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Cellular Automata and Nonperiodic Orbits

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Abstract. The periodic behavior of a particular class of dynamical systems, the cellular automata (CA), is studied. A large class of CAs is defined, (containing both additive and nonadditive CAs) in terms of the *sensitivity* of the local rule on which the CA is based. For this class, the set of configurations which enter a cycle after a finite number of iterations is completely characterized and it is proved that this set has measure zero according to every probability measure that assigns measure zero to single configurations.

1. Introduction

Cellular automata (CA) are dynamical systems consisting of a regular lattice of variables, any of which can take a finite number of discrete values. The state of the CA, specified by the values of the variables at a given time, evolves in synchronous discrete time steps according to a given local rule. CA have been widely used to model a variety of dynamical systems in physics, biology, chemistry, and computer science (e.g., [1, 8, 9, 10, 16]). Despite their apparent simplicity, many CA display a rich and complex behavior which is generally very hard, if not impossible, to predict. In particular, many properties of the temporal evolution of CA have been proved to be undecidable [4, 6, 14].

Informally, a CA is a pair (X, F), where X is the space of configurations and F, $F : X \to X$, is a map that governs the temporal evolution of the CA. In this paper we consider the following two problems.

1. Given any configuration c belonging to X, we want to know if there exist two integers $i, j \geq 0$, $i \neq j$, such that $F^i[c] = F^j[c]$. In other words, we want to know whether or not c enters a cycle after a finite number of iterations.

2. Measure the set of configurations that enter a cycle after a finite number of iterations.

The analysis of the periodic behavior of a dynamical system is central to the theory of chaos (e.g., [5, 15, 17]). Slightly different versions of problems 1 and 2 have been studied by several authors. The periodicity of the temporal sequences generated by a certain class of one-dimensional, binary, nearest-neighbor CA evolving from arbitrary finite initial configurations on an infinite lattice is studied in [13]. The periodicity of arbitrary configurations for the class of additive, one-dimensional, binary CA is studied in [3].

In this paper problems 1 and 2 are solved for a more general class of CA than those considered in [3] and for a more general set of configurations than those considered in [13]. More precisely, a class of CA is defined in terms of a particular property of the local rule; this is similar, in spirit, to the notion of sensitivity for continuous functions. Informally, a continuous function f is sensitive to one of its input variables if small modifications to the value of that variable cause large modifications to the output computed by f. In the case of discrete maps defined over finite sets, the definition above needs to be modified in order to fit some additional formal requirements, for example, one has to specify the meaning of "small modifications." The notion of sensitivity we use for discrete maps was introduced in [12] and called "permutivity." Here, the class of CA based on local rules that are permutive to the leftmost and/or to the rightmost variable is considered. This class of CA, which we call leftmost and/or rightmost permutive CA (L/R-CA), contains both additive and nonadditive CA. In particular, it contains all the additive CA defined over alphabets of prime cardinality. Loosely speaking, in a L/R-CA "information" moves through the lattice at each iteration without encountering any obstacle.

For the L/R-CA class of CA we prove the following.

- A configuration (finite or infinite) enters a cycle after a finite number of iterations if and only if it is spatially periodic.
- The measure of the set of configurations which enters a cycle after a finite number of iterations is zero for any measure function that assigns measure zero to single configurations.

Note that our results are independent of the number of input variables of the local rule on which the CA is based.

2. Notations and definitions

In this section some basic notations and definitions are reviewed. Let $\mathcal{A} = \{0, 1, \ldots, m-1\}$ be a finite alphabet and $f, f : \mathcal{A}^{2k+1} \to \mathcal{A}$, be any map. A one-dimensional CA based on the *local rule* f is a pair $(\mathcal{A}^Z, F)$, where

$$\mathcal{A}^Z = \{c \mid c : Z \to \mathcal{A}\} \tag{1}$$

is the space of configurations and F, $F : \mathcal{A}^Z \to \mathcal{A}^Z$, is defined as

$$F[c](i) = f(c(i-k), \ldots, c(i+k)), \qquad c \in \mathcal{A}^Z, \qquad i \in Z. \tag{2}$$

f depends on $2k+1$ variables, which will be denoted by $x_{-k}, \ldots, x_k$. For this reason, we say that k is the *radius* of f.

