linear algebra, (e.g., [14]). For example, any diagonal form for a transverse-diagonal rule would indicate that the global dynamics exhibits Class III behavior. This is the reason why we refer to diagonal bilinear CAs as main-diagonal rules.

We shall illustrate each class with a typical example. As most of these examples consist of *symmetric* bilinear CAs, we define symmetric CAs next.

Definition 4. A CA rule δ, is *symmetric* if, upon reversing the order of the neighborhood, then the next state remains unchanged, that is,

$$\delta(x_{i+r}, \ldots, x_{i+1}, x_i, x_{i-1}, \ldots, x_{i-r}) = \delta(x_{i-r}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+r}).$$

A bilinear CA, whose matrix of coefficients in given by $B = (b_{ij})$, is symmetric then if

$$\sum_{ij} b_{ij} x_i x_j = \sum_{ij} x_{n-i-1} x_{n-j-1},$$

that is, if the matrix B is symmetric about the *transverse diagonal.* This is a sufficient condition, but of course probably not a necessary condition, since there may be bilinear CA whose matrix does not have this property, but due to the particular values of the coefficients matched with the modulus, is nonetheless symmetric. Such cases occur with the elementary bilinear CA.

9	5	D	A			E	3	E	C	2	D	C	A	5
		7		3	3	A	6	9	6	7	9	9	A	A
		1		C	9	A	9		C	7			5	A
		7		C	9	A	6		6	1				
		1		3	6	A	9		3	D				
		A		C	9	A	6			A				
		A		9	3	A	C			A				
		A		6	C	A	3			A				
		A		6	C	A	3			A				
		A		6	C	A	3			A				
		A		6	C	A	3			A				

Figure 2: Typical evolution of a column rule.

3.1 Class I: Column rules

Example 3. The simple column rule:

$$P(\vec{x}) \quad = \quad x_i \sum_j x_j \quad (\text{mod } m).$$

The matrix of coefficients for the simple column rule consists of nonzero entries down column l, and 0s everywhere else:

$$\begin{pmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{i+n-1} \end{pmatrix}^T \begin{pmatrix} 0\cdots 0 & a_{0,l} & 0\cdots 0 \\ 0\cdots 0 & a_{1,l} & 0\cdots 0 \\ \vdots & \vdots & \vdots \\ 0\cdots 0 & a_{n-1,l} & 0\cdots 0 \end{pmatrix} \begin{pmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{i+n-1} \end{pmatrix}.$$

Qualitatively, the global dynamics of a column rule exhibit fixed barriers within which the behavior is cyclic, usually a fixed-point, that is, they fall in Class II under Wolfram's scheme.

Figure 2 provides a typical evolution in dimension one for $k = 15$ states and radius $r = 3$, $n = 7$ neighbors.

Note that the nonzero cells are isolated by *domain walls*, within which the behavior is usually fixed, (and not periodic). Due to the preponderance of fixed points, we say that the column rules fall in *an extended version* of Class I. In section 3.2, we show that the transverse-diagonal rules are very similar to column rules in exhibiting domain walls within which the nonzero behavior is restricted. However, transverse-diagonal rules are more likely to exhibit periodic behavior within the walls, and not fixed-point behavior. Hence, we say that transverse-diagonal rules are in a *restricted version* of Class II.

Analysis

A polynomial expression for a (centered) column rule is given by

$$P(\vec{x}) \;=\; x_{i+l}\sum_j a_{j,l}x_j \pmod{m}.$$

We may omit the subscript l, in the coefficient $a_{j,l}$, since l is fixed, and write the global dynamics $T(x)$ of a column rule as

$$T(x)_i \;=\; \sigma^l(x_i)\sum_{j=0}^{n} a_j x_{i+j} \pmod{m}.$$

Here, the superscript l, used with the shift σ, indicates shifting x_i by l places, resulting in $\sigma^l(x)_i = x_{i+l}$.

