A New Algorithmic Approach to the Minority Game

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In this paper a new approach for the study of the complex behavior of the minority game is introduced that uses the tools of algorithmic complexity, physical entropy, and information theory. It is shown that physical complexity and mutual information functions strongly depend on memory size of the agents. This yields more information about the complex features of the stream of binary outcomes of the game than volatility itself.

1. Introduction

In many natural and social systems agents establish among themselves a complex network of interactions. Often this structure reflects the competition for limited resources. In such systems successful agents are those which act in ways that are distinct from their competitors. There have been many attempts to understand the general underlying dynamics of systems in which agents seek to be different. Some of them have focused on the analysis of a class of simple games which have come to be known as *minority games* [1–3].

The minority game [1] was first introduced in the analysis of decision making by agents with bounded rationality, based on the "El Farol" bar problem [2]. It is a toy model of N interacting heterogeneous agents, which allows addressing their reaction to public information—such as price changes—and the feedback effects of these reactions. In some sense, the efficient market hypothesis [4] captures this issue assuming that all relevant information is instantaneously "incorporated" into the prices, but as some authors argue [5] it is exactly in the deviation of real markets from efficient markets that very interesting phenomena occurs.

The setup of the minority game is as follows. The N agents have to choose at each time step whether to go in room 0 or 1. The agents that choose the less crowded room (minority room) win, and the others lose. The agents have limited capabilities, and only "remember" the last m outcomes of the game. The number m is called memory size or brain size. Agents use strategies to decide what room to go into. A strategy is a choosing device, that is, an object that processes the outcomes of the winning room in the last m time steps and according to this information prescribes what room to go into next. The agents randomly pick s strategies at the beginning of the game. After each turn, the agents assign one (virtual) point to each of the strategies that would have predicted the correct outcome. At each turn of the game, they use whichever is the most successful strategy among the s in their possession, that is, the one that has gained the most virtual points is chosen.

As a dynamical system with many elements under mutual influence, the minority game is thought to underlie much of the phenomena associated with complexity. A group of measures have been defined in an attempt to understand this feature. Particular emphasis has been devoted to the mean square deviation of the number of agents making a given choice σ, which measures in the opinion of some authors [6] the efficiency of the system. According to [6–8] when the fluctuations are large (i.e., larger σ) the number of agents in the majority side (the number of losers) increases. In this way, the variance measures the degree of cooperation or mutual benefits of the agents. In the financial context, the observable σ is called *volatility*.

A lot of work has been done looking for relations among σ, s, and m. The main result, which emerges from numerical simulations [8, 9] is that when the number of strategies per agent s is small, the volatility σ exhibits a pronounced minimum as a function of the brain size m. Around this minimum, the volatility is substantially smaller than the value obtained when each agent makes their decision by tossing a coin. In that case $\sigma^2 = N/4$. However, it seems that this property does not depend on the memory size of the agents. According to [10] the only crucial requirement is that all the individuals must possess the same information, irrespective of the fact that the information is true or false. Starting from this, the results obtained in [8] and [9] are reproduced in [10]. Therefore, volatility σ has no sensitivity to memory size of the agents and the real history of the game.

In this paper we introduce a new approach for the study of the complex behavior of the minority game using the tools of algorithmic complexity, physical entropy, and information theory [11–19]. We will show that two measures, physical complexity [13] and the mutual information function [16, 17] strongly depend on brain size m and the real history of the game. Our results show that volatility is not a good measure of the complex behavior of the game and that other measures (two of them proposed by us) should be used to describe all the richness of behavior in the minority game. Our results in some sense extend the conclusion of [10]. All the claims are based on our belief that the binary string of the successive outcomes of the minority room, which is the whole history of the game, contains all the relevant information about the model. In some sense, to be explained later, it is a kind of generalization of the efficient market hypothesis for the minority game.

