Hexagonal Lattice Systems Based on Rotationally Invariant Constraints

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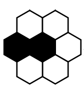
In this paper, periodic and aperiodic patterns on hexagonal lattices are developed and studied using a given set of constraints. On exhaustively enumerating all possible two-color patterns for up to a combination of four constraints, 106 distinct periodic tilings are obtained. Only one set of four constraints out of a total of 24 157 possible sets generated a pattern that is aperiodic in nature.

1. Introduction

The appearance of periodic patterns on a flat surface when subjected to constraints of various kinds is what is known as constraint tiling [1, 2]. When the applied constraints give rise to a pattern that shows no periodicity, the pattern is said to exhibit a non-repetitive behavior.

In this paper, the technique of constraint tiling has been applied to hexagonal lattices [3]. A single hex cell, along with its six nearest neighbors, forms a constraint. In this single constraint

,

the color of the middle hex cell is determined by the color of its six nearest neighbors. Starting from the top left in a clockwise direction, the colors of the surrounding hex cells are $\{0, 0, 0, 0, 0, 1\}$, where 0 is white and 1 is black. In this case, the color of the middle hex cell is black. However, it is seen that this constraint with rule number 16 777 216 is unable to tile a plane either periodically or aperiodically.

Given a constraint or a set of constraints, we use the backtracking algorithm to determine whether the constraint or the given set of constraints will form a pattern that is either periodic or aperiodic.

For each iteration, the backtracking algorithm starts with a single white hex cell as the initial condition. It then scans its six nearest neighbors. If the pattern of the six nearest neighbors matches any one of the given constraints, where the given constraints are allowed to rotate, a single white hex cell is added to the already existing hexagon

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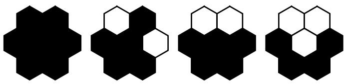
in a clockwise spiral. This process continues until the pattern of the six nearest neighbors fails to satisfy either of the constraints. Backtracking occurs at this stage, and the single white hex cell at this position is replaced with a black hex cell. This process continues and is a very useful way of determining whether a given set of constraints will form a pattern that may be either periodic or aperiodic.

With these constraints

,

the pattern shown in Figure 1 is seen to emerge.

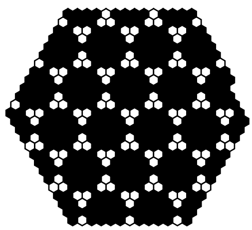


Figure 1. The periodic tiling generated on applying the four constraints shown.

In our endeavor to find and analyze aperiodic patterns, which is motivated by Stephen Wolfram's study of simple systems based on constraints [4], the backtracking algorithm is first applied to square lattices. The resulting graphic is then transformed into a hexagonal lattice.

The square lattices used here are very much like a Moore neighborhood after eliminating its two extreme sides.

In the Moore neighborhood, X is the innermost cell, whose color is to be determined. If we label the nearest neighbors as northwest (NW), north (N), northeast (NE), east (E), southeast (SE), south (S), southwest (SW), and west (W), the Moore neighborhood will look something like Table 1. The color of X depends on the colors of all its eight nearest neighbors.

NW	N	NE
\	|	|
W	-X-	E
|	|	\
SW	S	SE

Table 1. The Moore neighborhood.

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If we further replace the NW and SE sides of the already existing Moore neighborhood with _, where the color of _ can be either 0 or 1 (black or white), the Moore neighborhood changes as shown in Table 2.

If the above-mentioned procedure is then applied to hexagonal lattices where i, j indicates the position of the innermost hex cell whose color is to be determined, along with a function F that determines the color of the hex cell at that particular position by taking into consideration the colors of the six nearest neighbors, then

$$a_{(i,j)} = F\left[a_{(i,j+1)},\ a_{(i+1,j+1)},\ a_{(i+1,j)},\ a_{(i,j-1)},\ a_{(i-1,j-1)},\ a_{(i-1,j)}\right].$$

```
 _    N   NE
 \    |   |
 W  - X - E
 |    |   \
 SW   S   _
```

Table 2. The six nearest neighbor constraints.

2. Rule Space

For two colors, there can be a total of

340282366920938463463374607431768211456

or $\left(2^{2^7}\right)$ constraints on a single hexagonal lattice. After rotating all 128 constraints and taking the union of the resulting constraints, the rule space is left with the 28 constraints shown in Figure 2.

