Estimation of Multivariate Cumulative Processes

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This paper deals with the joint probability distribution of renewal and cumulative processes. Explicit formulas for the expectations and covariance matrix of the multivariate process are estimated.

1. Renewal and cumulative processes

Let us suppose that in a result of some action in a random time X we get a random receipt Y. If actions are repeated one-by-one, then in the time interval $[0, t]$ we are interested in the cumulative process of receipts. Let (X, Y) and (X_n, Y_n), with $n \geq 1$, be independent equidistributed random vectors. Let X be a positive random variable and Y be a nonnegative integer random variable. We assume that the considered random variables have two moments: $\mathrm{E}(X^i) = \mu_i$, $i = 1, 2$, $\mathrm{Var}(X) = \sigma^2$, $\mathrm{E}(Y) = \nu_1$, $\mathrm{Var}(Y) = \sigma'^2$, and $\mathrm{Cov}(X, Y) = c$. Notations for the distributions are introduced:

$$P(X \leq x, Y = m) = F(x, m), \qquad m \geq 0,$$

$$P(X \leq x) = F(x) = \sum_{m=0}^{\infty} F(x, m), \qquad x \geq 0.$$

We define the marked renewal process as $N = \{N(t), t \geq 0\}$, in which $S_n = X_1 + \cdots + X_n$ is the time of the nth event, $S_{N(t)} \leq t < S_{N(t)+1}$ defines the renewal process, and we let $T_{N(t)} = Y_1 + \cdots + Y_{N(t)}$ define the cumulative process. In this paper the explicit formulas for the expectations and covariance matrix of the process $\{N, T_N\} = \{(N(t), T_{N(t)}), t \geq 0\}$ are estimated. We complete the results of [2] where some formulas for renewal processes are established.

Let us introduce the generating function

$$\Psi(t, u, v) = \mathrm{E}(u^{N(t)} v^{T_{N(t)}}), \qquad |u| \leq 1, \quad |v| \leq 1,$$

and its Laplace–Stieltjes transform

$$\Psi^*(s, u, v) = \mathcal{L}\{\Psi(t, u, v), s\} = \int_0^{\infty} e^{-st} \Psi(dt, u, v), \qquad \mathrm{Re}(s) > 0.$$

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Theorem 1. The Laplace–Stieltjes transform of the generating function of $\{N, T_N\}$ is of the form

$$\Psi^*(s,u,v) = \frac{1-f^*(s)}{1-uf^*(s,v)}, \qquad |u| \le 1, \quad |v| \le 1, \quad \mathrm{Re}(s) > 0, \tag{1}$$

where $f^*(s,v) = \mathrm{E}(e^{-sX}v^Y)$ and $f^*(s) = \mathrm{E}(e^{-sX}) = f^*(s,1)$.

Proof. We have

$$(N(t), T_{N(t)}) \stackrel{\mathrm{d}}{=} \mathbf{1}_{\{X \le t\}}(1 + N'(t-X), Y + T'_{N'(t-X)}),$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution, $\{N', T'_{N'}\}$ is the probabilistic copy of $\{N, T_N\}$, and X and $\{N', T'_{N'}\}$ are mutually independent. Hence

$$\begin{aligned}\Psi(t,u,v) &= \int_t^\infty F(dx) + \int_0^t \sum_{m=0}^\infty \mathrm{E}\Big(u^{1+N(t-x)}v^{Y+T_{N(t-x)}}\Big)F(dx,m)\\ &= 1 - F(t) + u\int_0^t \Psi(t-x,u,v)\Big(\sum_{m=0}^\infty v^m F(dx,m)\Big).\end{aligned}$$

Passing to the Laplace–Stieltjes transforms we have

$$\Psi^*(s,u,v) = 1 - f^*(s) + u\Psi^*(s,u,v)f^*(s,v),$$

hence we obtain equation (1). ■

2. Asymptotical estimation

Theorem 1 implies the Laplace–Stieltjes transform of the generating functions of the boundary processes. By using the Cox method [1] the asymptotical estimation of the moments may be established.