2. Physical complexity of the minority game

The study of complex systems has enjoyed tremendous growth in recent years in spite of the fact that the concept of complexity is vaguely defined. In searching for an adequate measure for the complexity of a binary string one could expect that the two limiting cases (e.g., regular strings and random strings) have null complexity, while the "intermediate" strings that appear to have information encoded are thought to be complex. A good measure of physical complexity should have these properties.

Contrary to the intuition that the regularity of a string is in any way connected to its complexity, as in Kolmogorov–Chaitin theory ([14, 15] and references therein) we agree with [12] and [13] that classifying a string in the absence of an enviroment within which it is to be interpreted is quite meaningless. In other words the complexity of a string should be determined by analyzing its correlation with a physical enviroment. In reference to the minority game the only physical record one gets is the binary string of the successive outcomes which is the whole history of the game. The determination of the complexity of every substring should depend on the whole history of the game. Comprehension of the complex features of such substrings has high practical importance. First, every agent in the game uses only this kind of information to decide the next outcome, which has some weight in the formation of future substrings to be used by the agents themself in a future decision. Second, the well known complex behavior of financial indexes before crashes. The minority game as a toy model of financial markets captures some relevant features of those markets. Hence studying the complexity of substrings from the stream of outcomes of the game should throw light over some important properties of the crashes. As will be shown, some kind of loss of correlation appears in the stream of outcomes of the game.

In this section we introduce a measure (first developed in [12, 13] and called *physical complexity*) defined as the number of binary digits that are meaningful in a string η with respect to the enviroment ϵ. Here it is also proved that physical complexity depends inversely on memory size m of the agents as well as on the whole history of the game. The larger the brain size m, the smaller the number of binary digits that are meaningful in a string with respect to the enviroment. As we show later, for random strings it tends to be null.

We first introduce some concepts. A natural way of measuring the complexity of the state of a system is the size of the smallest prescription required to specify it with some assumed accuracy.

Kolmogorov–Chaitin theory [14, 15] provides a measure for the regularity of a symbolic string. A string is said to be regular if the algorithm necessary to produce it on a Turing machine is shorter than the string itself. For a string η the Kolmogorov–Chaitin complexity is defined as the length of the shortest program π producing η when run on the universal Turing machine T:

$$K(\eta) = \min\{|\pi| : \eta = T(\pi)\} \tag{1}$$

where $|\pi|$ represents the length of π in bits, $T(\pi)$ is the result of running π on Turing machine T, and $K(\eta)$ is the Kolmogorov–Chaitin complexity. For the details see [12–14] and references therein. As stated previously, the interpretation of a string should be done in the framework of an enviroment. Therefore, imagine a Turing machine that takes an infinite string ϵ as input (represented here by the whole history of the game). We can define the conditional complexity $K(\eta/\epsilon)$ [12] as the length of the smallest program that computes η on a Turing machine having ϵ as input:

$$K(\eta/\epsilon) = \min\{|\pi| : \eta = C_T(\pi, \epsilon)\} \tag{2}$$

where $C_T(\pi, \epsilon)$ denotes the result of running program π on Turing machine T given input string ϵ. As remarked in [13] $K(\eta/\epsilon)$ represents those bits in η that are random with respect to ϵ.

The physical complexity can be defined as the number of bits that are meaningful in η with respect to ϵ:

$$K(\eta : \epsilon) = |\eta| - K(\eta/\epsilon). \tag{3}$$

Notice that $|\eta|$ also represents (e.g., [13]) the unconditional complexity of string η, that is, the value of complexity if the input would be $\epsilon = \emptyset$. Of course, the measure $K(\eta : \epsilon)$ as defined in equation (3) has little practical application, mainly because it is impossible to know how information about ϵ is coded in η. However, if we are given multiple copies of a symbolic sequence, or more generally, if a statistical ensemble of strings is available to us, then the determination of complexity becomes an exercise in information theory. It can be proved (see [12] or [13] for the details) that the average values $C(|\eta|)$ taken over an ensemble Σ of strings of length $|\eta|$ could be approximated by:

$$C(|\eta|) = \langle K(\eta : \epsilon)\rangle_\Sigma \approx |\eta| - K(\Sigma/\epsilon) \tag{4}$$

where:

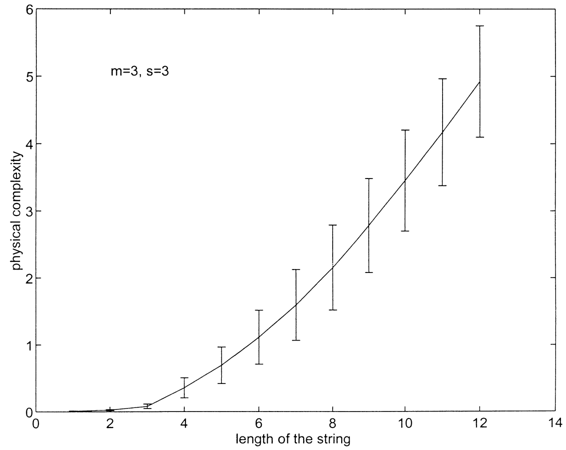
$$K(\Sigma/\epsilon) = -\sum_{\eta\in\Sigma} p(\eta/\epsilon)\log_2 p(\eta/\epsilon) \tag{5}$$

and the sum is taken over all the strings η in the ensemble Σ. In a population of N strings in enviroment ϵ, the quantity $n(\eta)/N$, where $n(s)$

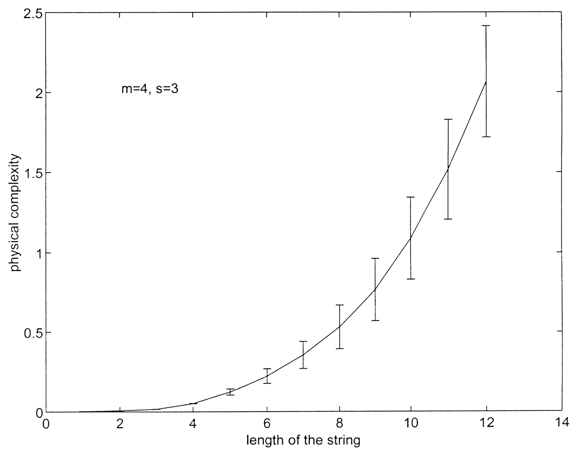
denotes the number of strings equal to η in Σ, approximates $p(\eta/\epsilon)$ as $N \to \infty$.

Let $\epsilon = a_1 a_2 a_3 \ldots a_n \ldots;\ a_i \in \{0, 1\}$ be the stream of outcomes of the game and l a positive integer $l \geq 2$. Let Σ_l be the ensemble of sequences of length l built up by a moving window of length l, that is, if $\eta \in \Sigma_l$ then $\eta = a_i a_{i+1} \ldots a_{i+l-1}$ for some value of i.

We calculate the values of $C(l)$ using this kind of ensemble Σ_l. Figure 1 shows the graph of $C(l)$ for different values of memory size m and for a fixed value of s. Notice that when m increases, the value of $C(l)$ for every

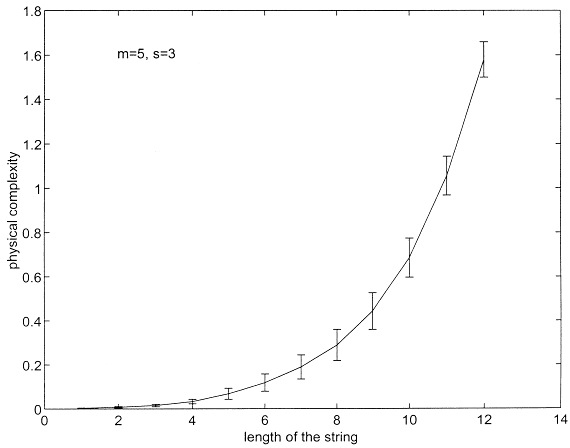


(a)



(b)

Figure 1. Values of $C(l)$ *versus* l for different values of m. Notice that when m increases the values of $C(l)$ decrease for every length l; (a) $m = 3$ and $s = 3$, (b) $m = 4$ and $s = 3$, and (c) $m = 5$ and $s = 3$.



