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Misspecifying GARCH-M Processes

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Abstract. We consider the relationships between ARCH-type and stochastic volatility models. A new class of volatility models, called generalized bilinear stochastic volatility, is described following an approach that transforms an initial GARCH-M process. The focus here is on the interpretation of some simulation results, with a special care devoted to model misspecification.

1. Introduction

The strong impulse that motivated the field of financial time series volatility models,[1] after the seminal work on *autoregressive conditional heteroskedastic* (ARCH) models [11], soon materialized in a wide spectrum of methodological proposals and applications, making this field a highly desired ground for testing statistical inferential techniques. Nevertheless, some of the aspects related to ARCH-type models, and initially pointed out in [11], have not yet captured the same interest. In this paper we will deal with some of these aspects, which automatically lead us to look at ARCH-type models not as structural features of the data-generating process (DGP) for which we have temporal observations, but as potentially incomplete models, that is, models with some latent structure or features that are not immediately revealed but can be possibly discovered in a subsequent stage of the specification analysis.

We propose here a new class of stochastic volatility processes and we compare them to the models from which they originate. We present some simulation experiments. Our goal is to verify the potential of this class of models for discovering features of the underlying DGP that were not eventually accounted for by the original model because of misspecification. Therefore, the assumption that some misspecification can possibly occur for GARCH-M processes with time-varying parameters represents a sort of master hypothesis for our results. We think that this is not a bizarre hypothesis, and we offer some motivations for it.

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[1]See, for example, [5, 23] for discussion and [4] for an extensive review.

The paper proceeds as follows. In section 2 we briefly introduce ARCH-type and stochastic volatility models; we also report on the relationships between them and bilinear processes. In section 3 we describe the model chosen here for analysis (i.e., the GARCH-M), and show how to switch to a new and more complex class of stochastic processes. Section 4 deals with model interpretation and misspecification issues in the light of other authors' investigations as well. Section 5 reports about some simulation experiments, and section 6 gives the conclusions.

2. Volatility models

The simple ARCH model is given by the following specification for the observed returns y_t and the related volatility process h_t:

$$y_t = x_t^T b + \epsilon_t \tag{1}$$

where $\epsilon_t/\mathcal{F}_{t-1} \sim N(0, h_t)$,[2] and

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_q \epsilon_{t-q}^2. \tag{2}$$

The order of the model is q and the α_i coefficients must be nonnegative in order to satisfy the nonnegativity constraint required by the conditional variance h_t. A well-known generalization of the ARCH is the GARCH model [3], and independently [22], which allows for the presence of the lags h_{t-i} in the specification of the conditional variance:

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1} \tag{3}$$

resulting in this case the GARCH(1,1) model.[3]

An alternative and also popular class of models is known as *stochastic volatility* (SV). Here a stochastic mechanism is introduced, that is, a shock that is random and independent (from the past information) drives the volatility process, together with other predeterminated variables. Now h_t is no longer observable because of the contemporaneous random variability added by the separate noise.

In [1] a GARCH(1,1) is generalized to a SV model as follows:

$$\epsilon_t = h_t^{1/2} z_t \tag{4}$$

with $z_t \sim$ i.i.d.$(0,1)$ and the volatility process given by

$$h_t = w + \beta h_{t-1} + \alpha h_{t-1} u_t \tag{5}$$

with $u_t \sim$ i.i.d.$(1, \sigma_u^2)$, $u_t > 0$; (z_t) and (u_t) are independent of each other and with respect to $\mathcal{F}_{t-1}$.[4] Without the substitution, the same equation defines an example of *stochastic autoregressive volatility* (SARV) models.

[2] $\mathcal{F}_{t-1}$ is the information available through the observed data up to time $t-1$.

[3] The volatility process is not considered stochastic so far, but it changes according to the past information set $\mathcal{F}_{t-1}$.