Now $T(x)$ is just the Hadamard (pointwise) product of a shift $\sigma^l(x)_i = x_{i+l}$ and the *linear rule* $L(x)_i = \sum_{j=0}^{n} a_j x_{i+j}$, written

$$T(x) \;=\; \sigma^l(x) \otimes L(x).$$

Now $\sum a_j x_{i+j} = 0 \pmod{m}$ for dk^{n-1} values of $(x_i, x_{i+1}, \ldots, x_{i+n-1})$, where $d = \gcd(a_0, a_1, \ldots, a_{n-1}, k)$. And since $x^t_{i+l} = 0 \rightarrow x_i^{t+1} = 0$, $T(x)$ must be "shrinking," that is, the number of nonzero sites must be decreasing. This implies that column rules behave much like the monomial CAs investigated in [4].

Now consider the following fixed-point equation over a finite field $\mathbf{Z}_p$, (which affords us a necessary cancellation law), for $l = 0$:

$$\begin{aligned} x &= x \otimes L(x), \\ x_i &= x_i \sum_{j=0}^{n} a_j x_{i+j} \pmod{p}, \\ 1 &= \sum_{j=0}^{n} a_j x_{i+j} \pmod{p}. \end{aligned}$$

This equation has dn^{p-1} solutions, where $d = \gcd(a_0, a_1, \ldots, a_{n-1}, p)$.

In order to investigate the possibility of periodic orbits, consider an isolated $x_i \neq 0$, that is, $x_{i+j} = 0$ for $j = 1, \ldots, n-1$. Then for $l = 0$, $L(x) = a_0 x_i$, and therefore the equation for an m-cycle is given by

$$\begin{aligned} x_i &= a_0^m x_i^2, \\ 1 &= a_0^m x_i, \end{aligned}$$

which always has a solution over $\mathbf{Z}_p$. (Of course, the cycle length is actually a divisor of m.)

Remark 1. Any bilinear CA T_B, can be expressed as a quasilinear combination of column rules, that is,

$$T_B(x)_i = [x_i \otimes \text{Ł}_0(x)_i] \oplus [\sigma(x_i) \otimes \text{Ł}_1(x)_i] \oplus \cdots \oplus \left[\sigma^{n-1}(x_i) \otimes \text{Ł}_{n-1}(x)_i\right].$$

3.2 Class II: Transverse-diagonal rules

Example 4. The simple transverse-diagonal rule:

$$P(\vec{x}) = \sum x_i x_{n-1-i} \pmod{m}.$$

The matrix of coefficients for the simple transverse-diagonal rule consists of 1s down the *transverse diagonal*, and 0s everywhere else:

$$\begin{pmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{i+n-1} \end{pmatrix}^T \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ & \vdots & & & \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{i+n-1} \end{pmatrix}.$$

Qualitatively, the global dynamics of a transverse-diagonal rule exhibit fixed barriers within which the behavior is cyclic, that is, Class II behavior under Wolfram's scheme. Figure 3 provides a typical sample evolution in dimension one for $k = 6$ states and radius $r = 3$, $n = 7$ neighbors.

Analysis

In an attempt to analyze the transverse-diagonal rules, we iterated the simple transverse-diagonal rule algebraically:

$$\begin{aligned} T(x)_i &= x_i^2 + x_{i-1}x_{i+1}, \\ T^2(x)_i &= (x_i + x_{i-1}x_{i+1})^2 + (x_{i-1}^2 + x_{i-2}x_i)(x_{i+1}^2 + x_i x_{i+2}) \\ &= (x_i^4 + 2x_{i-1}x_i^2 x_{i+1} + x_{i-1}^2 x_{i+1}^2) \\ &\quad +(x_{i-1}^2 x_{i+1}^2 + x_{i-1}^2 x_i x_{i+2} + x_{i-2}x_i x_{i+1}^2 + x_{i-2}x_i^2 x_{i+2}) \\ &= x_i^4 + 2x_{i-1}x_i^2 x_{i+1} + 2x_{i-1}^2 x_{i+1}^2 \\ &\quad +x_{i-1}^2 x_i x_{i+2} + x_{i-2}x_i x_{i+1}^2 + x_{i-2}x_i^2 x_{i+2}. \end{aligned}$$

