Study of All the Periods of a Neuronal Recurrence Equation

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We characterize the structure of the periods of a neuronal recurrence equation. First, we give a characterization of k-chains in 0–1 periodic sequences. Second, we characterize the periods of all cycles of some neuronal recurrence equation. Third, we explain how these results can be used to deduce the existence of the generalized period-halving bifurcation.

1. Introduction

The human brain can be viewed as a set of interconnected neurons. Caianiello [1, 2] suggested modeling the brain using the following threshold automata network:

$$x_i(t+1) = \mathbf{1}\left(\sum_{j=1}^{n}\sum_{s=1}^{k} a_{ij}(s)x_j(t+1-s) - \theta_i\right) 1 \le i \le n,\ \ t \ge k-1 \tag{1}$$

where:

- $x_j(t+1-s)$ is the state of the neuron j at time $t+1-s$.
- $a_{ij}(s)$ represents the influence of the neuron j at time $t+1-s$ on the neuron i at time $t+1$.
- θ_i is the threshold of the excitation of the neuron i.
- $\sum_{j=1}^{n}\sum_{s=1}^{k} a_{ij}(s)x_j(t+1-s)$ is the potential of the neuron i at time t.
- n is the number of the neurons of the network.
- k is the size of the memory.
- $\mathbf{1}[u] = 0$ if $u < 0$, and $\mathbf{1}[u] = 1$ if $u \ge 0$.

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The dynamics of this model have been studied in some particular cases:

1. In equation (1), when $k = 1$, we obtain the following equation:

$$x_i(t+1) = \mathbf{1}\left(\sum_{j=1}^{n} a_{ij} x_j(t) - \theta_i\right) 1 \leq i \leq n, \tag{2}$$

which models the dynamic behavior of n interconnected neurons of memory size 1. These networks were introduced by McCulloch and Pitts [3] and are quite powerful.

2. In equation (1), when $n = 1$, we obtain the following equation:

$$x(n) = \mathbf{1}\left(\sum_{j=1}^{k} a_j x(n-j) - \theta\right), \tag{3}$$

introduced by Caianiello and de Luca [4], which models the dynamic behavior of a single neuron with a memory, and which does not interact with other neurons.

Neural networks are usually implemented by using electronic components or are simulated by software on a digital computer. One way in which the collective properties of a neural network may be used to implement a computational task is through the *energy minimization* concept. The Hopfield network is a well-known example of such an approach. It has attracted wide attention in literature as a *content-addressable memory* [5].

Caianiello networks have been studied by Goles [6] and Ndoundam [7]. Many studies have been devoted to McCulloch and Pitts's neural networks [5, 8–13]. Matamala [12] studied McCulloch and Pitts's reverberation neural networks (i.e., neural networks of McCulloch and Pitts where each state of the system, after a finite number of steps, comes back to itself, also called hypercube permutation).

Cosnard, Moumida, Goles, and St. Pierre [14] showed the following result in the case of palindromic memory:

Proposition 1. [14] If the interacting coefficients $(a_1, a_2, \ldots, a_k)$ verify

$$a_i = a_{k+1-i} \forall\, i \in \mathbb{N},\ 1 \leq i \leq k,$$

then the length of each cycle is a divisor of $k+1$.

In the case of j-palindromic memory, they also showed:

Proposition 2. [14] If the interacting coefficients $(a_1, a_2, \ldots, a_k)$ are j-palindromic, that is, verify

$$a_1 = a_2 = \cdots a_j = 0,$$
$$a_i = a_{k+j+1-i} \forall\, i \in \mathbb{N},\ j+1 \leq i \leq k,$$

then the length of each cycle is a divisor of $k+j+1$.

When the memories are geometric sequences, they showed the following results:

Proposition 3. [14] If the interacting coefficients verify $a_i = -(b^i)$ with $b \in \left]0, 1/2\right]$, then the length of each cycle is less than or equal to $k+1$.

In the case of positive geometric sequences, they showed:

Proposition 4. [14] If the interacting coefficients verify $a_i = (b^i)$ with $b \in \left]0, 1/2\right]$, then the length of each cycle is 1.

Other results have been established on neuronal recurrence equations modeling neurons with memory [8, 14–22]. From the point of view of the period:

- In [15, 18–21], the authors did not study all the cycles generated by the neuronal recurrence equation.
- In this paper, we are studying all the cycles generated by the neuronal recurrence equation $\{y(n) : n \geq 0\}$.