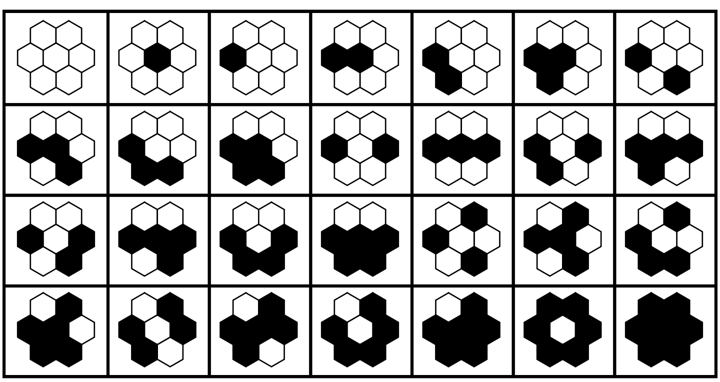

Figure 2. The set of all 28 possible constraints in the rule space.

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The numbering scheme for any rule is developed in the following way. Initially, there exists a list of 28 zeros. For any given constraint or group of constraints, the position of the given constraints/constraint in the template shown in Figure 2 is noted, and the zeros at those positions in the list are changed to ones. The resulting binary number converted to its decimal form gives the rule number.

For example, with the constraints

 and ,

the corresponding rule number will be 40.

3. Two-Color Periodic Tilings

Out of 28 constraints shown in Figure 2, 26 of them fail to generate any periodic tiling. The remaining two shown in Figure 3 generate tilings that are either all black or all white.

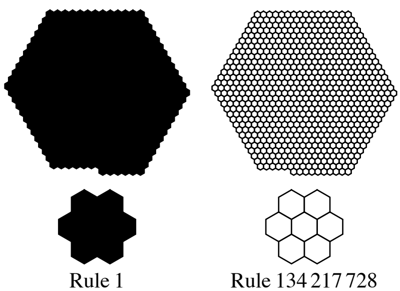

Figure 3. On taking all possible combinations of two constraints from the two-color rule space, there was a total of 378 combinations. Out of these, only 64 could tile the plane periodically. The remaining 314 combinations of two constraints were unable to tile the plane either periodically or aperiodically.

On enumerating all 64 possible combinations of two constraints, only the 11 distinct patterns shown in Figure 4 were obtained.

On taking all possible combinations of three constraints from the two-color rule space, there was a total of 3276 combinations. Out of these, only 968 could tile the plane periodically. The remaining 2308 combinations of three constraints were unable to tile the plane either periodically or aperiodically. On enumerating all 968 possible combinations of three constraints, the 20 distinct periodic tilings shown in Figure 5 were obtained, along with the complete set of periodic tilings obtained for two constraints as shown in Figure 4.

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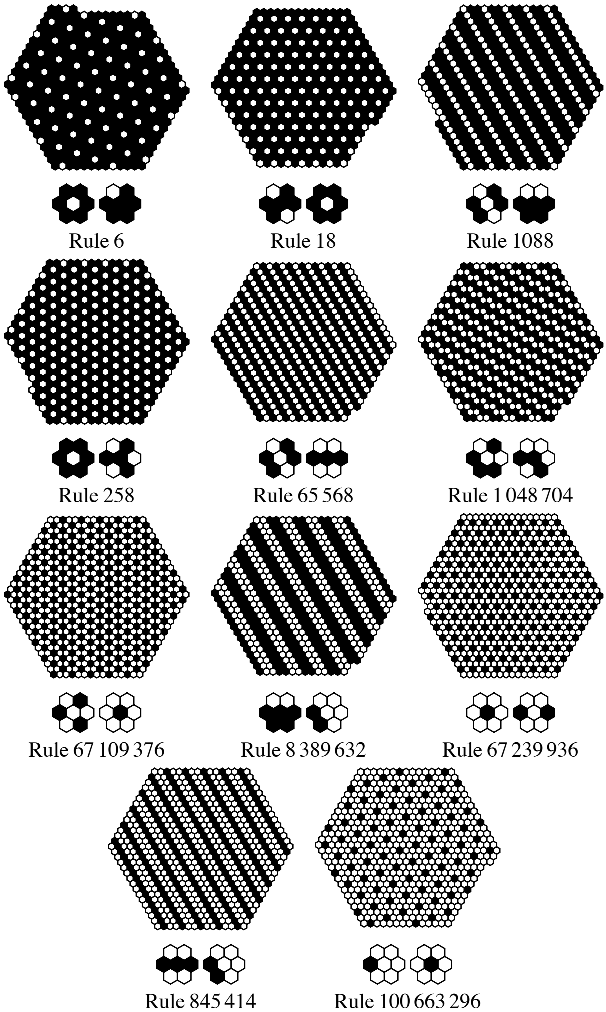