Proposition 1. The Laplace–Stieltjes transform of the generating function of N is of the form

$$\Psi_1^*(s,u) = \frac{1-f^*(s)}{1-uf^*(s)}, \qquad |u| < 1, \quad \mathrm{Re}(s) > 0.$$

The Laplace–Stieltjes transform of the generating function of T_N is of the form

$$\Psi_2^*(s,v) = \frac{1-f^*(s)}{1-f^*(s,v)}, \qquad |v| < 1, \quad \mathrm{Re}(s) > 0. \tag{2}$$

Hence, if $0 < \sigma^2 < \infty$, then under $t \to \infty$

$$\mathrm{E}(N(t)) = \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 + o(1) \tag{3}$$

$$\operatorname{Var}(N(t)) = \frac{\sigma^2 t}{\mu_1^3} + o(t) \tag{4}$$

$$\operatorname{E}(T_{N(t)}) = \frac{\nu_1 t}{\mu_1} + \frac{1}{2\mu_1^2}\Big(\nu_1\mu_2 - 2\mu_1 \operatorname{E}(XY)\Big) + o(1) \tag{5}$$

$$\begin{aligned}\operatorname{Var}(T_{N(t)}) &= \frac{t}{\mu_1^3}\Big(\nu_1^2\sigma^2 - 2\mu_1\nu_1 c + \mu_1^2\sigma'^2\Big)\\ &= \frac{t}{\mu_1^3}\operatorname{Var}(\nu_1 X - \mu_1 Y) + o(t)\end{aligned} \tag{6}$$

$$\begin{aligned}\operatorname{Cov}(N(t), T_{N(t)}) &= \frac{t}{\mu_1^3}\Big(\nu_1\sigma^2 - \mu_1 c\Big) + o(t)\\ &= \frac{t}{\mu_1^3}\operatorname{Cov}(\nu_1 X - \mu_1 Y, X) + o(t).\end{aligned} \tag{7}$$

The proof of Proposition 1 is given in the appendix.

3. Asymptotical normality

Assuming $0 < \sigma^2 < \infty$, $\sigma'^2 < \infty$, the asymptotical normality of the renewal process and the cumulative process are proved in [4]. Also the asymptotical normality of $\{N, T_N\}$ is expected.

Example. Let us suppose that X, Y have the three-point probability distributions: $P(X = 1, Y = 0) = 1/2$, $P(X = 2, Y = 1) = 1/4$, and $P(X = 2, Y = 2) = 1/4$. Then we have $\mu_1 = 3/2$, $\nu_1 = 3/4$, $\sigma^2 = 1/4$, $\sigma'^2 = 11/16$, and $c = 3/8$.

In this case $T_{N(t)} = t - N(t) + B(1/2, t - N(t))$ for $t - S_{N(t)} = 0$, and $T_{N(t)} = t - N(t) - 1 + B(1/2, t - N(t) - 1)$ for $t - S_{N(t)} = 1$, where $B(1/2, M) = \sum_{i=1}^{M} \delta_i$ for a random M and δ_i, $i \geq 1$, are the Bernoulli 0-1 trials with $P(\delta_i = 1) = 1/2$, where δ_i, $i \geq 1$, and M are mutually independent.

From the renewal theory under $t \to \infty$ we have $(1/t)N(t) \to 2/3$ (in probability) and the asymptotical normality: $(N(t) - 2t/3)/\sqrt{2t/27} \overset{\mathrm{d}}{\to} U \sim N(0, 1)$. From the central limit theorem a suitably standardized $B(1/2, t)$ is asymptotically normal. Consider, for simplicity, $\hat{T}_N(t) = t - N(t) + B(1/2, t - N(t))$. Then $2(\hat{T}_{N(t)} - t/2)/\sqrt{t}$ is asymptotically $N(0, 1)$ because

$$\begin{aligned}&\frac{2}{\sqrt{t}}\Big(t - N(t) + B(1/2, t - N(t)) - t/2\Big)\\ &\quad = \frac{B(1/2, t - N(t)) - (t - N(t))/2}{\sqrt{t - N(t)}/2}\;\frac{\sqrt{t - N(t)}/2}{\sqrt{t}} - 3\;\frac{N(t) - 2t/3}{\sqrt{t}}\\ &\quad \overset{\mathrm{d}}{\to} \frac{1}{\sqrt{3}}U - \sqrt{\frac{2}{3}}V \overset{\mathrm{d}}{=} Z, \qquad t \to \infty,\end{aligned}$$

where U, V, and Z are $N(0, 1)$, and U and V are mutually independent.