From the point of view of bifurcation:

- In [23], we studied the dynamics of the sequence $\{z(n) : n \geq 0\}$ from one and only one initial configuration. We characterized only one cycle of the sequence $\{z(n) : n \geq 0\}$.
- In [24], for any d $(0 \leq d \leq \rho(m) - 1)$, we studied the dynamics of the sequence $\{z(n, d) : n \geq 0\}$ from one and only one initial configuration. We characterized only one cycle of the sequence $\{z(n, d) : n \geq 0\}$.
- In this paper, for any d $(0 \leq d \leq \rho(m) - 1)$, we show how to study the dynamics of the sequence $\{z(n, d) : n \geq 0\}$ from any initial configuration. We show how to characterize the length of all cycles of the sequence $\{z(n, d) : n \geq 0\}$.

Our work is similar to the work of Matamala [12] in the sense that we study all the periods.

The paper is organized as follows: in Section 2, some previous results are presented. Section 3 presents a characterization of k-chains in 0–1 periodic sequences. Section 4 is devoted to the characterization of the period length of all the cycles. In Section 5, we study a bifurcation. Concluding remarks are stated in Section 6.

2. Previous Results

Given a finite neural network, the configuration assumed by the system at time t is ultimately periodic. As a consequence, there is an integer $p > 0$ called the period (or the length of a cycle) and another integer $T \geq 0$ called the transient length such that:

$$Y(p+T) = Y(T),$$
$$\nexists\ T' \text{and } p'(T', p') \neq (T, p) T \geq T' \text{and } p \geq p' \text{such that } Y(p'+T') = Y(T'),$$

where $Y(t) = (x(t), x(t-1), \ldots, x(t-k+2), x(t-k+1))$. The period and the transient length of the sequences generated are good measures of the complexity of the neuron. A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden "qualitative" or topological change in its behavior. A period-halving bifurcation in a dynamic system is a bifurcation in which the system switches to a new behavior with half the period of the original system from some initial configuration. A generalized period-halving bifurcation is a period-halving bifurcation from any initial configuration.

Cosnard, Tchuente, and Tindo [15] show the following lemma:

Lemma 1. [15] If there is a neuronal recurrence equation with memory length k that generates sequences of periods $p_1, p_2, \ldots, p_r$, then there is a neuronal recurrence equation with memory length kr that generates a sequence of period $r\ \operatorname{lcm}(p_1, \ldots, p_r)$.

Lemma 1 does not take into account the study of the transient length. Lemma 1 can be amended to obtain the following lemma:

Lemma 2. [24] If there is a neuronal recurrence equation with memory length k that generates a sequence $\{x^j(n) : n \geq 0\}$, $1 \leq j \leq g$ of transient length T_j and of period p_j, then there is a neuronal recurrence equation with memory length kg that generates a sequence of transient length $g \max(T_1, T_2, \ldots, T_g)$ and of period Per. Per is defined as follows:

First case: $\exists\ j$, $1 \leq j \leq g$ such that $p_j \geq 2$

$$\text{Per} = g \operatorname{lcm}(p_1, \ldots, p_g).$$

Second case: $p_j = 1$; $\forall\ j$, $1 \leq j \leq g$.

Per is a divisor of g.

Cosnard and Goles [16] studied the bifurcation in two particular cases of neuronal recurrence equations.

Case 1: Geometric coefficients and bounded memory. Cosnard and Goles completely described the structure of the bifurcation of the following equation:

$$x_{n+1} = \mathbf{1}\left(\theta - \sum_{i=0}^{k-1} b^i x_{n-i}\right)$$

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when θ varies. They showed that the associated rotation number is an increasing number of the parameter θ.

Case 2: Geometric coefficients and unbounded memory. Cosnard and Goles completely described the structure of the bifurcation of the following equation:

$$x_{n+1} = \mathbf{1}\left(\theta - \sum_{i=0}^{n} b^i x_{n-i}\right)$$

when θ varies. They showed that the associated rotation number is a Devil's staircase.

Section 3 is devoted to the study of k-chains.