Throughout this paper, $F[c]$ will denote the result of the application of the map F to the configuration c and $c(i)$ will denote the ith element of the configuration c. We recursively define $F^n[c]$ by $F^n[c] = F[F^{n-1}[c]]$, where $F^0[c] = c$. The set $\mathrm{SP}(n)$ of spatially periodic configurations of period n is defined as

$$\mathrm{SP}(n) = \{c \mid \forall i \in Z,\ c(i) = c(i+n)\}. \tag{3}$$

The set SP of periodic configurations is now defined by

$$\mathrm{SP} = \bigcup_{n \in N} \mathrm{SP}(n). \tag{4}$$

A configuration $c \in \mathcal{A}^Z$ is of time period n for the map F if and only if $F^n[c] = c$. When no confusion arises, it can be said that a configuration is of period n instead of time period n. Let $(\mathcal{A}^Z, F)$ be a CA. The set EP of eventually periodic configurations for F is defined as

$$\mathrm{EP} = \left\{c \mid \exists i, j \in Z,\ i, j \geq 0,\ i \neq j,\ F^i[c] = F^j[c]\right\}. \tag{5}$$

We now give definitions for the permutive and additive local rule.

Definition 1. From [12]; f is permutive in x_i, $-k \leq i \leq k$, if and only if, for any given sequence $\overline{x}_{-k}, \ldots, \overline{x}_{i-1}, \overline{x}_{i+1}, \ldots, \overline{x}_k \in \mathcal{A}^{2k}$, we have

$$\{f(\overline{x}_{-k}, \ldots, \overline{x}_{i-1}, x_i, \overline{x}_{i+1}, \ldots, \overline{x}_k)\}_{x_i \in \mathcal{A}} = \mathcal{A}. \tag{6}$$

Definition 2. f is leftmost permutive (rightmost permutive) if and only if there exists an integer i, $-k \leq i \leq k$, such that

- $i < 0$ $[i > 0]$,
- f is permutive in the ith variable, and
- f does not depend on x_j, $j < i$, $[j > i]$.

We denote by L/R-CA the set of CA which are leftmost and/or rightmost permutive.

Definition 3. From [18]; f is additive if and only if it can be written as

$$f(x_{-k}, \ldots, x_k) = \left(\sum_{i=-k}^{k} \lambda_i x_i\right) \bmod m, \tag{7}$$

where $\lambda_i \in \mathcal{A}$.

From now on, we say that a CA is permutive or additive if the local rule on which it is based is permutive or additive.

Let g, $g : \mathcal{A} \to \mathcal{A}$, be any map. We say that a local rule f, $f : \mathcal{A}^{2k+1} \to \mathcal{A}$, is *trivial* if it satisfies $f(x_{-k}, \ldots, x_k) = g(x_0)$. The evolution of a trivial CA can be easily determined and it is not interesting neither from the topological nor from the metric point of view. The following remark can be easily verified.

Remark 1. If $\mathcal{A}$ is an alphabet of prime cardinality and $(\mathcal{A}^Z, F)$ is a non-trivial additive CA, then $(\mathcal{A}^Z, F)$ is a L/R-CA.

As an example, the class of additive, one-dimensional, binary CA studied in [3] are L/R-CA. L/R-CA also satisfy other interesting properties. It has been proved in [11] that L/R-CA are topologically transitive dynamical systems. Loosely speaking, a dynamical system (X, F) is topologically transitive if it cannot be broken into two or more subsystems that do not interact under iterations of F. Moreover, it has been proved that in the class of elementary CA (one-dimensional binary CA with radius 1) L/R-CA are the only transitive CA. In [2], it has been proved that topologically transitive CA are sensitive to initial conditions. Intuitively, if a map possesses sensitive dependence to initial conditions; then, for all practical purposes, its dynamics defies numerical approximation. Small errors in computation introduced by round-off may become magnified upon iteration. The results of the numerical computation of an orbit, no matter how accurate, may be completely different from the real orbit. Note that many definitions of chaos for general dynamical systems are based on these two properties (e.g., [7]).