Figure 4. These 11 periodic tilings, along with the tilings that generate an all-white or all-black pattern, are the complete collection of tilings that are needed to satisfy two constraints. If none of these periodic tilings satisfy a particular pair of two constraints, then it follows that no tiling, either periodic or aperiodic, will satisfy that pair of two constraints. Here the backtracking algorithm uses a single white hex cell as the initial condition.

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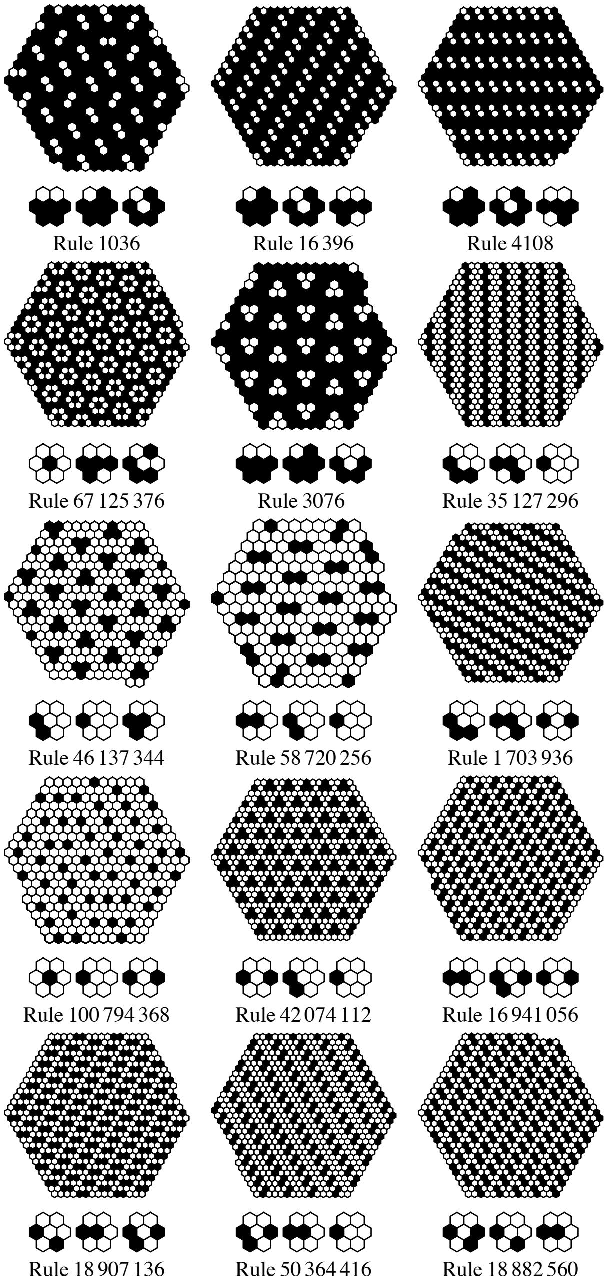

Figure 5. (*continues*).

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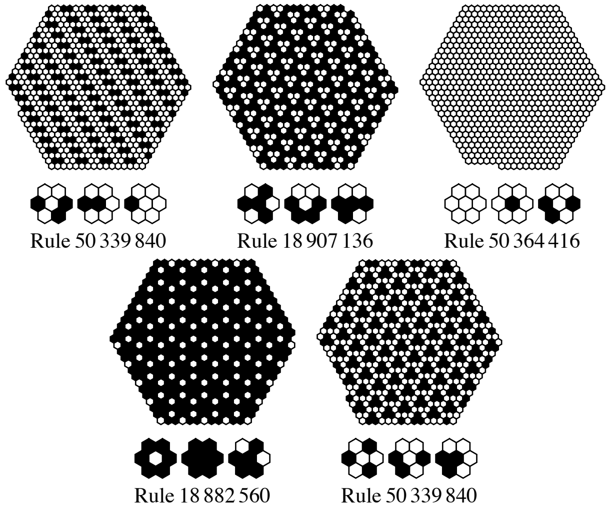

Figure 5. These 20 periodic tilings, together with the previously shown collection of one- and two-constraint tilings, are complete for three constraints. If none of these periodic tilings satisfy a particular combination of three constraints, then it follows that no tiling, either periodic or aperiodic, will satisfy that combination of three constraints. The initial condition used is a single white cell.

