A Symbolic Dynamics Perspective of the Game of Three-Dimensional Life

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The games of three-dimensional life are the extension models of Conway's Game of Life. Under the framework of symbolic dynamics, we undertake an analysis of the complexity of gliders in games of three-dimensional life rules by the directed graph representation and transition matrix. More specifically, the gliders here are topologically mixing and possess positive topological entropy on their concrete subsystems. Finally, the method presented in this paper is also applicable to other gliders in different *D*-dimensions.

1. Introduction

Conway's Game of Life, devised by John Horton Conway in 1970, is a two-dimensional cellular automaton (CA) endowed with the emergence of self-organization [1–4]. Ever since its inception, the Game of Life has attracted much interest because of the surprisingly evolutionary patterns. For some of the many research results regarding the Game of Life, see [5–16] and references therein. In particular, Bak et al. [5–7] used some concepts of statistical mechanics to study its evolution and claimed it is a system presenting self-organized criticality without any conserved quantity, Garcia et al. [8] explored some statistical properties of its dynamics, Rendell [9, 10] conceived its spe-

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cific configurations that can simulate the special universal Turing machines, and Reia and Kinouchi [11] applied the single-site mean-field approximation to explain the critical density. In addition, a series of its variants are designed according to the diversified tilings (aka tessellations) or evolution rules [12–16].

With the development of computer technology, a growing number of three-dimensional cellular automata (CAs) are conceived for the sake of practical problems, such as prediction of solidification grain structure [17], photoresist-etching process simulation [18], HIV infection analysis [19], effect analysis of microstructure on quasi-brittle properties [20], prediction of granular shear flow [21], encryption algorithm design of digital images [22], and others.

Interestingly, from the perspective of theoretical research, Bays [23–25] has unearthed a host of rules endowed with gliders by introducing the Game of Three-Dimensional Life. A particular rule defines a "Game of Life" if it satisfies two criteria: first, at least one glider exists and must occur "naturally" under random initial configurations of live cells; second, all patterns in the rule must exhibit bounded growth. Among the numerous CA rules, the ones exhibiting plentiful gliders and glider collisions have received special attention. They display complex behaviors via the interactions of gliders from random initial conditions. In general, gliders are localized structures of non-quiescent and non-ether patterns (ether represents a periodic background) translating along the automaton's lattice.

In this paper, we focus on providing an analytical method that is applicable to the gliders in three-dimensional CAs. The rest of this paper is organized as follows: Section 2 introduces some dynamical properties of symbolic space and presents the definitions of chaos and topological entropy. Section 3 demonstrates the chaotic symbolic dynamics of gliders in the Game of Three-Dimensional Life. Finally, Section 4 highlights the main results.

2. The Preliminaries

For D-dimensional coordinate space Z^D, each coordinate is marked as a vector of integers $\vec{l} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_D)$. Its maximum absolute value of the components $\|\vec{l}\| = \max\{|\varepsilon_1|, |\varepsilon_2|, \ldots, |\varepsilon_D|\}$. Then, the D-dimensional symbolic space is defined by $S^{Z^D} = \left\{X = (X_{\vec{l}}) \mid X_{\vec{l}} \in S, \vec{l} \in Z^D\right\}$, $S = \{0, 1, \ldots, k-1\}$. A feasible metric d on S^{Z^D} is defined as

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$$d(X, X') = \max_{i,j \in Z} \left\{ \frac{1}{\|\vec{l}\| + 1} \,\middle|\, X_{\vec{l}} \neq X'_{\vec{l}} \right\},$$

where $X = (X_{\vec{l}})$, $X' = \left(X'_{\vec{l}}\right) \in S^{Z^D}$.