Over $\mathbf{Z}_2$, this reduces to

$$\begin{aligned} T^2(x)_i &= x_i + x_{i-1}x_i x_{i+2} + x_{i-2}x_i x_{i+1} + x_{i-2}x_i x_{i+2} \\ &= x_i + x_i\left(x_{i-1}x_{i+2} + x_{i-2}x_{i+1} + x_{i-2}x_{i+2}\right). \end{aligned}$$

In comparison, a similar transverse-diagonal rule on a neighborhood of radius 2 is given by

$$T(x)_i = x_i + x_{i-2}x_{i+2} + x_{i-1}x_{i+1} \pmod{2}.$$

It is still not clear from this analysis why transverse-diagonal rules should exhibit the observed behavior. However, there is the following observation.

Remark 2. For $k = 2$ states, the transverse-diagonal rule given in Example 4 is the identity CA map.

5	4	1	2		1			1					2	1
1	2	1	4	4	1			1			2		4	1
1		1	4		1	2		5				4	4	1
1	2	1	4	2	1	4	2	1	4	4	4	4		1
1		1	4		3	2		3		2	2		2	1
1	2	1	4		3	4		5		4	4		4	1
1		1	4		3	4	4	3	4	4	2	4	4	1
1	2	1	4		5		4	1	4	4	2	4		1
1		1	4	4	3		2	5	4	4	2		2	1
1	2	1	4	4	1			5	4	4		4	4	1
1		1	4		1		4	1	2	2	2			1
1	2	1	4	2	3		2	3				4		1
1		1	4	4	1		4	3				4	2	1
1	2	1	4		3	2	4	3		4		4		1
1		1	4		5	4	4	1	4	4	2		2	1
1	2	1	4	4	3	4	4	1	4		4	4	4	1
1		1	4	4	1	2		3	2		2			1
1	2	1	4		5	2	4	1	4	2	4			1
1		1	4	4	3	4		3	4			4		1
1	2	1	4	4	5			3				4	2	1
1		1	4	2	1			3				4		1
1	2	1	4		1			3				4	2	1
1		1	4	2	1			3				4		1
1	2	1	4		1			3				4	2	1
1		1	4	2	1			3				4		1
1	2	1	4		1			3				4	2	1

Figure 3: Typical evolution of a transverse-diagonal rule.

3.3 Class III: Main-diagonal rules

Example 5. The simple main-diagonal rule:

$$P(\vec{x}) \;=\; \sum x_i^2 \pmod{m}.$$

The matrix of coefficients for the simple main-diagonal rule is the identity matrix, that is, 1s down the main diagonal and 0s everywhere else:

$$\begin{pmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{i+n-1} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{i+n-1} \end{pmatrix}.$$

Qualitatively, the global dynamics of this main-diagonal rule exhibit unlimited growth for every modulus m, with space-time trajectories reminiscent of Class III linear CA, such as rule 90: $x_i^{t+1} = x_{i-1}^t + x_{i-1}^t \pmod{2}$. Indeed,

```
                  1
                1 1 1
              1 2 3 2 1
            1 1 2 1 2 1 1
          1 2 2 2 1 2 2 2 1
        1 1 1   1 1 1   1 1 1
      1 2 3 2 2 2 3 2 2 2 3 2 1
    1 1 2 1 1   1 1 1   1 1 2 1 1
  1 2 2 2 2 2 2 2 3 2 2 2 2 2 2 2 1
1 1 1           1 1 1           1 1 1
```

Figure 4: Typical evolution of a main-diagonal rule.

the bilinear CA rule $x_i^{t+1} = (x_{i-1}^t)^2 + (x_{i-1}^t)^2 \pmod 2$, is precisely rule 90. Figure 4 provides an example of a typical evolution of an one-dimensional case for $k = 4$ states and radius $r = 1$, $n = 3$ neighbors.