[4] When the volatility equation is written as $h_t = w + \beta h_{t-1} + [\gamma + \alpha h_{t-1}]u_t$, replacing z_{t-1}^2 for u_t and given $\gamma = 0$, a GARCH(1,1) is easily obtained. Now h_t is measurable with respect to $\mathcal{F}_{t-1}$ and the model is therefore conditionally heteroskedastic.

Quite interestingly, ARCH-type models have revealed connections with other general stochastic processes too; for example, the *bilinear processes* considered in [15, 24, 25]. In [25] it is shown how difficult it is to try to separate the dynamics involved in the first two conditional moments of the distribution of interest when ARCH and bilinearity are contemporaneously present in a model such as

$$\phi(B)(y_t - \mu) = \theta(B)\epsilon_t + \sum_{i=1}^{P}\sum_{j=1}^{Q}\beta_{ij}y_{t-i}\epsilon_{t-j} \tag{6}$$

where $E(\epsilon_t/\mathcal{F}_{t-1}) = 0$ and

$$h_t = \alpha_0 + \sum_{i=1}^{R}\alpha_i\epsilon_{t-i}^2 + \delta_0(\hat{y}_t - \mu)^2 + \sum_{j=1}^{S}\delta_j(y_{t-j} - \mu)^2 \tag{7}$$

with $\hat{y}_t$ as the forecast for y_t calculated at time $t-1$. In [15] a *state space formulation* is given for a bilinear process obtained from an ARCH model. Starting from

$$\begin{aligned} y_t &= \epsilon_t h_t^{1/2} & (8)\\ h_t &= \alpha_0 + \alpha_1(y_{t-1}^2 - \alpha_0) + \cdots + \alpha_q(y_{t-q}^2 - \alpha_0) & (9) \end{aligned}$$

and given

$$y_t^2 = \epsilon_t^2 h_t \tag{10}$$

with $\epsilon_t^2 = \eta_{t-1} + 1$ and $y_t^2 = x_t + \alpha_0$, after some calculations it is easy to obtain the bilinear state space representation

$$\begin{aligned} y_t^2 &= x_t + \alpha_0 & (11)\\ x_t &= \sum_{i=1}^{p}\alpha_i x_{t-i} + \sum_{i=1}^{p}\alpha_i x_{t-i}\eta_{t-1} + \alpha_0\eta_{t-1} & (12) \end{aligned}$$

where η_t is distributed as a χ_1^2 random variable that is not centered and has a variance equal to 2.

In these examples the models considered were quite simple. We will study the case where we have a more complicated model setup, given by the GARCH-M process, and we transform this process to obtain a new volatility characterization with several stochastic features that differ from the initial ones.

3. A new class of stochastic volatility processes

3.1 The general framework

Our candidate model for the analysis is the *generalized autoregressive conditional heteroskedastic with effects in mean* (GARCH-M) [12]. The importance of the relationships between market risk and expected returns is crucial in finance theory, and this model takes into account the risk premium allowing for the introduction of the volatility process into the conditional mean equation. Following [9] we represent the GARCH-M model with a time varying mean-variance ratio:

$$y_t = b_t h_t + e_t \tag{13}$$
$$b_t = b_{t-1} + v_t \tag{14}$$
$$h_t = a_0 + a_1 \eta_{t-1}^2 + a_2 h_{t-1} \tag{15}$$
$$\eta_t = y_t - E_{t-1}(y_t) \tag{16}$$

where e_t and v_t represent zero-mean uncorrelated white noises whose variances are respectively h_t and Q_t, h_t measures the volatility, and b_t is the so-called price of volatility. The novelty here, compared to the original model, is the time-varying coefficient that relates the first two conditional moments, which is assumed to be distributed as a random walk instead of being a constant term. Thus, an additional source of randomness is introduced. Use of the Kalman filter algorithm is proposed in [9] in order to estimate the model in the given state space representation; it is well known that a likelihood function can be obtained via the prediction error decomposition, and the same function must be maximized with respect to the parameters of interest. From a methodological point of view, the recursivity of this estimation procedure seems a suitable and convenient property to exploit in order to avoid the difficulty of operating a separate estimation of the conditional mean and variance parameters in the way usually done for other ARCH-type models. With the state space framework and the Kalman filter built on it, the estimate of the two equations can be executed in two sequential steps at each observation in the available sample. First, at time t, an estimate of the conditional variance h_t is obtained and a new innovation value η_t can be computed from the Kalman filter; then, this last value is used to calculate the updated quantity h_{t+1}, and the process is repeated as before at each observation.