On taking all possible combinations of four constraints from the two-color rule space, there was a total of 20 475 combinations. Out of these, only 4834 could tile the plane periodically. The remaining 15 641 combinations of four constraints were unable to tile the plane either periodically or aperiodically. On enumerating all 4834 possible combinations of four constraints, the 73 distinct patterns shown in Figure 6 were obtained.

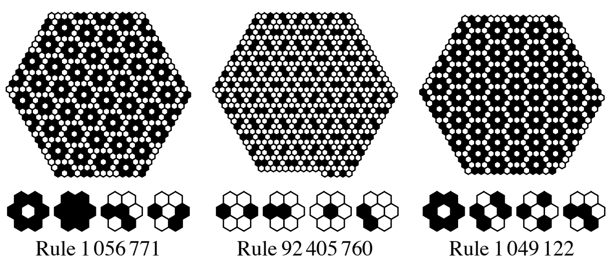

Figure 6. (*continues*).

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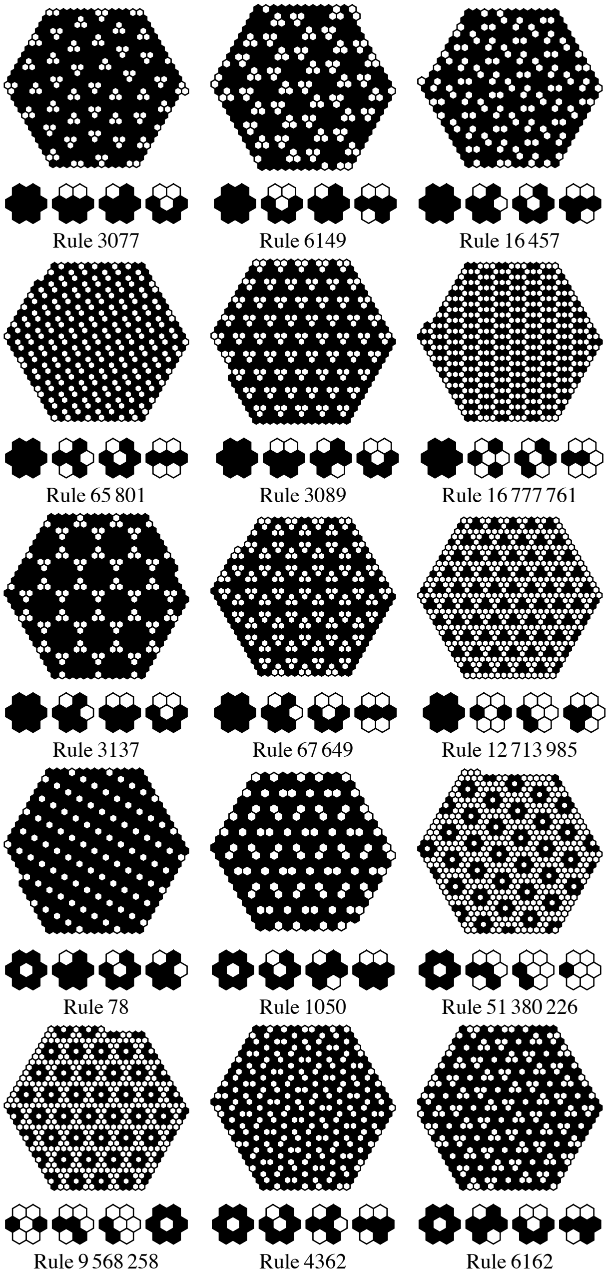

Figure 6. (*continues*).