Linear cellular automata in bilinear form

It seems appropriate at this point to consider when a bilinear CA is in fact, a linear CA. In order for a bilinear CA to be linear, we must have

$$\sum_{ij} b_{ij} x_i x_j \quad = \quad \sum_i a_i x_i \pmod m.$$

Such linear bilinear CAs are rare. In fact, a cursory study indicated that the set of bilinear CA rules lie at the maximum hamming distance from the set of linear CA rules.

However, when $b_{ij} + b_{ji} = 0$, we would only need

$$\sum_i b_{ii} x_i^2 \quad = \quad \sum a_i x_i \pmod m.$$

Furthermore, if $b_{ii} = a_i$ then we would only need $x_i^2 = x_i$. A general condition such as this one is clearly true for $\mathbf{Z}_2$, but only partially true for other moduli. That is, there may be a subset of states $S \subset \mathbf{Z}_m$ with the property that $\forall s \in S$, $s^2 = s$, so that certain bilinear CAs are linear on configurations whose sites have states in S.

Quasilinear bilinear cellular automata

In [12] the basin volumes, maximum cycle lengths, and *etcetera* have been calculated for rules 18 and 126, which are both noted to be Class III rules. "Intriguing properties of global structure" are found between them. Indeed, in [8, 9] it is noted that under certain block transformations, rule 18 is similar to the linear rule 90. The idea of block transformations are used in [13] to find exact solutions to the forward problem for rules 18 and 126, which are bilinear CAs, and rule 146, which is not a bilinear CA.

Rule 18 is given by:

$$T_{18}(x)_i \quad = \quad x_{i-1} + x_{i+1} + x_{i-1}x_i + x_ix_{i+1} \pmod 2.$$

Two equivalent matrices for rule 18 are:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Rule 126 is given by:

$$T_{126}(x)_i = x_{i-1} + x_i + x_{i+1} + x_{i-1}x_i + x_{i-1}x_{i+1} + x_ix_{i+1} \pmod 2.$$

Two equivalent matrices for rule 126 are:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Other Class III bilinear cellular automata

In addition to the linear and quasilinear CAs, there are other Class III bilinear CAs. In particular, in [21] it is suggested that the elementary bilinear CA rule 30, given by:

$$T_{30}(x)_i \quad = \quad x_{i-1} + x_i + x_{i+1} + x_ix_{i+1} \pmod 2$$

is an excellent random number generator. Two equivalent matrices for rule 30 are:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.4 Class IV: Random rules

In [6] "Candidates for the Game of Life in Three Dimensions," are investigated and the following two criteria for rules that are "life-like" is given.

1. All primordial soup configurations must exhibit *bounded growth*.

2. A *glider* must exist and must occur "naturally" in the evolution from primordial soup configurations.

In addition, it is conjectured that these criteria, (which we refer to simply as *bounded growth* and *gliders*), are sufficient grounds for T-universality. Now, since the bilinear CA over $\mathbf{Z}_p^p$ are T-universal, and rule 54 is a Class IV suspect, there are already ample grounds for the conjecture that the bilinear CA are T-universal. Here we present more evidence. We have found separate

									3	3	2	3	2	3					
								1	1			1		3	2	2			
							1	3	1		1	3		3					
						3	2	1	1		2			1		2			
						1			1	1	2	2		1		2			
					2	1	3	2	1	3	1		1		2				
				2	3	1	2		1	1		2	1	1		2			
			2	3	1	3	2	2	1	3		2	3	1	1	2			
		2	3		1			2	1	1	2		2	1	2	3			
	2	2	1		1	2		1	1	3	1		2	3		3	2	2	
	2	2	1		3	3	2		3	3	2		3	1		3			
	2	2	1			1		2	3	1			1	2	2	1		2	
	2	2	1	1	1	1	3	2	3	3	2	3	1	1	3	1	2	2	
	2	3		1	1	1	3			2		1	3	1	3		3		
2	2	1		1	1	1	2	3			1		1	3	1		1		2
2	2	1	2	3	1	3	2	3	3	3	3	2	3	1	1		1		2
2		3		1	1	1		3	1	3	1		1	1	3		1		2
	2	3		1	1	3		1	1	3	1		1	1	1	2	1		2
2	2	1		3	1	1		3	3	3	1		3	1	1		3	2	2
2	2	1		3	1	1		1	1	1	1		3	1	1	2	3		
2	2	1		3	1	3		3	1	3	3		3	1	1	2	3	2	2