Let $L = \left[\theta_1, \theta_{n_1}\right] \times \left[\theta_1^{(2)}, \theta_{n_2}^{(2)}\right] \times \cdots \times \left[\theta_1^{(D)}, \theta_{n_D}^{(D)}\right]$ be the interval of $\vec{l}$, where $1 \leq i \leq D$, $\varepsilon_i \in \left[\theta_1^{(i)}, \theta_{n_i}^{(i)}\right]$ and $n_i \in Z$ is the length of each interval of ε_i. One $n_1 \times n_2 \ldots \times n_D$-word block in S^{Z^D} is a D-dimensional matrix, denoted by

$${}_{\left(\theta_1, \theta_1^{(2)}, \ldots, \theta_1^{(D)}\right)}a_L \doteq \left\{ a_{\left[\theta_1, \theta_{n_1}\right]} \middle/ a_{\left[\theta_1^{(2)}, \theta_{n_2}^{(2)}\right]} \middle/ \cdots \middle/ a_{\left[\theta_1^{(D)}, \theta_{n_D}^{(D)}\right]} \right\},$$

where $a_{\left[\theta_1^{(i)}, \theta_{n_i}^{(i)}\right]} = \left(a_{\theta_1^{(i)}}, a_{\theta_2^{(i)}}, \ldots, a_{\theta_{n_i}^{(i)}}\right)$, $1 \leq i \leq D$. For ${}_{\left(\theta_1, \theta_1^{(2)}, \ldots, \theta_1^{(D)}\right)}a_L$, the parameter $\left(\theta_1, \theta_1^{(2)}, \ldots, \theta_1^{(D)}\right)$ is regarded as the initial coordinate of the first vertex of a D-dimensional matrix, and the matrix increases its area to L along the direction of each axis. For convenience, ${}_{\left(\theta_1, \theta_1^{(2)}, \ldots, \theta_1^{(D)}\right)}a_L$ is expressed simply as a_L in the following. A point $X_{\vec{l}} \in a_L$ if and only if $X_{\varepsilon_i} \in a_{\left[\theta_1^{(i)}, \theta_{n_i}^{(i)}\right]}$, $1 \leq i \leq D$. In S^{Z^D}, the cylinder set of an $n_1 \times n_2 \ldots \times n_D$-word block $[a]_L \in S^{Z^D}$ is $[a]_L = \left\{X \in S^{Z^D} \,\middle|\, X_L = a_L\right\}$. Thus, the cylinder sets generate a topology on S^{Z^D} and form a countable basis for this topology. Therefore, each open set is a countable union of cylinder sets. Endowed with this topology, S^{Z^D} is compact, totally disconnected, and Hausdorff space. For any nonzero vector $\vec{n} = (n_1, n_2, \ldots, n_D) \in S^{Z^D}$, the shift map σ is introduced as

$$[\sigma_{\vec{n}}(X)]_{\vec{l}} = X_{\vec{l}+\vec{n}} = X_{\varepsilon_1 + n_1, \varepsilon_2 + n_2, \ldots, \varepsilon_D + n_D}$$

for any $X \in S^{Z^D}$. In S^{Z^D}, let the map $F : S^{Z^D} \to S^{Z^D}$ be a Boolean function of CA. Following [26, 27], some terminology and notations are presented as follows.

Definition 1. The map F is chaotic on S^{Z^D} in the sense of Li–Yorke if

1. $\lim\limits_{n \to \infty} \sup d(F^n(x), F^n(y)) > 0$, $\forall\, x, y \in S^{Z^D}$, $x \neq y$;
2. $\lim\limits_{n \to \infty} \inf d(F^n(x), F^n(y)) = 0$, $\forall\, x, y \in S^{Z^D}$.