In this more complicated setup, given the presence of various stochastic influences coacting to drive the underlying observed process, it seems interesting to analyze the possible consequences of misspecifying, for instance, the price-of-volatility formulation, (i.e., b_t), and/or the market risk premium (i.e., $b_t h_t$). Both of them represent a very important subject of discussion in finance, given that no unifying theory exists and there are no homogeneous results from the empirical side (e.g., [2]) to justify recourse to a particular functional form for the risk premium or a given price-of-volatility specification.

3.2 A new formulation

New interesting formulations can be obtained through a transformation of the GARCH-M process, whose complexity depends on the variable selected to represent the volatility process. We show here one of the possible derivations.[5]

[5] A more detailed explanation is offered in [8].

Consider the following GARCH(p,q)-M model:

$$
\begin{aligned}
y_t &= b_t h_t^{\frac{1}{2}} + \gamma_t && (17)\\
h_t &= \sum_{i=1,p} \alpha_i \eta_{t-i}^2 + \sum_{j=1,q} \beta_j h_{t-j} && (18)
\end{aligned}
$$

where $\gamma_t \sim \text{NID}(0, h_t)$ and $\eta_t = y_t - E_{t-1}(b_t h_t^{1/2})$ is the innovation of the model. The coefficient b_t can be described by an AR(p) process:[6]

$$\phi(B)b_t = r_t \tag{19}$$

or, equivalently,

$$b_t = \phi^{-1}(B)r_t \tag{20}$$

where $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ and $r_t \sim WN(0, \sigma_r^2)$.

As an assumption regarding the squared transform of the coefficient in equation (20), we allow the following relation to hold:

$$b_t^2 = \epsilon_{t-1} + k_t \tag{21}$$

where $\epsilon_t = \phi^2(B)\epsilon_{t-1} + \phi^2(B)k_t$, with k_t that takes only positive values. This process could be seen as a *control input vector* whose nature is exogeneous and therefore independent from the variable of interest (i.e., volatility) in our context. To verify the last relation, just define $\epsilon_t = r_t^2$ and substitute first for r_t^2 in $b_t^2 = \phi^{-2}(B)r_t^2$ and then for ϵ_t in its AR formulation. More generally, a complete signal-plus-noise model related to the process b_t^2 is given by $b_t^2 = \epsilon_{t-1} + k_t$ and $\epsilon_t = F\epsilon_{t-1} + Gk_t$, and the framework built up so far should be interpreted as follows. The first important assumption is that the risk premium is not time-invariant but varies with an agent's perceptions of the underlying economy uncertainty, that is, according to the opportunities and preferences of the investors toward risk. Another assumption concerns the choice of relaxing the usually retained random walk hypothesis for the distribution of the coefficient b_t of risk aversion, that is, the slope coefficient in the conditional mean linear equation relating the excess returns to their variance.

The following relation is easily shown to hold:

$$b_t^2 h_t = (\epsilon_{t-1} + k_t)\left(\sum_{i=1,p} \alpha_i \eta_{t-i}^2 + \sum_{j=1,q} \beta_j h_{t-j}\right) \tag{22}$$

and thus we can obtain h_t as

$$
\begin{aligned}
h_t &= k_t \sum_j \beta_j h_{t-j} b_t^{-2} + \sum_j \beta_j h_{t-j}\epsilon_{t-1} b_t^{-2}\\
&\quad + \sum_i \alpha_i \eta_{t-i}^2 \epsilon_{t-1} b_t^{-2} + k_t \sum_i \alpha_i \eta_{t-i}^2 b_t^{-2}. && (23)
\end{aligned}
$$

[6]Note that it is possible to generalize the process for b_t to a stationary and invertible ARMA(p,q).