3. Characterization of *k*-Chains in 0–1 Periodic Sequences

We recall the concept of k-chains in 0–1 periodic sequences [8], which is useful in the study of the limit orbits. Let $Y = (y(t) : t \in \mathbb{N})$ be a periodic sequence of zeros and ones; suppose that the period $\gamma(Y)$ (which is a priori unknown) divides T. Thus $y(t) \in \{0, 1\}$ for any $t \in \mathbb{Z}$, and $y(t) = y(t')$ when $t \equiv t' (\text{mod } T)$.

In studying period lengths, we will deal with sets invariant under translations [8], so the following notation will be useful: if $\Gamma \subset \mathbb{Z}_T$, $l \in \mathbb{Z}$, we write:

$$\Gamma + l = \{t + l (\text{mod } T) : t \in \Gamma\}.$$

Let us partition the set $\mathbb{Z}_T$ into $\Gamma^0(Y) = \{t \in \mathbb{Z}_T : y(t) = 0\}$ and $\Gamma^1(Y) = \{t \in \mathbb{Z}_T : y(t) = 1\}$, which is called the support of Y. The period of the set $\Gamma^1(Y)$ is the smallest positive number γ such that $\Gamma^1(Y) + \gamma = \Gamma^1(Y)$. The following result was established in [8]: the period of the sequence (i.e., $\gamma(Y)$) is equal to the period of $\Gamma^1(Y)$. It is shown in [8] that:

$$\gamma(Y) \text{ divides } k \text{ if and only if } \Gamma^1(Y) + k = \Gamma^1(Y).$$

Now let us define k-chains (for $k \geq 1$) contained in the support $\Gamma^1(Y)$. A subset $C \subset \Gamma^1(Y)$ is called a k-chain if and only if it is of the form $C = \{t + kl (\text{mod } T) : 0 \leq l \leq s - 1\}$ for some $s \geq 1$. So a k-chain is a subset $C = \{t + kl \in \mathbb{Z}_T : 0 \leq l \leq s - 1\}$ such that $y(t') = 1$ for any $t' \in C$.

We characterize the 0–1 sequence, which contains two different chains.

Lemma 3. If a 0–1 sequence $\{u(n) : n \geq 0\}$ contains an ℓ_1-chain and an ℓ_2-chain such that ℓ_1 and ℓ_2 are relatively prime, then $\exists\, t \in \mathbb{N}$ such that $u(t) = 1$, $u(t + \ell_1) = 1$, and $u(t + \ell_2) = 1$.

We use Lemma 3 to characterize all the periods of all the attractors.

4. Characterization of the Periods of All the Cycles

Let us consider a positive integer m and a positive real number $\theta \geq 2m$; we note:

Notation 1. $p_0, p_1, \ldots, p_{s-1}$ are prime numbers taken between $2m$ and $3m$ such that $p_i < p_{i+1}$, $\alpha_i = 3m - p_i$, $0 \leq i \leq s-1$, $k = 6m$, and we define the coefficients as follows:

$$\operatorname{coef}_1(j) = \begin{cases} (\theta/2) - \alpha_i & \text{if } j = 3m - \alpha_i,\ 0 \leq i \leq s-1; \\ (\theta/2) + \alpha_i & \text{if } j = 2(3m - \alpha_i),\ 0 \leq i \leq s-1; \\ -k(\theta + m) & \text{otherwise.} \end{cases} \tag{4}$$

The coefficients defined in equation (4) are analog to those defined in [21]. For each i, $0 \leq i \leq s-1$, the first k terms of the sequence $\{x^{\alpha_i}(n) : n \geq 0\}$ are defined as follows:

$$x^{\alpha_i}(0)x^{\alpha_i}(1)\ldots x^{\alpha_i}(k-1) = \underbrace{00\ldots0}_{2\alpha_i}\underbrace{100\ldots0}_{3m-\alpha_i}\underbrace{100\ldots0}_{3m-\alpha_i}. \tag{5}$$

$\forall\, n \geq k$, the term $x^{\alpha_i}(n)$ of the sequence $\{x^{\alpha_i}(n) : n \in \mathbb{N}\}$, is defined as follows:

$$x^{\alpha_i}(n) = \mathbf{1}\left(\sum_{i=1}^{k} \operatorname{coef}_1(i) x^{\alpha_i}(n-i) - \theta\right).$$

By using the technique developed by Tchuente and Tindo [21], it is easy to prove the following lemma:

Lemma 4. The sequence $\{x^{\alpha_i}(n) : n \in \mathbb{N}\}$ describes a cycle of length $3m - \alpha_i$ of the following form:

$$\underbrace{00\ldots0}_{2\alpha_i}\underbrace{100\ldots0}_{3m-\alpha_i}\underbrace{100\ldots0}_{3m-\alpha_i}\underbrace{100\ldots0}_{3m-\alpha_i}\cdots\underbrace{100\ldots0}_{3m-\alpha_i}\cdots.$$

In [21], the authors did not study all the cycles generated by the neuronal recurrence equation. One of our aims is to study all the cycles generated by some neuronal recurrence equation.

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We construct the sequence $\{u(n) : n \geq 0\}$ generated by the neuronal recurrence equation

$$u(n) = \mathbf{1}\left(\sum_{j=1}^{k} \mathrm{coef}_1(j) u(n-j) - \theta\right), n \geq k \tag{6}$$

such that the initial terms are defined as follows:

$$u(0)u(1)\ldots u(k-1) \in \{0, 1\}^k. \tag{7}$$

Let us characterize the period of the sequence $\{u(n) : n \geq 0\}$ by showing the following proposition:

Proposition 5. The sequence $\{u(n) : n \geq 0\}$ converges to the null sequence, that is, to 00…00…00…, or to one of the sequences $\{x^{\alpha_i}(n) : n \geq 0\}$, $0 \leq i \leq s-1$.

Example 1. In the aim of giving an idea of the basin of attraction of the sequence $\{u(n) : n \geq 0\}$, we choose the following parameters:

$$m = 5,\ \theta = 12,\ p_0 = 11,\ p_1 = 13, \text{ and } k = 30.$$

We build from the preceding parameters the following neuronal recurrence equation

$$u(n) = \mathbf{1}\left(\sum_{j=1}^{30} \mathrm{coef}_1(j) u(n-j) - \theta\right), n \geq 30 \tag{8}$$

where:

$$\mathrm{coef}_1(j) = \begin{cases} 2 & \text{if } j = 11 \\ 4 & \text{if } j = 13 \\ 10 & \text{if } j = 22 \\ 8 & \text{if } j = 26 \\ -510 & \text{otherwise.} \end{cases} \tag{9}$$

Let us note:

$$\mathrm{config}(i) = \{u(0)u(1)\ldots u(28)u(29) : \forall\, i, 0 \leq i \leq 29, u(i) \in \{0, 1\}\},$$

and the neuronal recurrence equation defined by equation (8) from the initial terms

$$u(0)u(1)\ldots u(28)u(29)$$

converges to a cyle of length i.

We also note $\chi(i) = \mathrm{card}(\mathrm{config}(i))$. By numerical simulations, the values of the sequence $\chi(i)$ are:

$$\chi(i) = 0 \text{ if } i \notin \{1, 11, 13\},$$
$$\chi(1) = 1\,073\,713\,157,$$
$$\chi(11) = 4094,$$
$$\chi(13) = 24\,573.$$

Notation 2. Let us define the memory length of some neuronal recurrence equations as follows:

$$h = sk = 6ms.$$

Let $\{y(n) : n \geq 0\}$ be the sequence whose first h terms are defined as follows:

$$y(i) \in \{0, 1\},\ 0 \leq i \leq -1 + h, \tag{10}$$

and the other terms are generated by the following neuronal recurrence equation:

$$y(n) = \mathbf{1}\left[\sum_{f=1}^{h} \mathrm{coef}_2(f) y(n - f) - \theta_1\right];\ n \geq h \tag{11}$$

where

$$\mathrm{coef}_2(f) = \begin{cases} \mathrm{coef}_1(j) & \text{if } f = sj,\ 1 \leq j \leq k \\ 0 & \text{otherwise;} \end{cases} \tag{12}$$

$$\theta_1 = \theta. \tag{13}$$

The parameters $\mathrm{coef}_1(j)$ are those defined in equation (4).

Remark 1. (*a*) The first h terms of the sequence $\{y(n) : n \geq 0\}$ are obtained by taking any element of the set $\{0, 1\}^h$.

(*b*) The coefficients $\mathrm{coef}_2(f)$ of neuronal recurrence equation (11) are obtained by applying the construction of Lemma 1 to the parameters defined by equation (4).