Figure 5: Example of bounded growth in a bilinear CA.

bilinear CAs that satisfy at least one of the two criteria. However, we have yet to find a single bilinear CA rule that satisfies both criteria.

Figure 5 provides an example of the *bounded growth* criteria, based on the following matrix of coefficients modulo 4:

$$\begin{pmatrix} 2 & 2 & 1 & 2 & 2 \\ 3 & 0 & 3 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 3 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Figure 6 provides an example of a *glider*, based on the following matrix of coefficients modulo 6:

$$\begin{pmatrix} 4 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Actually, the *glider* behaves more like a *soliton* (e.g., [3, 18]).

4					3					
2	4				3					
		4			3					
		2	4		3					
				4	3					
				2	1					
					3	4				
					3	2	4			
					3			4		
					3			2	4	
					3					4

4				3	3	3				
2	4			3	3	3				
		4		3	3	3				
		2	4	3	3	3				
				1	3	3				
				5	1	3				
				3	3	1				
				3	3	5	4			
				3	3	3		4		
				3	3	3		2	4	
				3	3	3				4

Figure 6: Example of solitons in a bilinear CA.

Rule 54

Perhaps the most important bilinear CA, due to a combination of its simplicity and its candidacy for Class IV, is rule 54, given by:

$$T_{54}(x)_i = x_{i-1} + x_i + x_{i+1} + x_{i-1}x_{i+1} \pmod 2.$$

Two of the possible matrices for rule 54 are:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Note that the only nonzero entries off the main diagonal appear on the transverse diagonal.

To investigate the dynamics of rule 54, we examine the fixed-point equation:

$$x_i = x_{i-1} + x_i + x_{i+1} + x_{i-1}x_{i+1} \pmod 2,$$

$$\begin{array}{c}
\bar{0}\ \underline{1}\ \bar{0} \\
\bar{0}\ 1\ \underline{1}\ 1\ \bar{0} \\
\bar{0}\ 1\ 0\ \underline{0}\ 0\ 1\ \bar{0} \\
\bar{0}\ 1\ 1\ 1\ \underline{0}\ 1\ 1\ 1\ \bar{0} \\
\bar{0}\ 1\ 0\ 0\ 0\ \underline{1}\ 0\ 0\ 0\ 1\ \bar{0} \\
\bar{0}\ 1\ 1\ 1\ 0\ 1\ \underline{1}\ 1\ 0\ 1\ 1\ 1\ \bar{0} \\
\bar{0}\ 1\ 0\ 0\ 0\ 1\ 0\ \underline{0}\ 0\ 1\ 0\ 0\ 0\ 1\ \bar{0} \\
\bar{0}\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ \underline{0}\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ \bar{0} \\
\bar{0}\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \underline{1}\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ \bar{0} \\
\vdots
\end{array}$$

Figure 7: Perturbation $\bar{0}\underline{1}1\bar{0}$ of $\bar{0}$ under rule 54.

$$0 \quad = \quad x_{i-1} + x_{i+1} + x_{i-1}x_{i+1} \quad (\text{mod } 2).$$

This means $x_i \equiv 0$ for every cell i, or $x = \bar{0}$ is the only fixed-point. (Here, $\bar{0}$ is used to indicate the *equivalence* class of configurations, $\ldots 00\underline{0}00 \ldots$, modulo the center cell, e.g., [4].)

Perturbing $\bar{0}$ by $\bar{0}1\bar{0}$, and applying T, we get the evolution in Figure 7, which tends to the 4-cycle $\overline{0001}$.