The evidence here is that h_t represents in fact a process with respectively an autoregressive term, a bilinear part, a mixed input,[7] and a not necessarily gaussian-colored noise.

Defining $x_t^2 = (y_t - \gamma_t)^2$, with $\gamma_t \sim \mathrm{NID}(0, h_t)$, where h_t is distributed as in equation (18), we do not observe x_t^2 exactly; only a noisy version of it can be measured. Since h_t represents a state variable, once an appropriate transformation of equation (23) is taken, a complete state space model then results. As a first step, we aim at representing a *markovian form* for the model. This allows for standard specification and estimation techniques to be applied. However, it is shown in [21] that in order to do this, some assumptions are required in cases where nonlinear predictor spaces are considered. In particular, we must fix the state vector dimension in a way that both the available information and most of the structure of the original model are retained.

Therefore we first find a so-called *vectorial bilinear form* by fixing the vector $\beta = [\beta_1 \cdots \beta_j]^T$ and the state vector $H_t^* = [H_t^*(1) \cdots H_t^*(m)]^T$, where the new state variable is indicated by $H_t^*(j) = h_{t-m+j}$, for $j = 1, \ldots, m$. We are thus assuming that the dimension of the state vector is m.[8] Then, we have to apply consequent transformations to the other variables and coefficients involved. Given the scalar control input, we just adjust for the coefficients α_i (i.e., it becomes a component of the vector α) and the variables η_{t-i}^2 (which are transformed in η_t^*). As a last step, we define some new time-varying quantities, according to $p_t = \frac{k_t}{b_t^2}$, $q_t = \frac{1}{b_t^2}$, $A_t = p_t\beta$, $B_t = q_t\beta$, $C_t = q_t\alpha'\eta_t^*$, and $D_t = p_t\alpha'\eta_t^*$. Thus, we obtain the following new transformed volatility specification:

$$\begin{aligned} H_t^* &= A_t H_{t-1}^* + B_t H_{t-1}^* \epsilon_{t-1} + C_t \epsilon_{t-1} + D_t && (24) \\ y_t &= G' H_t^* + v_t && (25) \end{aligned}$$

where $G = [0 \cdots \epsilon_{t-1}]^T$ and v_t is an additive noise derived from the representation obtained under the conditions established by equation (21). Note that this is a *quasi-markovian bilinear representation* for the stochastic volatility process (also called GeBiSV in [8]), where the word *quasi* stands for the fact that the usually retained gaussian assumption about the noise's distribution is not necessarily satisfied here. Moreover, the original volatility is recovered by $h_t = F' H_t^*$, where $F = [0\ 0\ \cdots\ 1]^T$. The new measurement equation is the following:

$$z_t = x_t^2 + v_t = b_t^2 h_t + v_t \tag{26}$$

where the new process z_t measures with a noise v_t the previous process $b_t^2 h_t$ and $v_t \sim \mathrm{NID}(0, h_t)$, or alternatively, we assume that $v_t = h_t^{1/2} u_t$, with $u_t \sim$

[7] In this term different forces are acting together: the lagged squared model innovations, η_t^2, the inverted time-varying squared coefficients b_t^{-2}, and the lagged process ϵ_{t-1}.

[8] This assumption, commonly retained, is crucial for some aspects. We follow here the common practice of fixing a finite dimension, but how to fix it is an open problem. We are considering this topic for a companion paper.

NID(0, 1).[9] In a more compact form, the observations from the described stochastic system are represented by $z_t = H_t h_t + G_t u_t$, where $H_t = b_t^2$ is the measurement (stochastic) matrix and $G_t = I$, in the regression case, or $G_t = h_t^{-1/2}$.