(c)

Figure 1. (*continued*).

fixed l decreases. The explanation of this fact is as follows. Consider the following two "histories" of the game:

$$h_1 = 1010\ldots \qquad h_2 = 1011\ldots.$$

If the brain size is $m = 3$, then the agents cannot differentiate the given histories. Hence, they act in both cases as their best performing strategy suggests. If $m = 4$ they can differentiate, and in general have different responses to, histories h_1 and h_2. Therefore, as m increases the perception of the agents becomes less "coarse" and global response is more unpredictable.

Then, there is a loss of information as the brain size increases. More precisely, for every value of l the corresponding value of $C(l)$ decreases as the memory size m increases. Figure 2 shows three curves concerning the loss of information when m changes from 3 to 4, from 4 to 5, and from 5 to 6. Notice that as m increases the curves are more flat. This means that the loss of information (or correspondingly the increase of randomness) is slower when the values of m are larger.

In Figure 3 the mentioned $C(l)$ curves are compared with those calculated from random sequences (see Figure 1). The bottom plot corresponds to the mean values of 10 random sequences. It confirms our claim that physical complexity tends to be null in random sequences. Hence the whole history of the game encodes some information that seems to be insensitive to volatility according to [10]. Therefore, physical complexity is a better measure than volatility itself. Further, this shows that the information content of strings of length l decrease as the memory size increases for every fixed l.

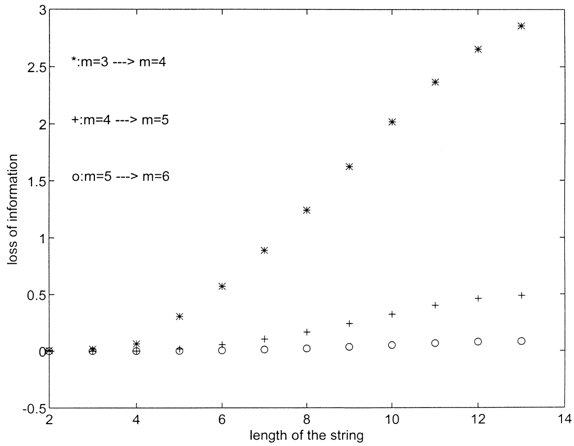


Figure 2. Values of the loss of information *versus* l when the memory size m changes from 3 to 4 ($*$) from 4 to 5 ($+$), and from 5 to 6 ($\circ$).

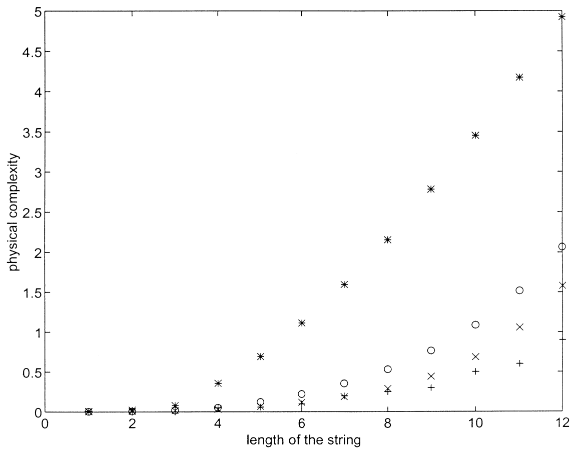


Figure 3. Graphs of $C(l)$ *versus* l for different values of m. From top to bottom the plots correspond to $m = 3,\ 4,\ 5$. The lowest plot is the mean value of $C(l)$ over 10 random sequences.