3. Main results

In this section we consider the class L/R-CA and prove the following two results.

1. A configuration c belongs to EP if and only if it is spatially periodic, that is, EP = SP.

2. For any probability distribution P defined over the space of the configurations A^Z that assigns probability 0 to single configurations, we have $P(\mathrm{SP}) = 0$.

The following theorem proves that each configuration which lies on a cycle of a L/R-CA must be spatially periodic.

Theorem 1. *Let $(\mathcal{A}^Z, F)$ be a L/R-CA based on a local rule f with radius k. Let $c \in \mathcal{A}^Z$ be a periodic configuration for F, then c is spatially periodic.*

Proof. Without loss of generality, we assume that F is rightmost permutive. Since c is a periodic configuration for F, then there exists an integer number n such that $F^n[c] = c$. We partition c into a sequence $\{c^j\}_{j\in Z}$, of subconfigurations each of length $2kn$ defined by

$$c^j(i) = c(2knj + i), \qquad 0 \le i < 2kn, \qquad j \in Z. \tag{8}$$

Let $a_1 \in \mathcal{A}^{n_1}$ and $a_2 \in \mathcal{A}^{n_2}$ be two finite configurations of length n_1 and n_2, respectively. We use a_1a_2 to denote the configuration of length $n_1 + n_2$ defined by

$$a_1a_2(i) = \begin{cases} a_1(i) & \text{if } 0 \le i < n_1, \\ a_2(i) & \text{if } n_1 \le i < n_2. \end{cases} \tag{9}$$

We define the directed graph $G_c = (V, E)$ as

$$V = \{\delta \in \mathcal{A}^{2kn} \mid \delta = c^j \text{ for some integer } j\}, \tag{10}$$

this is the set of all configurations of length $2kn$ which appear at least once in the partition of c. The arc (δ_1, δ_2) belongs to E if and only if there exists an integer number i such that $\delta_1\delta_2 = c^ic^{i+1}$. One can easily verify that G is connected and that each node of G has in-degree at least 1. We now prove that each node has out-degree 1. Assume that both (δ, δ_1) and (δ, δ_2) belong to E with $\delta_1 \ne \delta_2$. This means that c contains both $\delta\delta_1$ and $\delta\delta_2$. Assume that c contains $\delta\delta_1$ starting at position p and $\delta\delta_2$ starting at position q.

Let δ', δ'', δ'_1, δ''_1, δ'_2, and δ''_2 be configurations of length kn such that $\delta'\delta'' = \delta$, $\delta'_1\delta''_1 = \delta_1$, and $\delta'_2\delta''_2 = \delta_2$. At least one of the following two inequalities holds: $\delta'_1 \ne \delta'_2$ or $\delta''_1 \ne \delta''_2$. Assume that $\delta'_1 \ne \delta'_2$. Let $t < kn$ be such that $\delta'_1(i) = \delta'_2(i)$, $0 \le i < t$, and $\delta'_1(t) \ne \delta'_2(t)$. Since F is rightmost permutive, we have that

$$F^n[c](p + kn + t) \ne F^n[c](q + kn + t). \tag{11}$$

Since c is periodic of period n and it contains δ starting both at position p and at position q, we have that

$$F^n[c](p + i) = F^n[c](q + i), \qquad i = 0, \ldots, 2kn - 1. \tag{12}$$

From equations (11) and (12) we have a contradiction. Thus, $\delta'_1 = \delta'_2$. Assume now that $\delta''_1 \ne \delta''_2$. Let $t < kn$ be such that $\delta''_1(i) = \delta''_2(i)$, $0 \le i < t$, and $\delta''_1(t) \ne \delta''_2(t)$. Since F is rightmost permutive, we have that