4. Interpreting the new models

In [11], the seminal work on ARCH models, some misspecification aspects were discussed for which clear answers have not been found yet. ARCH models could theoretically be the result of a misspecified underlying regression model with nonARCH errors, thus implying that some omitted variables or structural changes are possible features of the DGP which result unobserved or latent at the first stage of the analysis. In [10,16] this aspect is emphasized, likewise the fact that ignoring a residual conditional mean nonlinearity would have important consequences for the theoretical understanding of the dynamics involved and the empirical analysis, particularly with regard to point prediction.

Recent work on nonlinear time series models strongly directs the attention to this fact, and various important contributions seem to justify the hypothesis that some nonlinearity is not fully accounted for by ARCH-type models. For example, in [13, 14] an approximation of the joint density for conditioning and conditioned variables is introduced using a Hermite series expansion about the gaussian density, which allows the deviations from it to be captured by the higher-order terms. ARCH processes can be modeled too, with particular insight for exchange rates because of residual nonlinearity evidence, even after the conditional heteroskedasticity has been accounted for.

Other interesting papers, related to the BDS test [19, 20], extended parameterizations involving the conditional density of normalized errors [17], and Markov switching structures [6, 16], respectively, indicated that (1) extra nonlinearity is present in the model after the routine ARCH-type analysis; (2) there is statistical significance of shape parameters in the error distribution; and (3) regime shifts have impact on the observed time series.

Therefore, we could reasonably look at ARCH effects more as a phenomenon with a strong misspecification nature than as a structural aspect of the DGP at hand, at least in many cases and under various conditions. The present paper suggests what changes could occur in the model once it proves to be misspecified and it becomes evident, from the graphs of simulations shown in the next section, that the possible misspecification may be harder to detect than is expected. While in [25] the intervent of bilinearity is explicit as a feature of the DGP, here second- and higher-order interactions show up in the volatility process from a transformation of an original GARCH-M process. In our setup, at a first glance, it appears more problematic to try to

[9] Simply assuming that the disturbance in the conditional mean equation is a white noise, we obtain a regression model with stochastic regressors h_t and time-varying coefficients b_t^2.

separate the different sources that drive the observed DGP, thus leaving the misspecification causes not easily identifiable.

Ignoring the question of which is the best approximating DGP governing the mean-variance ratio can have important consequences in terms of a correct model specification. Therefore, a first contribution here given is to show interesting consequences that derive from a possibly incorrect choice of the functional form of the market risk premium term for the expected volatility, (i.e., $g(b_t)h_t^p$). In our example we assume an AR specification for the mean-variance ratio; then, to choose the functional form of the volatility term characterizing the risk premium term is important, since we could find two completely different volatility processes once we deviate slightly from the initial model conditions. In the following section we complement our technical results with some simulation experiments that show—not surprisingly perhaps—the similarity existing between different classes of SVM. This is not a surprise in the light of the many possible empirical phenomena that can influence volatility and the observations at hand, but nevertheless it is worthwhile to mention that this same fact can have some relevance for understanding covariance-nonstationary settings, which, according to some recent literature, are gaining more and more credit among researchers in the field (e.g., [18]).

5. Simulation experiments

We generate data from a GARCH(1,1)-M process with time-varying parameters, obtained as a special case of the general framework here presented. Then another simulated time series is derived from a GeBiSV(1,1) process obtained in correspondence with the first one. We investigate the degree of misspecification that we could possibly face when we consider a risk premium as in the GARCH-M setup without pursuing the investigation of a different functional form for it.

The general structure of the models we want to generate is given by

$$y_t = \mathrm{ERP}_{1,t} + e_t$$

with $\mathrm{ERP}_{1,t} = b_t h_t^{1/2}$, $e_t \sim \mathrm{NID}(0, h_t)$, and h_t GARCH(1,1)-M specified, and by

$$z_t = \mathrm{ERP}_{2,t} + v_t$$

with $\mathrm{ERP}_{2,t} = b_t^2 h_t$, $v_t \sim \mathrm{NID}(0, h_t)$, and h_t given as in the GeBiSV specification.