The following calculation shows that there are no 2-cycles:

$$\begin{aligned}
T_{54}^2(x)_i &= T_{54}(x)_{i-1} + T_{54}(x)_i + T_{54}(x)_{i+1} + T_{54}(x)_{i-1}T_{54}(x)_{i+1} \\
&= (x_{i-2} + x_{i-2}x_{i+1} + x_{i-2}x_{i+2} + x_{i+2}x_{i-1} + x_{i+2}) \\
&\quad + (x_{i-2} + x_{i-2}x_{i+1} + x_{i-2}x_{i+2} + x_{i+2}x_{i-1} + x_{i+2})\, x_i \\
&\quad + (x_{i-1} + x_{i+1})\, x_i.
\end{aligned}$$

If $x_i = T^2(x)_i$, then

$$\begin{aligned}
x_i = 0 &\ \rightarrow\ (x_{i-2} + x_{i-2}x_{i+1} + x_{i-2}x_{i+2} + x_{i+2}x_{i-1} + x_{i+2}) = 0, \\
x_i = 1 &\ \rightarrow\ x_{i-1} + x_{i+1} = 0.
\end{aligned}$$

But neither of these conditions can be satisfied.

Further investigations of this kind have indicated that the phase portrait of rule 54 may consist solely of the apparently repelling fixed-point $\bar{0}$, and four apparently *strange attracting* 4-cycles: $\overline{0001}$, $\overline{0101}$, $\overline{0110}$, and $\overline{0111}$. (These are periodic configurations obtained by repeating the given period infinitely in both directions.) However, the true phase portrait for rule 54 may well be uncomputable.

4. Diagonalization

Following the line of classical work in linear algebra (e.g., [14]), it is natural to attempt a classification of the bilinear CAs based upon the diagonalized local coefficient matrix. The idea is to find a nonsingular matrix P, such that $D = PBP^T$ is a diagonal matrix. This defines an equivalence relation $\sim$, on the set of $n \times n$ matrices over a given field, ($\mathbf{Z}_p$ in the present case). That is, $B \sim D$ if and only if $(\exists P)(PBP^T = D)$. Then, the set of matrices

$$\mathbf{C}(D) \quad = \quad \left\{B \,:\, (\exists P)(PBP^T = D)\right\}$$

is an equivalence class.

Put in terms of bilinear CAs, we would like it if

$$T_D(T_P(x)) \quad = \quad T_P(T_B(x))$$

where

$$T_M(x)_i \quad = \quad \vec{x}_i M \vec{x}_i.$$

However, the overlapping neighborhoods of a CA prevent this possibility. As an example, consider the simple transverse-diagonal rule in one dimension with radius $r = 1$ over $\mathbf{Z}_5$. The matrix of coefficients is given by:

$$B \quad = \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

B is diagonalized by the matrix:

$$P \quad = \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 3 & 0 & 1 \end{pmatrix}.$$

Applying $PBP^T = D$ results in the identity matrix:

$$D \quad = \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As indicated in section 3, the identity matrix corresponds to the simple main-diagonal rule, whose dynamics are similar to the unlimited growth of linear rules. On the other hand, the dynamics of the simple transverse-diagonal rule is that of periodic behavior within definite walls. Since these two bilinear CAs are quite dissimilar dynamically, it is apparent that diagonalization will fail in general to produce a topological conjugacy between the corresponding dynamical systems. Yet it is still possible to "diagonalize" a bilinear CA in a more general sense, which we now show.

Consider the following scenario, where T_P is a linear transformation, (using some P that diagonalizes B), that *lifts* a "horizontal" n-tuple to a "vertical" n-tuple, reminiscent of the *higher block presentation* of a subshift (e.g., [15]).