Our aim is to characterize the structure of all the periods of the sequence $y(n)$ from a qualitative point of view. The next theorem gives the period of the sequences $\{y(n) : n \geq 0\}$.

Theorem 1. From any initial term, the sequence $\{y(n) : n \geq 0\}$ converges to a cycle of length $s\,\mathrm{lcm}(\mathrm{elt}_1, \mathrm{elt}_2, \ldots, \mathrm{elt}_s)$ where $\mathrm{elt}_i \in \{p_0, p_1, \ldots, p_{s-1}\}$ for any $i \in \{0, 1, 2, \ldots, s - 1\}$, or p where p is equal to 1.

In Section 5, we show how to apply the previous technique to the study of bifurcation of the neuronal recurrence equation $z(n, d)$.

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5. Generalized Bifurcation of the Neuronal Recurrence Equation

Let us define the neuronal recurrence equation $\{\tilde{x}(n) : n \geq 0\}$ by the following recurrence:

$$\tilde{x}(t) = \mathbf{1}\left(\sum_{j=1}^{k_2} \mathrm{coef}_3(j)\tilde{x}(t-j) - \overline{\theta}\right);\ t \geq k_2 \tag{14}$$

where $\mathrm{coef}_3(j)$ is defined as follows:

First case: s is even and $\forall\ i_2 \in \mathbb{N},\ 0 \leq i_2 \leq s-1$:

$$\mathrm{coef}_3(j) = \begin{cases} 2 & \text{if } j \in \mathrm{R1}(\alpha_{i_2}) \text{ and } j \leq (3sp_{i_2})/2, \\ -2 & \text{if } j \in R1(\alpha_{i_2}) \text{ and } j > (3sp_{i_2})/2, \\ -4k_2 & \text{otherwise.} \end{cases} \tag{15}$$

Second case: s is odd, $s \geq 3$, and $\forall\ i_2 \in \mathbb{N},\ 0 \leq i_2 \leq s-1$:

$$\mathrm{coef}_3(j) = \begin{cases} 2 & \text{if } j \in \mathrm{R1}(\alpha_{i_2}) \text{ and } j \leq ((3s)-1)/2p_{i_2}, \\ -2 & \text{if } j \in \mathrm{R1}(\alpha_{i_2}) \text{ and } ((3s)+1)/2 \\ & p_{i_2} \leq j \leq (2s-2)p_{i_2}, \\ -1 & \text{if } j \in \{(2s-1)p_{i_2},\ 2sp_{i_2}\}, \\ -4k_2 & \text{otherwise.} \end{cases} \tag{16}$$

The parameters $R1(\alpha_i)$, $\overline{\theta}$, and k_2 are defined as follows:

$$\begin{aligned} R1(\alpha_i) = \{jp_i : j = 1, \ldots, 2s\} &= \\ \{p_i, 2p_i, \ldots, (-1+2s)p_i, 2sp_i\},&\ \ 0 \leq i \leq -1+s; \end{aligned} \tag{17}$$

$$R2 = \{i : i = 1, \ldots, k_2\} = \{1, 2, \ldots, -1+k_2, k_2\}; \tag{18}$$

$$R3 = \bigcup\nolimits_{i=0}^{-1+s} \mathrm{R1}(\alpha_i); \tag{19}$$

$$R4 = R2 \backslash R3; \tag{20}$$

$$\overline{\theta} = 2s; \tag{21}$$

$$k_2 = (6m-1)s. \tag{22}$$

By applying the technique developed in Sections 2 and 3 and the one developed in [24] to the neuronal recurrence equation defined by equation (14), it is easy to construct a family of neuronal recurrence equations $\{z(n, d) : n \geq 0\}$ that verify the following theorem.

Theorem 2. For all m $\left(m \geq e^2\right)$ and $d \in \mathbb{N}$ such that $0 \leq d \leq s-2$, we construct a set of neuronal recurrence equations whose behavior has the following characteristics:

- From any initial configuration, the neuronal recurrence equation $\{z(n, d) : n \geq 0\}$ converges to a cycle of length s $\mathrm{lcm}(\mathrm{elt}_1, \mathrm{elt}_2, \ldots, \mathrm{elt}_s)$ where $\mathrm{elt}_i \in \{p_{d+1}, p_{d+2}, \ldots, p_{s-1}\}$ for any $i \in \{1, 2, \ldots, s\}$, or to a cycle of length 1.
- From any initial configuration, the neuronal recurrence equation $\{z(n, s-1) : n \geq 0\}$ converges to a fixed point (i.e., the period of a cycle is 1).