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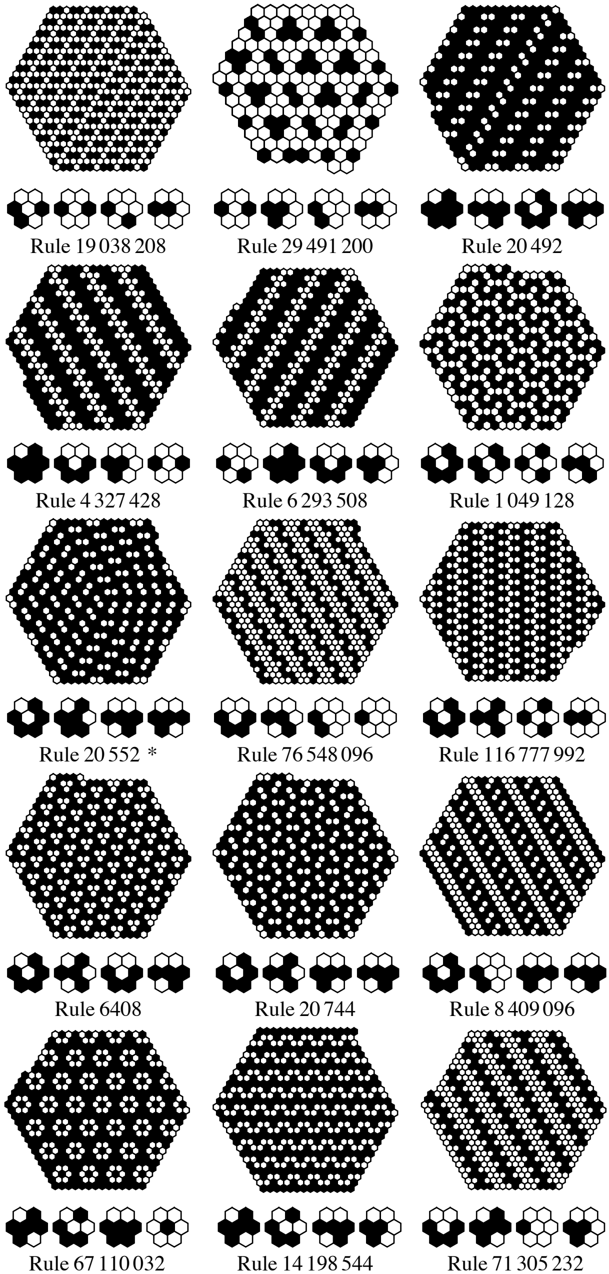

Figure 6. (*continues*).

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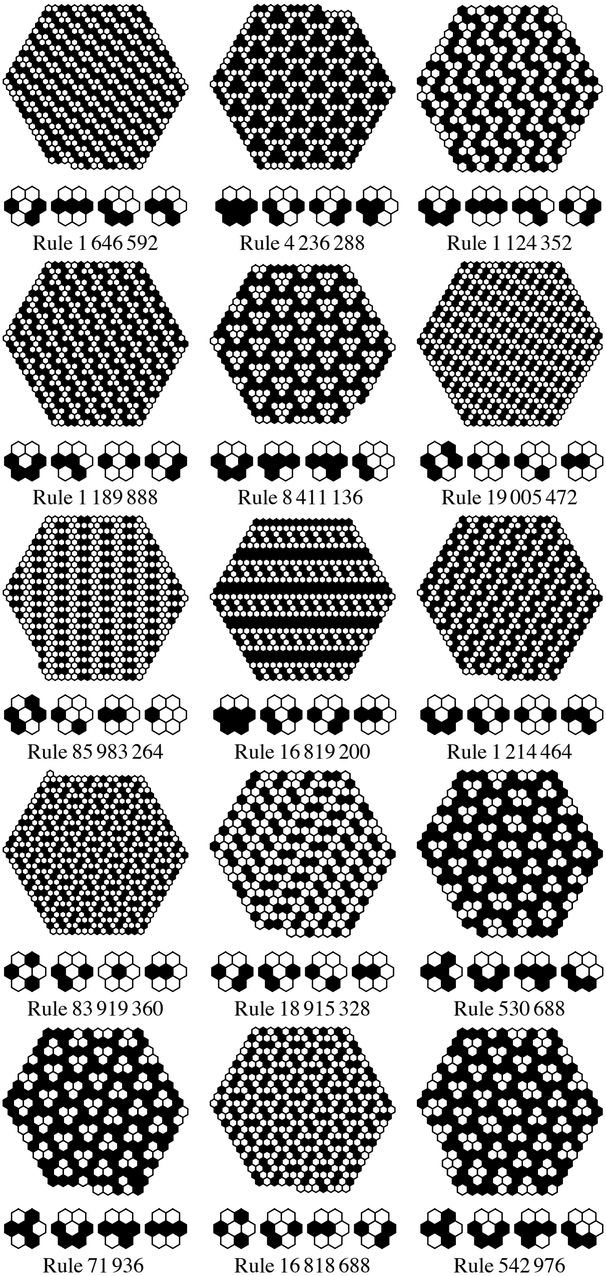

Figure 6. (*continues*).