The calculated values of physical complexity are more "stable" as the length of the strings increase. Figure 4 shows the ratio (standard deviation/mean) for several $C(l)$ curves for different values of memory size m and number of strategies s. Notice that as the length l increases this ratio decreases, indicating that the standard deviation is a smaller fraction of the mean when the values of l grow. Interestingly, the bottom curve corresponds to the physical complexity of random sequences. This

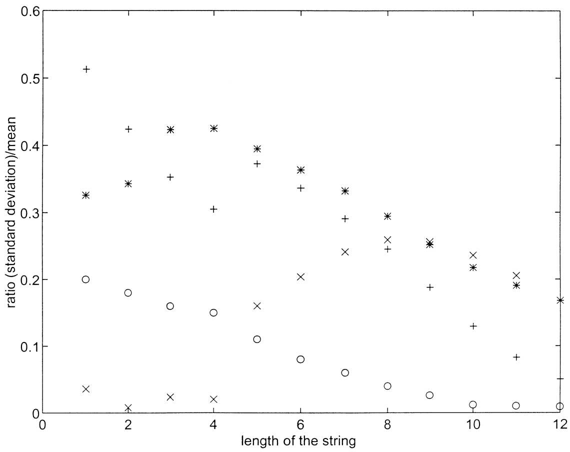


Figure 4. The ratio (standard deviation)/mean for numerical simulations with different values of brain size and number of strategies per agent. The mean values were calculated over 10 runs with the same parameters. The curves intersect for $2 \leq l \leq 6$, but for $6 < l$ they are ordered. In the interval $6 < l$, from top to bottom in the graph: $m = 3$ and $s = 6$; $m = 4$ and $s = 6$; $m = 3$ and $s = 3$. The lowest curve corresponds to random sequences.

statement enforces our claim about the relevance of memory and the lack of sensitivity to σ.

3. Mutual information function of the minority game

In section 2 we showed how the memory size of agents affects the degree of randomness of the stream of successive outcomes of the minority game. However, nothing has been said about the correlation of the outcomes with time. Note that the distance between two binary symbols in the stream of data represents the number of time iterations between them. Therefore, a measure of the degree of correlation between elements in a symbolic string could be useful in understanding the behavior of the game.

The quantities often used to statistically characterize the arrangement of symbols in a sequence are the correlation function and the mutual information function [16–19]. The correlation function is defined as the correlation between two symbols as a function of the distance between them [17]:

$$\Gamma(d) = \sum_{\alpha,\beta} \alpha\beta P_{\alpha\beta}(d) - \left(\sum_{\alpha} P_{\alpha}\right)^2 \tag{6}$$

where $P_{\alpha\beta}(d)$ is the probability of having a symbol α followed d sites away by a symbol β and P_α the density of the symbol α.

With these definitions the mutual information function is defined as:

$$M(d) = \sum_{\alpha,\beta} P_{\alpha\beta}(d) \log_2 \left[\frac{P_{\alpha\beta}(d)}{P_\alpha P_\beta} \right]. \tag{7}$$

It can be proved [17] that zero $M(d)$ at some distance d implies zero $\Gamma(d)$ at that distance, but the reverse may not be true. Therefore, the mutual information function is a more sensitive measure of correlation than the correlation function and hence is adopted here.

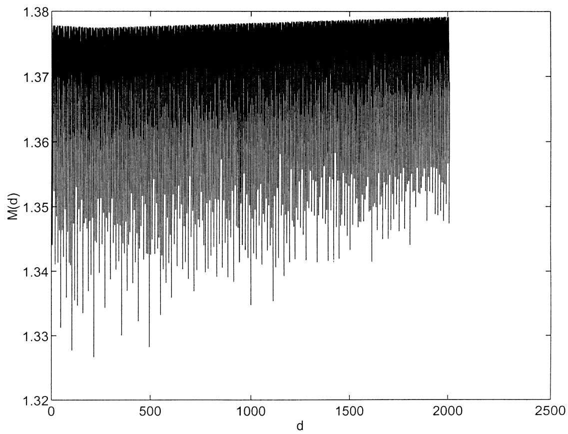
Fourier spectra (e.g., [20]) is widely used in time series analysis, although it does not provide any new information which is not described by the time series itself. Nevertheless, the visual representation in the frecuency domain can more easily reveal patterns which are hard to discern in the primary data, for example, intricate periodic behavior. We use a Fourier transform of the mutual information function to detect some periodical features of that function when applied to outcomes of the game. From now on, we call the power spectra of a mutual information function the product of a Fourier transform of that function by its complex conjugate:

$$\hat{S}(k) = \theta \left| \sum_{d=1}^{L} M(d) e^{-i2\pi \frac{k}{L} d} \right|^2 \tag{8}$$

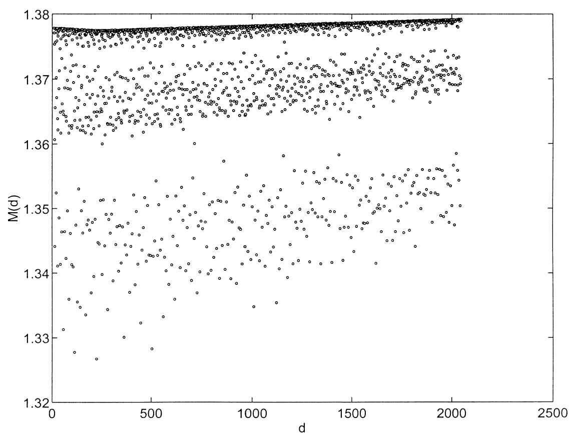
where θ is a constant related with the sample frecuency and L is the amount of data available for $M(d)$, see [20] for details.

The most important feature of the mutual information function of the string ϵ of outcomes of the game is the remarkable persistence of correlation at some distances and its periodic behavior. Figure 5 shows that function and its power spectra for a simulation of the game. The value of brain size is $m = 3$ and the number of strategies per agent $s = 3$. Figure 5(a) represents the mutual information function plotted using a solid line, while Figure 5(b) is the same function using points. We have done this in order to enhance the periodic features of that function. Figure 5(c) corresponds to a power spectra of the mutual information function. Notice that there are many values of d for which $M(d)$ is high, while for some d' a bit bigger than d the value of $M(d')$ is low. Hence, there are abrupt changes in the correlation of symbols along the ϵ string for certain distances. This implies a lack of coordination in the actions of agents. More than that, this behavior is periodic and depends on m as can be concluded from the power spectra of $M(d)$. Note that this loss of correlation reflected in $M(d)$ is translated into loss of predictability of the agents of the game.

Another interesting fact is the behavior of the power spectra as memory size m increases. Figure 6 shows this function for several values of



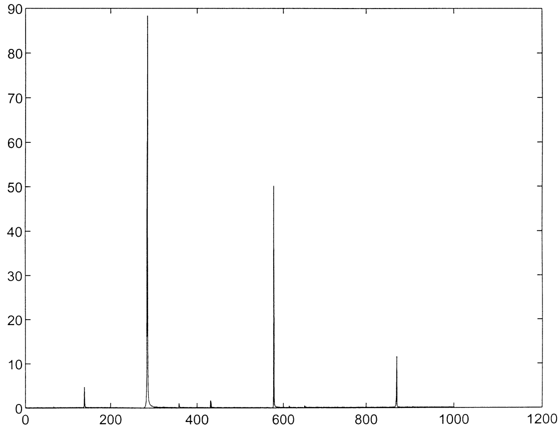
(a)



(b)

Figure 5. The mutual information function of the stream of outcomes and its power spectra. The value of brain size $m = 3$ and number of strategies per agent $s = 3$. (a) Represents the mutual information function plotted using lines. (b) Is the same function, but plotted using points. This enhances the periodic behavior of the mutual information function. (c) Represents the power spectra of the mutual information function.