We call $X \in S^{Z^D}$ an n-period point of F if there exists the integer $n > 0$ such that $F^n(X) = X$. Let $P(F)$ be the set of all n-period points; that is, $P(F) = \left\{X \in S^{Z^D} \,\middle|\, \exists\, n > 0, F^n(X) = X\right\}$. In particular, if

$F(X) = X$ for some $X \in S^{Z^D}$, then X is called a fixed point. The map F is said to be topologically transitive if for any non-empty open subsets U and V of S^{Z^D} there exists a natural number n such that $F^n(U) \bigcap V \neq \emptyset$. The set $P(F)$ is called a dense subset of S^{Z^D} if, for any $X \in S^{Z^D}$ and any constant $\varepsilon > 0$, there exists an $X' \in P(F)$ such that $d(X, X') < \varepsilon$. The map F is sensitive to initial conditions if there exists a $\delta > 0$ such that, for $X \in S^{Z^D}$ and for any domain $B(X)$ of X, there exists an $X' \in B(X)$ and a natural number n such that $d(F^n(X), F^n(X')) > \delta$.

Definition 2. The map F is chaotic on S^{Z^D} in the sense of Devaney if: (1) F is topologically transitive; (2) $P(F)$ is a dense subset of S^{Z^D}; and (3) F is sensitive to initial conditions.

The set $R \subset S^{Z^D}$ is called an (n, ε)-spanning set if and only if for any $X \in S^{Z^D}$ and any constant $n > 0$, $\varepsilon > 0$ there exists an $X' \in R$ such that $d\left(F^i(X), F^i(X')\right) \leq \varepsilon$, $i = 0, 1, \ldots, n-1$. $R_n\left(\varepsilon, S^{Z^D}, F\right)$ denotes the infimum of the cardinal number of an (n, ε)-spanning set of F. Similarly, $T \subset S^{Z^D}$ is called an (n, ε)-disjoint set if and only if for any $X, X' \in T$ and $X \neq X'$ there exists $0 \leq i < n$ such that $d\left(F^i(X), F^i(X')\right) > \varepsilon$. $T_n\left(\varepsilon, S^{Z^D}, F\right)$ denotes the supremum of the cardinal number of an (n, ε)-disjoint set with F. Bowen's topological entropy is defined as follows:

$$\mathrm{ent}(F) = \lim_{\varepsilon\to\infty} \lim_{n\to\infty} \sup \frac{1}{n} \log R_n\left(\varepsilon, S^{Z^D}, F\right) =$$
$$\lim_{\varepsilon\to\infty} \lim_{n\to\infty} \sup \frac{1}{n} \log T_n\left(\varepsilon, S^{Z^D}, F\right).$$

In addition, F is topologically mixing if there exists a natural number N such that $F^n(U) \bigcap V \neq \emptyset$ for all $n \geq N$.

Theorem 1.

1. The chaos of F in the sense of Li–Yorke can be deduced from positive topological entropy.
2. The chaos of F in the sense of both Li–Yorke and Devaney can be deduced from topologically mixing.

A set $U \subseteq S^{Z^D}$ is F-invariant if $F(U) \subseteq U$ and strongly F-invariant if $F(U) = U$. If U is closed and F-invariant, then (U, F) or simply U is called a subsystem of F. A set $U \subseteq S^{Z^D}$ is an attractor if there exists a nonempty clopen F-invariant set U' such that $\bigcap_{n\geq O} F^n(U') = U$. For instance, the fixed-point set and the periodic-point set are two types of simple attractors. Furthermore, the limit set of F actually defines a global attractor $\Omega = \bigcap_{n \geq O} F^n\left(S^{Z^D}\right)$. Let $\mathcal{B}$ denote a set of some finite word blocks over S. And $\Lambda_{\mathcal{B}}$ is the set of the $X \in S^{Z^D}$, which is composed of the whole elements in $\mathcal{B}$. Thus, $\mathcal{B}$ is called the determinative system of $\Lambda_{\mathcal{B}}$ and $\Lambda_{\mathcal{B}}$ is a subsystem of $\left(S^{Z^D}, \sigma\right)$. For a closed invariant subset $\Lambda_{\mathcal{B}} \subseteq S^{Z^D}$, $\Lambda_{\mathcal{B}}$ is called a subshift of σ. If there exist finite word blocks in $\mathcal{B}$, $\Lambda_{\mathcal{B}}$ is called a finite-type subshift. It is of interest that a special 2-order finite-type subshift can be constructed according to any order finite-type subshift, and they are topologically conjugate. Subsequently, the directed graph representation and transition matrix can be introduced as the effective tools of $\Lambda_{\mathcal{B}}$.