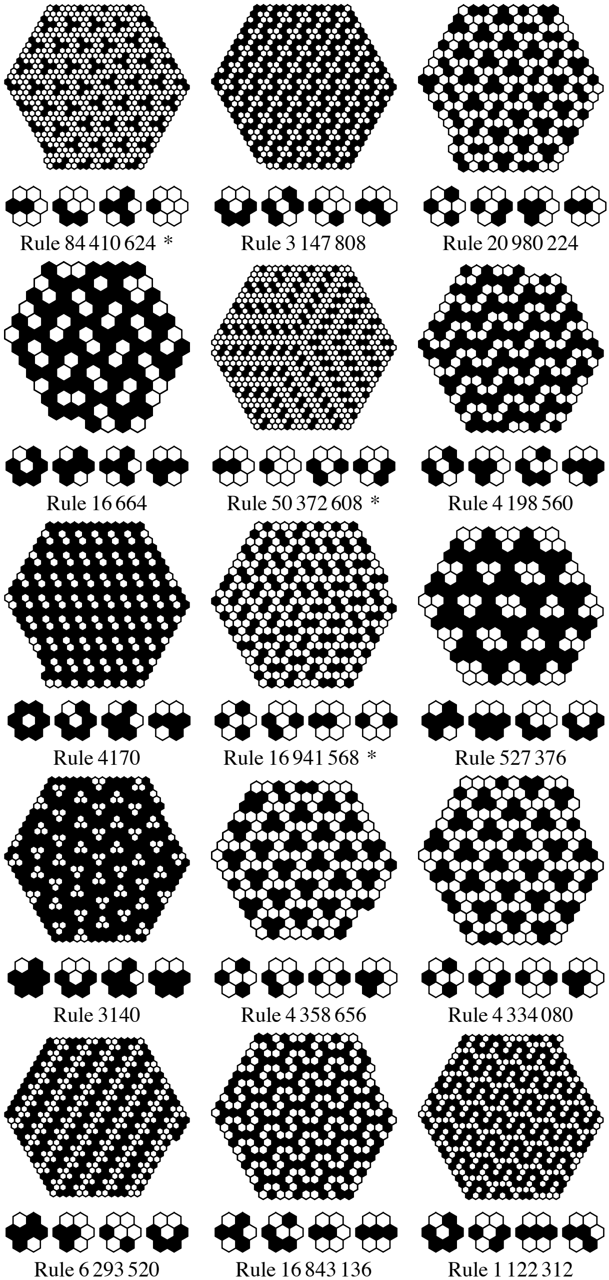

Figure 6. (*continues*).