Theorem 2. If $0 < \sigma^2 < \infty$, then under $t \to \infty$

$$\left(\left(N(t) - \frac{t}{\mu_1}\right)\left(\frac{\sigma^2 t}{\mu_1^3}\right)^{-1/2}, \left(T_{N(t)} - \frac{\nu_1 t}{\mu_1}\right)t^{-1/2}\right)$$
$$\stackrel{\mathrm{d}}{\to} \left(U, \sigma\sqrt{\mu_1^{-3}}(\nu_1 - a\mu_1)U + \sigma_0\sqrt{\mu_1^{-1}}V\right),$$

where U and V are $N(0,1)$ and mutually independent.

Proof. For N it is obvious (see [4]) that

$$\left(N(t) - \frac{t}{\mu_1}\right)\left(\frac{\sigma^2 t}{\mu_1^3}\right)^{-1/2} \stackrel{\mathrm{d}}{\to} U \sim N(0,1), \qquad t \to \infty.$$

Let $Y_n = \nu_1 + a(X_n - \mu_1) + \epsilon_n$, where $a = c/\sigma^2$ and $\epsilon = Y - \nu_1 - a(X - \mu_1)$. Then $\mathrm{E}(\epsilon) = 0$, $\mathrm{Var}(\epsilon) = \sigma_0^2 = \sigma'^2 - c^2/\sigma^2$, and ϵ is not correlated with X. Let $W = \{N(t) = u(t) \stackrel{\mathrm{df}}{=} t/\mu_1 + u\sigma\sqrt{t/\mu_1^3}\}$, $-\infty < u < \infty$. We prove that $T_{N(t)}|W$ is normal with the expected value linear of u and the common variance. Recall that $\sum_{i=1}^{N(t)} X_i = S_{N(t)} = t - \gamma(t)$, where $\gamma(t)$ is the residual time. If $0 < \sigma^2 < \infty$, then $\gamma(t)|W \stackrel{\mathrm{d}}{\to} \gamma$ holds under $t \to \infty$. Further we have

$$\sum_{i=1}^{N(t)} Y_i\Big|W = \left(\left(\frac{t}{\mu_1} + \sigma\sqrt{t/\mu_1^3}u\right)\nu_1 - a\left(\gamma(t) + \mu_1\sigma\sqrt{t/\mu_1^3}u\right) + \sum_{i=1}^{N(t)} \epsilon_i\right)\Big|W.$$

Hence

$$\frac{1}{\sqrt{t}}\left(T_{N(t)} - \frac{\nu_1 t}{\mu_1}\right)\Big|W = \frac{1}{\sqrt{t}}\left(\sigma\sqrt{t/\mu_1^3}\nu_1 u - a(\gamma(t) + \sigma\sqrt{t/\mu_1^3}\mu_1 u) + \sum_{i=1}^{N(t)} \epsilon_i\right)\Big|W$$
$$\stackrel{\mathrm{d}}{\to} \sigma\sqrt{\mu_1^{-3}}(\nu_1 - a\mu_1)u + \sigma_0\sqrt{\mu_1^{-1}}V,$$

where $V \sim N(0,1)$. After transformation we obtain

$$\left(T_{N(t)} - \frac{\nu_1 t}{\mu_1}\right)/\sqrt{t} \stackrel{\mathrm{d}}{\to} \sigma\sqrt{\mu_1^{-3}}(\nu_1 - a\mu_1)U + \sigma_0\sqrt{\mu_1^{-1}}V, \qquad t \to \infty,$$

where U and $V \sim N(0,1)$ and are mutually independent. ■

Remark.

From Theorem 1 it follows that

$$\mathrm{Var}(T_{N(t)}) = \frac{t}{\mu_1}\left(\sigma^2\mu_1^{-2}(\nu_1 - a\mu_1)^2 + \sigma_0^2\right) + o(t)$$
$$= t\mu_1^{-3}\mathrm{Var}(\nu_1 X - \mu_1 Y) + o(t),$$

and equation (6) is satisfied.

4. Multivariate extension

Let us suppose that r persons together take some action in a random time X and receive a random vector $\mathbf{Y} = (Y^{(j)}, 1 \le j \le r)$. If actions are repeated one-by-one, then we define the cumulate process

$$\mathbf{T}_{N(t)} = (T^{(j)}_{N(t)}, 1 \le j \le r) = (\mathbf{Y}_1 + \cdots + \mathbf{Y}_{N(t)}), \qquad t \ge 0.$$

We assume that the random variables in $\mathbf{Y}$ are mutually correlated. The interesting problem is the explicit asymptotical form of the correlation matrix into $\mathbf{T}_{N(t)}$. The expected values and variances of $\{N, \mathbf{T}_N\}$ are given in Proposition 1.