3. Dynamics of the Game of Three-Dimensional Life

Following [23–25], the notation of each rule is denoted by E_1, E_2, … /F_1, F_2, …; the E_i and F_i are listed in ascending order. Here, the E_i specifies the number of touching neighbors required to keep a living cell alive at the next generation (the "safe environment" range), and the F_i gives the number of touching neighbors required to bring a currently dead cell to life at the next generation (the "fertility" range). In the following, we focus our attention on the discussion of the symbolic dynamics of some representative gliders. Figure 1 illustrates their configuration patterns. In addition, the gliders belong to different rules; that is, the glider *a* belongs to rule 5,7/6, the glider *b* belongs to rule 2,3/5, the glider *c* belongs to rule 3,8/5, the glider *d* belongs to rule 8,5, the glider *e* belongs to rule 3,7/5, and the glider *f* belongs to rule 2,5/5.

The three-dimensional symbolic space is

$$S^{Z^3} = \left\{X = \left(X_{k,i,j}\right) \mid X_{k,i,j} \in S,\, k,\, i,\, j \in Z\right\}$$

and a metric d in S^{Z^3} is

$$d(X, X') = \max_{k,\, i,\, j \in Z}\left\{\frac{1}{\max\{|k|, |i|, |j|\} + 1} \,\middle|\, X_{k,i,j} \neq X'_{k,i,j}\right\},$$

where $X = \left(X_{k,i,j}\right)$, $X' = \left(X'_{k,i,j}\right) \in S^{Z^3}$. The shift map σ is

$$\left[\sigma_{p,\, s,\, t}(X)\right]_{k,\, i,\, j} = X_{k+p,\, i+s,\, j+t}$$

for any $X \in S^{Z^3}$, k, i, $j \in Z$.

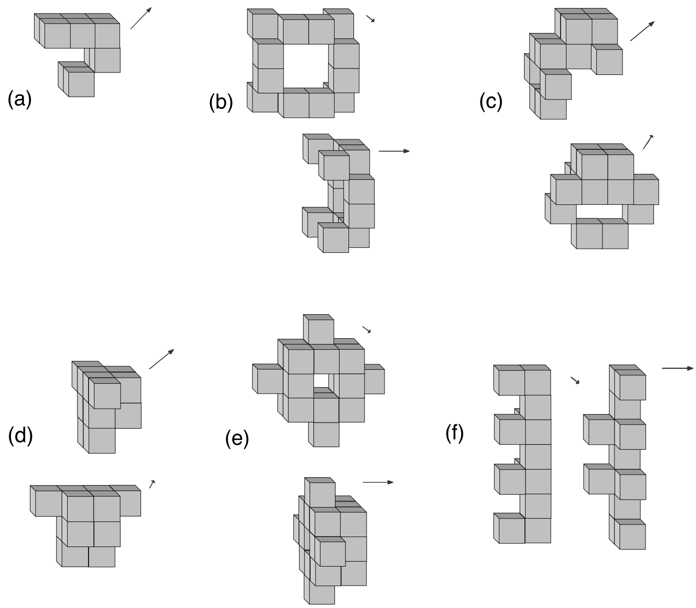

Figure 1. The configuration patterns of gliders. The same glider presents, if needed, in different directions of observing angles. The arrows (next to gliders) roughly display the shift directions of gliders.