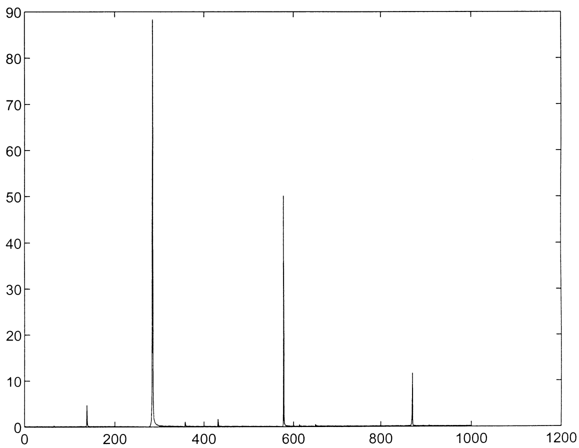
memory size. Notice that as m increases, more and more frecuencies enter the spectra. This means that abrupt changes in the mutual information function appear more often and when m changes from 3 to 4 a kind of phase transition appears. This is in perfect agreement with our results in section 2, because as shown there, the increase of m tends to decrease the predictability of agents as the behavior of the averaged physical complexity $C(l)$ shows (see Figures 1 and 3 and discussion in the text).



(c)

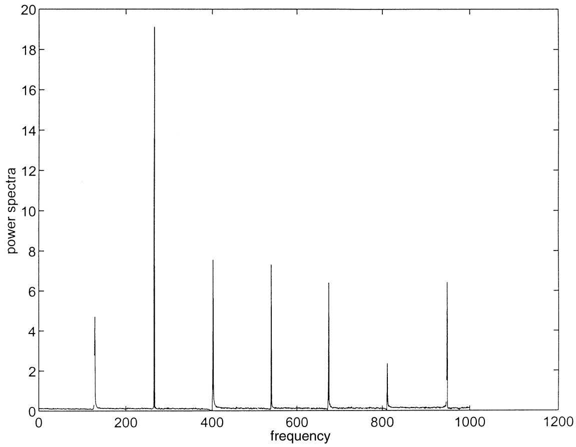
Figure 5. (*continued*).

The mutual information function for random sequences behaves very differently. Figure 7 shows the mutual information function for a random sequence as well as its power spectrum. We would like to emphasize the sharp descent of that function. It can be proved that it behaves as $1/d^{\alpha}$ where d is the distance between symbols and $\alpha > 2$ (see [18] and



(a)

Figure 6. Power spectra of the mutual information function for several values of m. Starting from (a) to (d) $m = 3, 4, 5, 6$. As the brain size increases, more and more frecuencies enter the signal. When m changes from 3 to 4 a kind of phase transition seems to appear. The number of strategies per agents in all simulations is $s = 3$.



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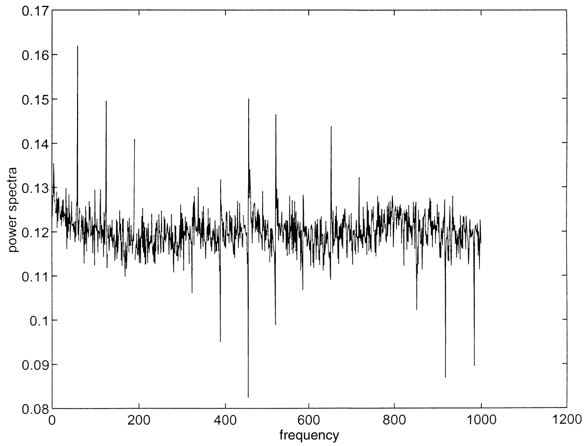