From the general model we have the following GARCH(1,1)-M model:

$$y_t = b_t h_t^{1/2}; \qquad h_t = a_1 \eta_{t-1}^2 + c_1 h_{t-1}$$

with $b_t = b_0 + \phi_1 b_{t-1} + r_t$. Then the correspondent GeBiSV(1,1) model is given by:

$$\begin{aligned} z_t &= \gamma_t h_t + v_t \\ h_t &= k_t \beta_1 h_{t-1} \gamma_t^{-1} + \beta_1 h_{t-1} \epsilon_{t-1} \gamma_t^{-1} + \alpha_1 \eta_{t-1}^2 \epsilon_{t-1} \gamma_t^{-1} + k_t \alpha_1 \eta_{t-1}^2 \gamma_t^{-1} \end{aligned}$$

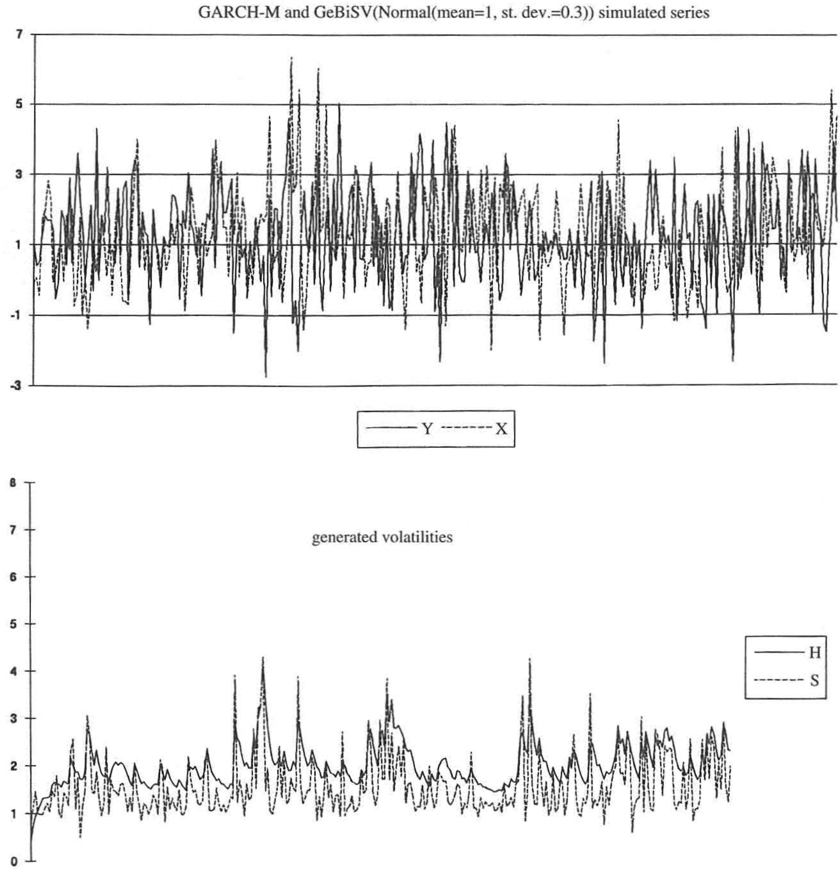


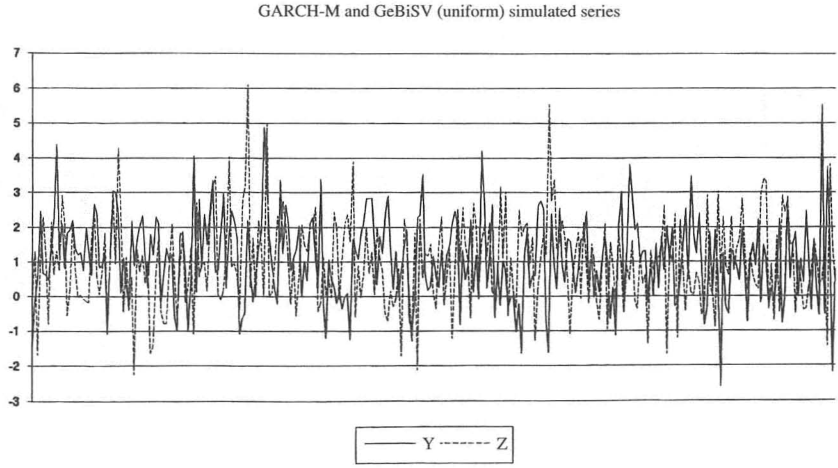
Figure 1: First experiment: gaussian case.