In other words, the first part of Theorem 2 can be interpreted as follows: in some cases, the length of the cycles of the neuronal recurrence equation $\left\{z\left(n, d-1\right) : n \geq 0\right\}$ is divided by p_d to obtain the length of the cycles of the neuronal recurrence equation $\left\{z\left(n, d\right) : n \geq 0\right\}$.

By perturbation, we can build the neuronal recurrence equation $\left\{z\left(n, d\right) : n \geq 0\right\}$ from the neuronal recurrence equation $\left\{z\left(n, d-1\right) : n \geq 0\right\}$.

Remark 2. The new contributions in this paper with respect to the previous works are:

First, from the point of view of the period:

- In [15, 18–21], the authors did not study all the cycles generated by the neuronal recurrence equation.
- In this paper, we studied all the cycles generated by the neuronal recurrence equation $\{y(n) : n \geq 0\}$.

Second, from the point of view of bifurcation:

- In [23], we studied the dynamics of the sequence $\{z(n) : n \geq 0\}$ from one and only one initial configuration. We characterized only one cycle of the sequence $\{z(n) : n \geq 0\}$.
- In [24], for any d $(0 \leq d \leq \rho(m) - 1)$, we studied the dynamics of the sequence $\{z(n, d) : n \geq 0\}$ from one and only one initial configuration. We characterized only one cycle of the sequence $\{z(n, d) : n \geq 0\}$.
- In this paper, for any d $(0 \leq d \leq \rho(m) - 1)$, we studied the dynamics of the sequence $\{z(n, d) : n \geq 0\}$ from any initial configurations. We characterized the length of all cycles of the sequence $\{z(n, d) : n \geq 0\}$.

6. Conclusion

We have given a characterization of k-chains in 0–1 periodic sequences. This characterization allows us to determine the periods of all cycles of some neuronal recurrence equations. From the structure of the periods of all cycles, we show how to build the family of neuronal recurrence equations $\{z(n, d) : n \geq 0\}$ that admit a generalized period-halving bifurcation. The structure of the configuration of neuronal recurrence equations can be used in steganography (see Second Approach and Third Approach in [25]).

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Appendix

A. Proof of Lemma 3

From the hypothesis that the sequence $\{u(n) : n \geq 0\}$ contains two chains, we can deduce that:

$$\begin{aligned}
&\exists\, a \in \mathbb{N},\ 0 \leq a < \ell_1,\\
&\exists\, b \in \mathbb{N},\ 0 \leq b < \ell_2,\\
&u(a + (i\ell_1)) = 1,\ \forall\, i \in \mathbb{N},\\
&u\big(b + (j\ell_2)\big) = 1,\ \forall\, j \in \mathbb{N}.
\end{aligned}$$

By the hypothesis that integers ℓ_1 and ℓ_2 are relatively prime, and from the definition of greatest common divisor, we can deduce:

$$\exists\, n_1, n_2 \in \mathbb{Z} \text{ such that } n_1\ell_1 + n_2\ell_2 = 1. \tag{A.1}$$

From equation (A.1), we can easily deduce:

$$n_1\big(b - a\big)\ell_1 + n_2\big(b - a\big)\ell_2 = b - a, \tag{A.2}$$

$$a + \big(n_1\big(b - a\big)\ell_1\big) = b - \big(n_2\big(b - a\big)\ell_2\big). \tag{A.3}$$

From equation (A.3), it follows that $\exists\, i_0, j_0 \in \mathbb{N}$ is defined as follows:

$$i_0 = n_1\big(b - a\big) + \big(\big(1 + \big|\, n_1\big(b - a\big)\big| + \big|\, n_2\big(b - a\big)\big|\big)\ell_2\big), \tag{A.4}$$

$$j_0 = -n_2\big(b - a\big) + \big(\big(1 + \big|\, n_1\big(b - a\big)\big| + \big|\, n_2\big(b - a\big)\big|\big)\ell_1\big), \tag{A.5}$$

such that:

$$a + (i_0\ell_1) = b + (j_0\ell_2).$$

It suffices to choose $t = a + (i_0\ell_1)$.