Theorem 3. Let $(X_n, \mathbf{Y}_n)$ be independent equidistributed random vectors and let N be the renewal process generated by X_n with $n \ge 1$. If $0 < \sigma^2 = \mathrm{Var}(X) < \infty$, then $((T^{(j)}_{N(t)} - (t/\mu_1)\mathrm{E}(Y^{(j)}))/\sqrt{\sigma^2 t/\mu_1^3}, 1 \le j \le r)$ is asymptotically multivariate normal with the expected vector zero and the covariance matrix

$$\begin{aligned} c_{ij} &= \mathrm{Cov}(T^{(i)}_{N(t)}, T^{(j)}_{N(t)}) \\ &= \frac{\sigma^2 t}{\mu_1^3}\mathrm{Cov}(\nu_1^{(i)} X - \mu_1 Y^{(i)}, \nu_1^{(j)} X - \mu_1 Y^{(j)}) + o(t), \quad 1 \le i, j \le r, \quad t \to \infty. \end{aligned}$$

Proof. Similar to the proof of Theorem 2 we define $\epsilon^{(k)} = Y^{(k)} - \nu_1^{(k)} - a^{(k)}(X - \mu_1)$, where $a^{(k)} = \mathrm{Cov}(X, Y^{(k)})/\sigma^2$, for $k = i, j$. Then we obtain

$$\begin{aligned} &\left(\left(T^{(i)}_{N(t)} - \frac{\nu_1^{(i)} t}{\mu_1}\right)/\sqrt{t}, \left(T^{(j)}_{N(t)} - \frac{\nu_1^{(j)} t}{\mu_1}\right)/\sqrt{t}\right) \\ &\quad \overset{\mathrm{d}}{\to} \Big(\sigma\sqrt{\mu_1^{-3}}(\nu_1^{(i)} - a^{(i)}\mu_1)U + \sigma_0^{(i)}\sqrt{\mu_1^{-1}}\,V^{(i)}, \\ &\qquad \sigma\sqrt{\mu_1^{-3}}(\nu_1^{(j)} - a^{(j)}\mu_1)U + \sigma_0^{(j)}\sqrt{\mu_1^{-1}}\,V^{(j)}\Big), \qquad t \to \infty, \end{aligned}$$

where $V^{(i)} \sim N(0,1)$, $V^{(j)} \sim N(0,1)$, $\mathrm{Cov}(V^{(i)}, V^{(j)}) = \mathrm{Cov}(\epsilon^{(i)}, \epsilon^{(j)})$, $(\sigma_0^{(k)})^2 = (\sigma^{(k)})^2 - \mathrm{Cov}^2(X, Y^{(k)})/\sigma^2$, and $\mathrm{Cov}(U, V^{(k)}) = 0$, with $k = i, j$.

Now we have

$$\begin{aligned} c_{ij} &= \frac{t}{\mu_1^3}\mathrm{Cov}\Big(\sigma(\nu_1^{(i)} - a^{(i)}\mu_1)U + \sigma_0^{(i)}\mu_1 V^{(i)}, \\ &\qquad\qquad \sigma(\nu_1^{(j)} - a^{(j)}\mu_1)U + \sigma_0^{(j)}\mu_1 V^{(j)}\Big) + o(t) \\ &= \frac{t}{\mu_1^3}\Big(\nu_1^{(i)} - a^{(i)}\mu_1)(\nu_1^{(j)} - a^{(j)}\mu_1)\sigma^2 + \mu_1^2\sigma_0^{(i)}\sigma_0^{(j)}\mathrm{Corr}(V^{(i)}, V^{(j)})\Big) + o(t). \end{aligned}$$

Because

$$\sigma_0^{(i)}\sigma_0^{(j)}\mathrm{Corr}(V^{(i)}, V^{(j)}) = \mathrm{Cov}(\epsilon^{(i)}, \epsilon^{(j)}) = \mathrm{Cov}(Y^{(i)}, Y^{(j)}) - a^{(i)}a^{(j)}\sigma^2$$

we have

$$c_{ij} = \frac{t}{\mu_1^3}\Big(\sigma^2 \nu_1^{(i)} \nu_1^{(j)} - \sigma^2 \mu_1 a^{(i)} \nu_1^{(j)} - \sigma^2 \mu_1 a^{(j)} \nu_1^{(i)} + \mu_1^2 \mathrm{Cov}(Y^{(i)}, Y^{(j)})\Big) + o(t).$$