and, defining $k_t\gamma_t^{-1} = m_t$, it follows that $\epsilon_{t-1}\gamma_t^{-1} = 1 - m_t$, and thus we have, after some more calculations, the following final form for the volatility:

$$h_t = \beta_1 m_t h_{t-1} + \beta_1 \frac{m_t}{k_t} h_{t-1}\epsilon_{t-1} + \alpha_1 \eta_{t-1}^2.$$

This last equation, together with the z_t variable, is the GeBiSV model that was simulated.

Three experiments are presented; for each of them the size of the simulated set of observations is $n = 300$. They differ for the distributional assumptions about the k_t innovation influencing the risk aversion coefficient γ_t that characterizes the GeBiSV formulation. In one case (Figure 1) we adopted $normal(\mu = 1, \sigma = 0.3)$ distribution; in another case (Figure 2) a $uniform(0, 1)$ distribution; and still other results (Figure 3) under the



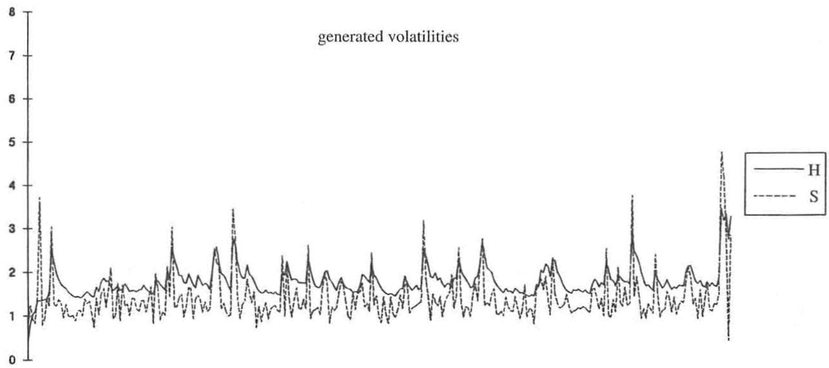


Figure 2: Second experiment: uniform case.

hypothesis of a *Poisson* distribution whose mean is 0.05. The parameter structure of the conditional mean, risk aversion coefficient, and volatility coefficients in the three experiments is fixed at the following values: $b_0 = 0.5$, $\phi_1 = 0.4$, $r \sim \text{NID}(\mu = 0, \sigma = 0.2)$, $h_0 = 0.4$, $a_1 = 0.1$, $c_1 = 0.7$, $s_0 = 0.8$, $\beta_1 = 0.2$, and $\alpha_1 = 0.2$.

From the visual inspection of these generated DGPs and the related volatilities we can notice that, under some conditions about the residual distributions and the model structure, stochastic processes with very similar characteristics but different nature and order of complexity can be derived from the variation of the functional form of the systematic term in the conditional mean equation. Even if the resulting data patterns are quite close in our models, important differences are introduced, since both the risk aversion coefficient and the volatility process present a different stochastic characterization compared to the starting GARCH-M model.