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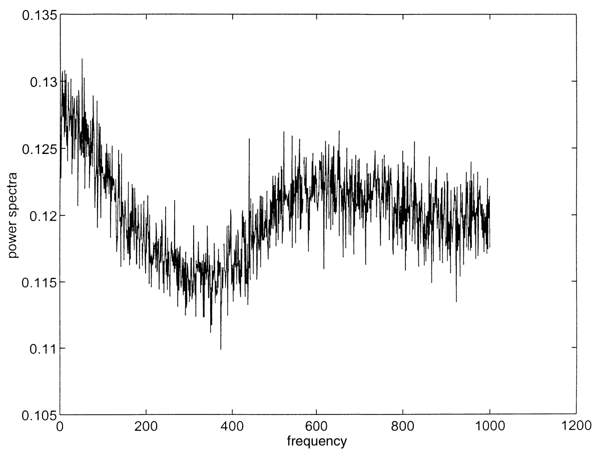
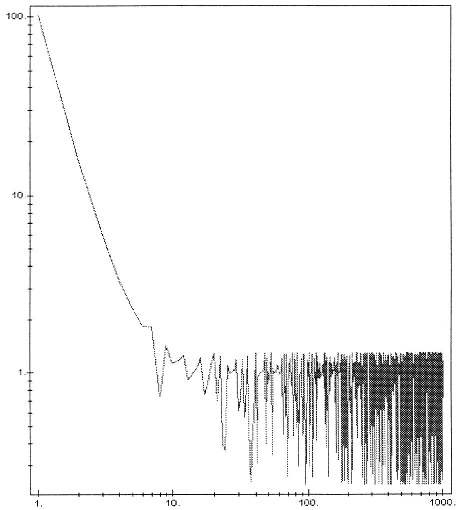
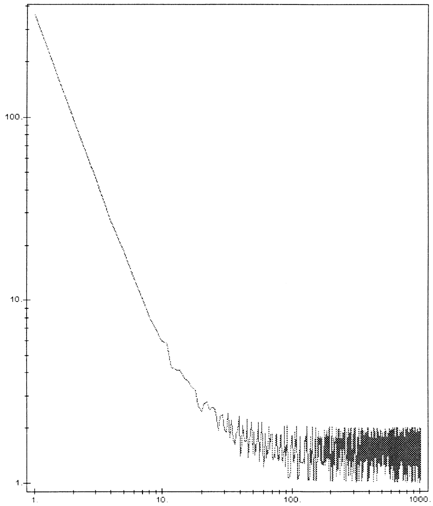


(d)

Figure 6. (*continued*).



(a)



(b)

Figure 7. The mutual information function (a) and power spectrum (b) of a random sequence.

references therein). Better structures, such as those drawn from DNA molecules, also have a similar mutual information function [18, 19]. Hence the behavior reported in this paper characterizes the outcome of the minority game. Furthermore, we would like to stress the relevance of the mutual information function for giving new insight about the game.

Few words have been said about the relevance of the behavior of the model with different numbers of strategies per agent. We can only add that the increase of s for fixed m yields results similar to those exposed previously. A closer look is needed to differentiate both behaviors.

Conclusions

This paper introduced a new approach for the study of the complex behavior of the minority game using the tools of algorithmic complexity, physical entropy, and information theory. This approach gives more insight than that obtained from the study of volatility σ. We have shown that the string ϵ of outcomes of the game is not quite random and contains some relevant information which depends on some parameter of the model, for example, the memory size of the agents. We also show that as m increases, the average number of bits that are meaningful in a substring of length l with respect to the whole history of the game ϵ decreases. It does not convert the history ϵ into a random string as shown in section 3 using the mutual information function and its power spectra. The way in which the average loss of information impinges on the whole series of outcomes yields sudden changes of correlation in the series. As m increases these changes appear more often and for some values of m a kind of phase transition seems to arise and the power spectra becomes continuous. This shows that the claim in [10] about irrelevance of memory in the minority game could be complemented. In our opinion, the right conclusion is that the volatility defined as the mean square deviation of the number of agents making a given choice is not a good measure for the study of complex behavior of the minority game.

Acknowledgments

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