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B. Proof of Proposition 5

Without loss of generality, let us choose the following initial terms:

$$u(0)u(1)u(2)\ldots u(k-1) \in \{0, 1\}^k.$$

We suppose that from the following initial terms:

$$u(0)u(1)u(2)\ldots u(k-1) \in \{0, 1\}^k \tag{B.1}$$

the sequence $\{u(n) : n \geq 0\}$ describes a transient of length T_1 and a cycle of length P_1. We define the sequence $\{w(n) : n \geq 0\}$ as follows:

$$\forall\ n \in \mathbb{N} w(n) = u(n + T_1).$$

In other words, the sequence $\{u(n) : n \geq 0\}$ converges to the attractor $\{w(n) : n \geq 0\}$. The proof is divided into two parts:

First, let us suppose that the sequence $\{w(n) : n \geq 0\}$ is not equal to one of the following three sequences:

- the null sequence, that is, to 0 0…0 0 0…
- one of the sequences $\{x^{\alpha_i}(n) : n \geq 0\}$, $0 \leq i \leq s-1$

We can extract from the sequence $\{w(n) : n \geq 0\}$ an ℓ-chain such that:

$$\ell \not\equiv 0 \bmod p_i,\ 0 \leq i \leq s-1. \tag{B.2}$$

Without loss of generality, let us assume that:

$$w(t_1) = 1. \tag{B.3}$$

From the fact that the sequence $\{w(n) : n \geq 0\}$ admits an ℓ-chain, we can deduce that:

$$w(t_1 + \ell) = 1. \tag{B.4}$$

From equation (B.2), we can easily deduce that:

$$\text{coef}_1(\ell) = -k(\theta + m). \tag{B.5}$$

From the fact that:

$$w(\ell + t_1) = \mathbf{1}\left(\sum_{j=1}^{k} \text{coef}_1(j) w(\ell + t_1 - j) - \theta\right),$$
$$\text{coef}_1(\ell) = -k(\theta + m),$$
$$w(t_1) = 1,$$

we deduce that $w(\ell + t_1) = 0$. It follows that we have a contradiction with equation (B.4). We can deduce that there is no ℓ-chain in the sequence $\{w(n) : n \geq 0\}$ that verifies equation (B.2).

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Second, let us suppose that on the sequence $\{u1(n) : n \geq 0\}$, there exist at least two different chains.

Without loss of generality, let us suppose that there exist on the sequence $\{w(n) : n \geq 0\}$ an ℓ_1-chain such that $\ell_1 = p_{i1}$, $0 \leq i1 \leq s-1$, an ℓ_2-chain such that $\ell_2 = p_{i2}$, $0 \leq i2 \leq s-1$, with $l_1 < l_2$, that is, $p_{i1} < pi2$.

From the fact that the sequence $\{w(n) : n \geq 0\}$ admits two chains: ℓ_1-chain and ℓ_2-chain, we deduce by application of Lemma 3 that there exists $t_1 \in \mathbb{N}$, which verifies:

$$w(t_1) = 1, \tag{B.6}$$

$$w(t_1 + \ell_1) = 1, \tag{B.7}$$

$$w(t_1 + \ell_2) = 1. \tag{B.8}$$

We have: $2m \leq p_{i1} < p_{i2} \leq 3m$. It follows that:

$$\imath = \ell_2 - \ell_1 = p_{i2} - p_{i1} \leq m. \tag{B.9}$$

From equation (4) and equation (B.9), we deduce that:

$$\text{coef}_1(\imath) = -k(\theta + m). \tag{B.10}$$

Based on the facts that:

$$\begin{aligned} &w(t_1 + \ell_1) = 1, \\ &\quad \text{coef}_1(\imath) = -k(\theta + m), \\ &w(t_1 + \ell_2) = \mathbf{1}\left(\sum_{j=1}^{k} \text{coef}_1(j) w(t_1 + \ell_2 - j) - \theta\right), \end{aligned}$$

we deduce easily that $w(t_1 + \ell_2) = 0$. This is a contradiction with equation (B.8). We easily deduce that the sequence $\{w(n) : n \geq 0\}$ contains one and only one chain.

C. Proof of Theorem 1

Based on Lemma 2 and Proposition 5, we deduce the result.

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