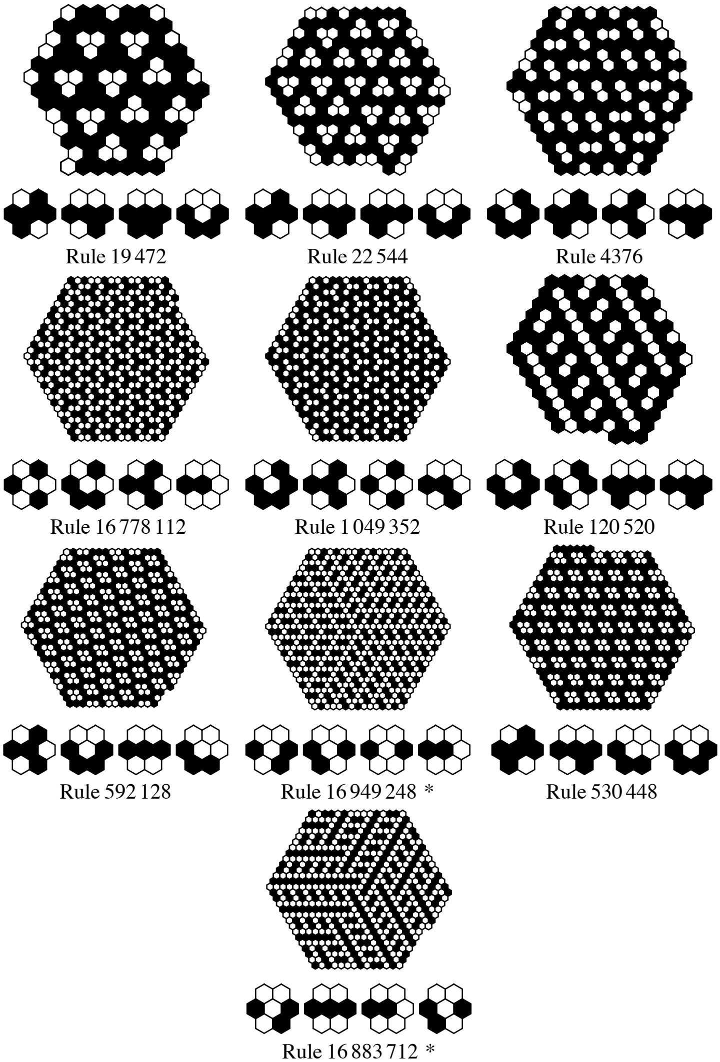

Figure 6. This collection of 73 periodic tilings, together with the previously shown collection of one-, two-, and three-constraint tilings, is complete for four constraints. If none of these periodic tilings satisfy a particular combination of four constraints, then it follows that no tiling, either periodic or aperiodic, will satisfy that combination of four constraints. Here the backtracking algorithm uses a single white hex cell as the initial condition.

The periodic tilings in Figure 6 marked with a $*$ are divided into separate regions and are repetitive in that particular section. Let us take the example of rule 16883712 in a square lattice shown in Figure 7.

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When looking closely at the edges of the hexagonal lattices generated after applying a given set of constraints, irregularities can be noticed at the boundaries for certain tilings. These irregularities arise from the fact that the backtracking algorithm used to generate patterns has been iterated for a certain number of steps and continues to be an ongoing process. When the number of steps is increased, the irregularities disappear from the old lattice sites and reappear at the newly generated boundary, owing to the action of the backtracking algorithm, which retraces its steps when a particular condition fails.

For further information on periodic tilings, see [1, 2, 5, 6].

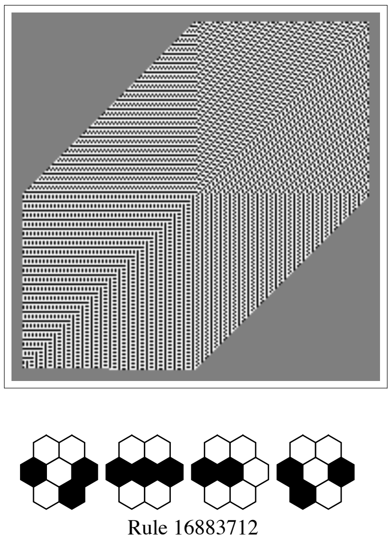
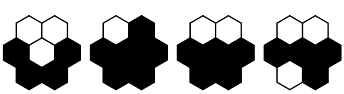


Figure 7. This pattern with rule 16 883 712 exhibits different repetitive behavior in different sections.

4. An Aperiodic Pattern

With the constraints

,

the aperiodic pattern that emerges is shown in Figure 8.

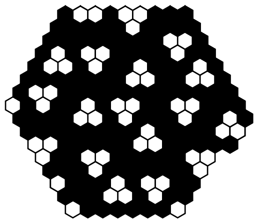


Figure 8. The only aperiodic pattern generated after evaluating all possible combinations up to four constraints.

After running the backtracking algorithm for 900 000 times, no periodicity is noticed in the pattern, as shown in Figure 9.

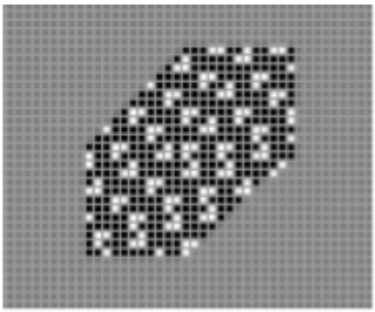


Figure 9. The four constraints of rule 7172 force this pattern that exhibits a nonrepetitive behavior [4]. On increasing the number of steps, the pattern continues to grow, although at a very slow rate. Gray is used to indicate cells whose colors have not been determined yet. The initial condition uses a single white hex cell.

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