After transformation we obtain Theorem 3. ■

Appendix

Proof of Proposition 1. For equations (3) and (4) see [4], for equation (6) see [3]. We verify equations (5) through (7) by the Cox method [1]. Let us introduce the auxiliary functions:

$$f^*(s) = \mathrm{E}(e^{-sX}) = f^*(s,1)$$
$$m_1^*(s) = \mathrm{E}(e^{-sX}Y) = \frac{\partial}{\partial v} f^*(s,v)\Big|_{v=1}$$
$$m_2^*(s) = \mathrm{E}(e^{-sX}Y(Y-1)) = \frac{\partial^2}{\partial v^2} f^*(s,v)\Big|_{v=1}.$$

From equations (1) and (2) we obtain the Laplace–Stieltjes transforms

$$\mathcal{L}\{\mathrm{E}(T_{N(t)}(T_{N(t)}-1)), s\} = 2\Big(\frac{m_1^*(s)}{1-f^*(s)}\Big)^2 + \frac{m_2^*(s)}{1-f^*(s)}$$
$$\mathcal{L}\{\mathrm{E}(N(t)T_{N(t)}), s\} = \frac{m_1^*(s)}{1-f^*(s)}\Big(\frac{2f^*(s)}{1-f^*(s)} + 1\Big).$$

Under $s \to 0$ we have

$$f^*(s) = 1 - \mu_1 s + \frac{1}{2}\mu_2 s^2 + o(s^2)$$
$$\frac{m_1^*(s)}{1-f^*(s)} = \frac{\nu_1}{s\mu_1} + \frac{1}{2\mu_1^2}\Big(\nu_1\mu_2 - 2\mu_1 \mathrm{E}(XY)\Big) + o(1)$$
$$\Big(\frac{m_1^*(s)}{1-f^*(s)}\Big)^2 = \Big(\frac{\nu_1}{s\mu_1}\Big)^2 + \frac{\nu_1}{s\mu_1^3}\Big(\nu_1\mu_2 - 2\mu_1 \mathrm{E}(XY)\Big) + o\Big(\frac{1}{s}\Big)$$
$$\frac{m_2^*(s)}{1-f^*(s)} = \frac{1}{s\mu_1}\mathrm{E}(Y(Y-1)) + o\Big(\frac{1}{s}\Big).$$

Hence we obtain the estimations

$$\mathcal{L}\{\mathrm{E}(T_{N(t)}(T_{N(t)}-1)), s\}$$
$$= \frac{2}{s^2}\Big(\frac{\nu_1}{\mu_1}\Big)^2 + \frac{2\nu_1}{s\mu_1^3}(\nu_1\mu_2 - 2\mu_1\mathrm{E}(XY)) + \frac{1}{s\mu_1}\mathrm{E}(Y(Y-1)) + o\Big(\frac{1}{s}\Big)$$
$$\mathcal{L}\{\mathrm{E}(N(t)T_{N(t)}), s\}$$
$$= \frac{2\nu_1}{s^2\mu_1^2} + \frac{1}{s\mu_1^3}\Big(2\nu_1\mu_2 - \nu_1\mu_1^2 - 2\mu_1\mathrm{E}(XY)\Big) + o\Big(\frac{1}{s}\Big).$$

Reversing formally the transforms we obtain the estimations under $t \to \infty$:

$$\mathrm{E}(T_{N(t)}) = \frac{\nu_1 t}{\mu_1} + \frac{1}{2\mu_1^2}\Big(\nu_1\mu_2 - 2\mu_1 \mathrm{E}(XY)\Big) + o(1)$$

$$\mathrm{E}(T_{N(t)}(T_{N(t)} - 1)) = \Big(\frac{\nu_1}{\mu_1}\Big)^2 t^2 + \frac{t}{\mu_1}\Big(2\frac{\nu_1}{\mu_1^2}(\nu_1\mu_2 - 2\mu_1\mathrm{E}(XY) + \mathrm{E}(Y(Y-1))\Big) + o(t)$$

$$\mathrm{E}(N(t)T_{N(t)}) = \frac{\nu_1}{\mu_1^2}t^2 + \frac{t}{\mu_1^3}\Big(2\nu_1\mu_2 - \nu_1\mu_1^2 - 2\mu_1\mathrm{E}(XY)\Big) + o(t).$$

From this we may obtain Proposition 1. ■

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