$$F^n[c](p + 2kn + t) \ne F^n[c](q + 2kn + t). \tag{13}$$

Since c is periodic of period n and $\delta'_1 = \delta'_2$, we have that

$$F^n[c](p + i) = F^n[c](q + i), \qquad i = 0, \ldots, 3kn - 1. \tag{14}$$

From equations (13) and (14) we have a contradiction. Note that if all the nodes of a finite graph have out-degree 1 and in-degree at least 1 then they have in-degree exactly 1.

Summarizing, we have that E is of the form

$$E = \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), \ldots, (\alpha_{h-1}, \alpha_h), (\alpha_h, \alpha_1)\} \tag{15}$$

for some integer h. Consider now the configuration $\alpha = \alpha_1\alpha_2 \cdots \alpha_h$. One can easily verify that

$$c(2hknj + i) = \alpha(i), \qquad 0 \leq i < 2hkn, \qquad j \in Z, \tag{16}$$

that is, c is spatially periodic of period $2hkn$. ■

Theorem 1 guarantees that if a configuration is periodic then it is also spatially periodic. This does not imply that a configuration which is not spatially periodic might enter a cycle after a finite number of iterations. The next theorem guarantees that this is not possible.

Theorem 2. *Let $(\mathcal{A}^Z, F)$ be a L/R-CA based on a local rule with radius k. Let $a \in \mathcal{A}^Z$ be a spatially periodic configuration for F and b is one of its predecessors. Then b is spatially periodic.*

Proof. Assume that there exists a spatially periodic configuration a that has a predecessor b which is not spatially periodic. Since a is spatially periodic, then there exists an integer $n \geq 1$ such that

$$a(i + nj) = a(i), \qquad 0 \leq i < n, \qquad j \in Z. \tag{17}$$

We partition b into subconfigurations b^j, $j \in Z$, of length n defined as

$$b^j(i) = b(i + nj), \qquad 0 \leq i < n, \qquad j \in Z. \tag{18}$$

Since the number of distinct b^j is no greater then m^n (where m is the cardinality of $\mathcal{A}$), b must contain two subconfigurations of the type b^ib^h and b^ib^l such that $b^h \neq b^l$. Assume that b contains b^ib^h and b^ib^l starting at positions p and q, respectively. Assume that $b^h(i) = b^l(i)$, $0 \leq i < t < n$, and $b^h(t) \neq b^l(t)$. Without loss of generality, assume that F is rightmost permutive. Thus,

$$F[b](p + n + t - k) \neq F[b](q + n + t - k). \tag{19}$$

Since $p = q + nj$ for some integer j, and $F[b] = a$, we have

$$a(q + nj + n + t - k) \neq a(q + n + t - k). \tag{20}$$

From equations (17) and (20) we get a contradiction. ■

Corollary 1. *Let $(\mathcal{A}^Z, F)$ be a L/R-CA and let $a \in \mathcal{A}^Z$ be any configuration. Then a enters a cycle after a finite number of iterations if and only if it is spatially periodic.*

Proof. If a is spatially periodic then it is clear that after a finite number of iterations a enters a cycle. If a is not spatially periodic, by Theorem 2 we know that $F^n[a]$, $n \geq 0$, is not spatially periodic and then, by Theorem 1 it cannot enter any cycle. ■

In Example 1 we exhibit a one-dimensional binary CA that is neither rightmost nor leftmost permutive and which does not satisfy Theorem 2.

Example 1. Let $(\mathcal{A}^Z, F)$ be the CA based on the following local rule

$$f(x_{-1}, x_0, x_1) = \begin{cases} 1 & \text{if } x_{-1} + x_0 + x_1 = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

One can then easily verify that the spatially nonperiodic configuration c defined by

$$c(i) = \begin{cases} 0 & \text{if } i = 0, \\ 1 & \text{otherwise,} \end{cases} \tag{22}$$

enters a cycle after one iteration, while the spatially periodic configuration c defined by $c(i) = 1$, $\forall i \in Z$, has infinitely many predecessors that are not spatially periodic.