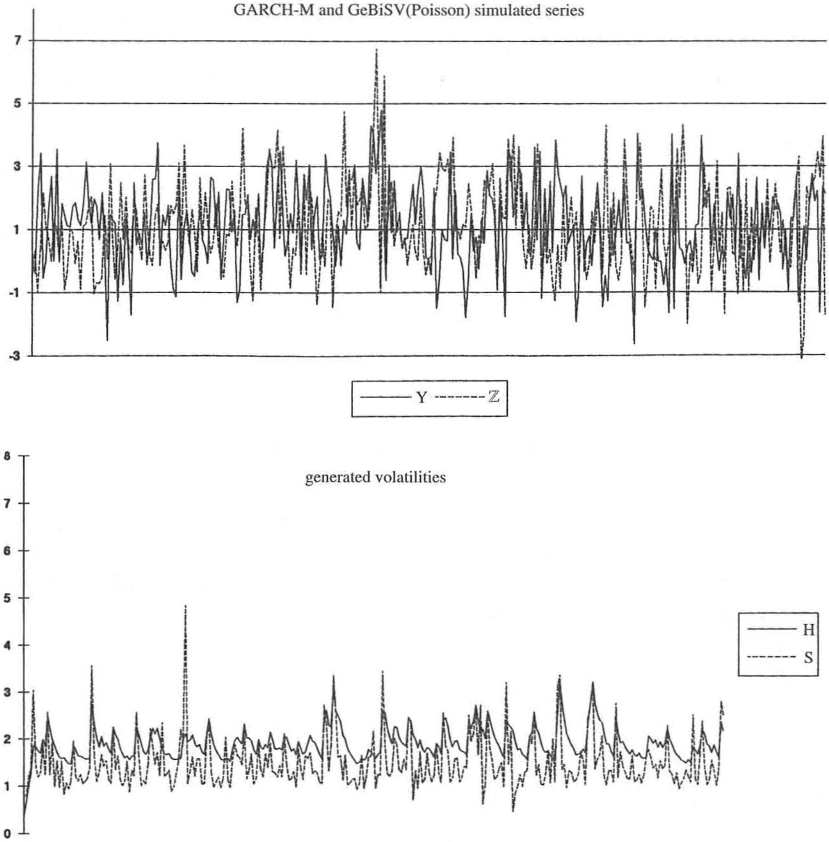


Figure 3: Third experiment: Poisson case.

Further experiments were done in order to verify the sensitivity of the generated DGPs and volatilities to the parameter model structure and innovation distributions (e.g., gamma innovations); the results suggested that the range of parameter values and distributional hypotheses that present the previously mentioned interesting features with regard to the simulated returns and volatilities is not particularly restrictive. In summary, under a value around 0.5 for the autoregressive coefficient in the risk aversion relation and controlling the GARCH-M and GeBiSV volatility coefficients in order to avoid contemporaneous (i.e., with regard to both the autoregressive and the exogenous model components) near-unitary situations, the framework proves to be quite stable. Two different level coefficients were adopted in the volatility equations so that it would be easier to recognize the similar pattern structure characterizing these two different volatility formulations.

A first consideration that derives from the given results is that if we consider the case where one or more regime shifts are encountered in the conditionally heteroskedastic data at hand, from the analysis of some subsample periods we could find that stock returns show volatility structures with different underlying dependence laws. In other words, according to the various possible regimes, one can observe different volatility shapes. As a result, we could misspecify our initial model and not be able to recognize the presence of covariance nonstationarity for the series at hand. In this important but (in the light of the previous results) possible case, it would be possible not only to misspecify the true model, because of the misacknowledgment of the presence of a switching regime stochastic nature for the GARCH-M process, but also to ignore the fact that different volatility processes could plausibly be suspected of coacting and thus characterizing the various regimes. Therefore, a shift between classes of volatility models, not simply between models in the ARCH class, could occur, according to the regime considered.

6. Conclusions and future research directions

The results given in this paper extend the analysis of the relationships between volatility stochastic processes and allow for interesting conjectures when the starting point of the analysis is a GARCH-M model. The structure of the GARCH-M model has not been unanimously and uniquely specified yet, particularly with regard to the choice of the risk premium functional representation. Thus, misspecification is possible if one deviates from the usually retained assumptions, and this comes to be relevant for the inferential aspects involved, since different classes of volatility stochastic processes can underly the observed time series.

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