We use the matrix P to *encode* x in the following way:

$$P \begin{pmatrix} x_{i-1} \\ x_i \\ x_{i+1} \end{pmatrix} \quad = \quad \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}.$$

$$
\begin{array}{ccc}
 & \hat{\mathbf{C}} & \\
T_P & \uparrow & \searrow \quad T_D \\
 & \mathbf{C} & \xrightarrow[T_B]{} \quad \mathbf{C}
\end{array}
$$

Figure 8: Standard diagonalization of bilinear CA.

$$
\begin{array}{ccccc}
 & \hat{\mathbf{C}} & \xrightarrow{T_D} & \hat{\mathbf{C}} & \\
T_P & \uparrow & & \uparrow & T_P \\
 & \mathbf{C} & \xrightarrow[T_B]{} & \mathbf{C} &
\end{array}
$$

Figure 9: Desired conjugacy for a bilinear CA.

This results in the following diagram:

$$
\begin{array}{ccccccc}
x & = & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots \\
 & & & & & & \\
 & & & u_{i-1} & u_i & u_{i+1} & \\
T_P(x) & = & \cdots & v_{i-1} & v_i & v_{i+1} & \cdots . \\
 & & & w_{i-1} & w_i & w_{i+1} &
\end{array}
$$

We may then define a diagonalization T_D, such that $T_B(x) = T_D(T_P(x))$. However, T_D is a strictly local rule, that is,

$$
T_D(P(x))_i = \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}^T D \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix}.
$$

We may summarize this as in the diagram in Figure 8.

Diagonalization over an extended field

Naturally, we would prefer that the diagonalization of the diagram in Figure 8 take triples to triples in $\hat{\mathbf{C}}$, so that we might have the diagram in Figure 9.

We have attempted such a diagonalization over an extended field $\mathbf{Z}_3[\alpha]$ via the irreducible polynomial $P(x) = x^3 - 2x^2 - 1$, (which leads to the reduction formulas $\alpha^3 = 2\alpha^2 + 1$, and $\alpha^4 = \alpha^2 + \alpha + 2$). An element of the extended field $\mathbf{Z}_3[\alpha]$ has the form $a_1\alpha^2 + a_2\alpha + a_3$, which we write (horizontally) as the triple (a_1, a_2, a_3). Unfortunately, the resulting system has no solution.

We next thought that perhaps the transverse-diagonal rule is somewhat special in its defying our attempts at diagonalization. So we tried to diagonalize the following simple bilinear CA:

$$
T_B(x)_i = x_i\,(x_{i-1} + x_{i+1})\,.
$$

The matrix of coefficients B, is given by:

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, there is apparently no matrix P to diagonalize B over $\mathbf{Z}_3$, or $\mathbf{Z}_5$. And based on the general equations for a diagonalization over $\mathbf{Z}_p$ for p prime, we conjecture that there is no diagonalization of B over $\mathbf{Z}_p$. Hence, it appears that even if there were *nice* (canonical) diagonal forms over $\mathbf{Z}_p$, not all bilinear CA would admit to a diagonalization.

5. Conclusions

We have established the π-universality of bilinear CAs over $\mathbf{Z}_p^p$, and have provided evidence for the T-universality of bilinear CA over $\mathbf{Z}_m$. However, a proof remains elusive. But if the bilinear CA over $\mathbf{Z}_m$ are indeed T-universal (as we suspect), then there is no possibility of a general purpose algorithm for predicting the global dynamical behavior of any nonlinear CA. The best we can do is to attempt a phenomenological classification along the lines of that presented herein.

On the other hand, if it turns out that the bilinear CA over $\mathbf{Z}_m$ are not T-universal, then the question becomes whether the quadratic CA over $\mathbf{Z}_m$ are T-universal. Barring that, is there a T-universal cubic CA over $\mathbf{Z}_m$? While these questions also remain open, the authors have found a quartic polynomial representation for Bank's computer, a known T-universal CA. However, Bank's computer is a two-dimensional CA, so another question arises as to whether there exists a T-universal CA in one dimension with degree less than 4. So far, the authors have not found a T-universal CA in one dimension whose polynomial representation has degree less than 18.

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