Note that the local rule of Example 1 differs from the rightmost and leftmost permutive local rule

$$g(x_{-1}, x_0, x_1) = \begin{cases} 1 & \text{if } (x_{-1} + x_0 + x_1) \bmod 2 = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

only if $x_{-1} = x_0 = x_1 = 1$.

Consider now the family $\mathcal{P}$ of probability measures (or, more generally, measures) defined over the set $\mathcal{A}^Z$ that satisfy the property

$$\forall P \in \mathcal{P}, \qquad \forall c \in \mathcal{A}^Z, \qquad P(c) = 0. \tag{24}$$

In Example 2, a probability distribution over the space $\mathcal{A}^Z$ which satisfies Property 24, the so-called Bernoulli distribution, is given.

Example 2. Let $\mathcal{A} = \{0, \ldots, m-1\}$ be a finite alphabet. We define the cylindric set $\text{Cyl}(\delta_{-n}, \ldots, \delta_n) \subset \mathcal{A}^Z$ by

$$\text{Cyl}(\delta_{-n}, \ldots, \delta_n) = \{c \in \mathcal{A}^Z \mid c(i) = \delta_i,\ \delta_i \in \mathcal{A},\ i = -n, \ldots, n\}. \tag{25}$$

Cylinders are open and closed sets that form a basis for the topology of $\mathcal{A}^Z$. We define the probability distribution on the set of cylinders of $\mathcal{A}^Z$ as follows:

$$P(\text{Cyl}(\delta_{-n}, \ldots, \delta_n)) = \prod_{i=-n}^{n} \text{Prob}(\delta_i), \tag{26}$$

where $\text{Prob}(\delta_i)$ stands for the probability of the singleton δ_i to occur. If Prob is the uniform probability distribution over the set $\mathcal{A}$, we have

$$P(\text{Cyl}(\delta_{-n}, \ldots, \delta_n)) = \prod_{i=-n}^{n} \frac{1}{m} = \frac{1}{m^{2n+1}}. \tag{27}$$

It is easy to check that for any configuration $c \in \mathcal{A}^Z$ we have $P(c) = 0$, that is, $P \in \mathcal{P}$.

We now prove that for L/R-CA the set of configurations that eventually enter a cycle has measure zero according to any probability measure $P \in \mathcal{P}$. Absence of periodicity can be interpreted as further evidence that L/R-CA are chaotic dynamical systems.

Theorem 3. *Let* $(\mathcal{A}^Z, F)$ *be a L/R-CA. Let* $P \in \mathcal{P}$ *be any probability distribution. Then* $P(\mathrm{EP}) = 0$.

Proof. By Corollary 1 we know that, starting with a configuration $a \in \mathcal{A}^Z$, a cycle is entered if and only if a is spatially periodic, that is, SP = EP. We now prove that $P(\mathrm{SP}) = 0$. Since $\mathrm{SP} = \{o_1, \ldots, o_n \ldots\}$ is a countably infinite set, we have that

$$P(\mathrm{SP}) = \sum_{i=1}^{+\infty} P(o_i) = 0. \blacksquare \tag{28}$$

4. Conclusions

In this paper a particular class of CA were defined according to the sensitivity of the local rule on which they are based: the leftmost and/or rightmost permutive CA (L/R-CA). For this class of CA, we prove that the measure of the set of configurations which enter a cycle after a finite number of iterations is zero for any probability distribution that assigns probability zero to single configurations. In [2] and [11] it has been proved that L/R-CA are chaotic dynamical systems according to Knudsen's definition of chaos and it has been conjectured that they are chaotic in the sense of Devaney as well. Since the absence of periodicity is a widely accepted feature of chaotic behavior, the results can be considered as more evidence that L/R-CA are chaotic dynamical systems.

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