In S^{Z^3}, one $q \times m \times n$-word block is a $q \times m \times n$-order matrix, denoted by

$$a_{q\times m\times n} = \{a_{1,m\times n} / a_{2,m\times n} / \cdots / a_{q,m\times n}\} = \left\{\begin{pmatrix} a_{1,1,1} & \cdots & a_{1,1,n} \\ & \cdots & \\ a_{1,m,1} & \cdots & a_{1,m,n} \end{pmatrix} \Big/ \begin{pmatrix} a_{2,1,1} & \cdots & a_{2,1,n} \\ & \cdots & \\ a_{2,m,1} & \cdots & a_{2,m,n} \end{pmatrix} \Big/ \cdots \Big/ \begin{pmatrix} a_{q,1,1} & \cdots & a_{q,1,n} \\ & \cdots & \\ a_{q,m,1} & \cdots & a_{q,m,n} \end{pmatrix}\right\},$$

where $a_{k,i,j} \in S$, $1 \le |k| \le q$, $1 \le |i| \le m$, $1 \le |j| \le n$. In S^{Z^3}, the cylinder set of a $q \times m \times n$-word block is $[a]_{q\times m\times n} \in S^{Z^3}$. The evolution function of a three-dimensional CA is specified as $F: S^{Z^3} \to S^{Z^3}$, $S = \{0, 1\}$. The local rule $F_{k,i,j}$ of the Game of Three-Dimensional Life has 27 inputs and one output; that is,

$$[F(X)]_{k,i,j} = N\big(X_{k-1,i-1,j-1}, X_{k-1,i-1,j}, X_{k-1,i-1,j+1}, X_{k-1,i,j-1}, X_{k-1,i,j}, X_{k-1,i,j+1}, X_{k-1,i+1,j-1}, X_{k-1,i+1,j},$$

$$X_{k-1,\,i+1,\,j+1},\, X_{k,\,i-1,\,j-1},\, X_{k,\,i-1,\,j},\, X_{k,\,i-1,\,j+1},\, X_{k,\,i,\,j-1},\, X_{k,\,i,\,j},\, X_{k,\,i,\,j+1},\, X_{k,\,i+1,\,j-1},\, X_{k,\,i+1,\,j},\, X_{k,\,i+1,\,j+1},\, X_{k+1,\,i-1,\,j-1},\, X_{k+1,\,i-1,\,j},\, X_{k+1,\,i-1,\,j+1},\, X_{k+1,\,i,\,j-1},\, X_{k+1,\,i,\,j},\, X_{k+1,\,i,\,j+1},\, X_{k+1,\,i+1,\,j-1},\, X_{k+1,\,i+1,\,j},\, X_{k+1,\,i+1,\,j+1}\big).$$

Obviously, for a concrete rule, there are 2^{27} output results of $[F(X)]_{k,\,i,\,j}$ in all.

Subsequently, the $q\times m\times n$-block transformation $B_{\langle q\times m\times n\rangle}$ is defined as

$$Y_{k,\,i,\,j} = \sum_{k'=1}^{q}\sum_{i'=1}^{m}\sum_{j'=1}^{n} X_{(k-1)q+k',\,(i-1)m+i',\,(j-1)n+j'}\cdot 2^{(k'-1)mn+(i'-1)n+j'-1}.$$

Let $\hat{S} = \{Y_{k,\,i,\,j}\}$ be a new symbolic set. After introducing the extended space $\hat{S}^{Z^3} = \{Y \mid Y_{k,\,i,\,j} \in \hat{S},\, k,\, i,\, j \in Z\}$ and the particular distance, it is demonstrated that the new three-dimensional CA has 2^{qmn} states. Let G refer to the new evolution function, and $Y_{k,\,i,\,j} = [G(X)]_{k,\,i,\,j}$ has 2^{27qmn} output results. Furthermore, the block transformation $B_{\langle q\times m\times n\rangle}$ is a homeomorphism, and the evolution function G is topologically conjugate with F.

In the following, we explore in detail the symbolic dynamics of the glider d. For different directions of observing angles, the glider d can be engineered to 23 other versions whose moving directions may be different. As they possess the same dynamical behaviors (topologically conjugate), we only select one case, whose two periodic patterns are presented in Figure 2.

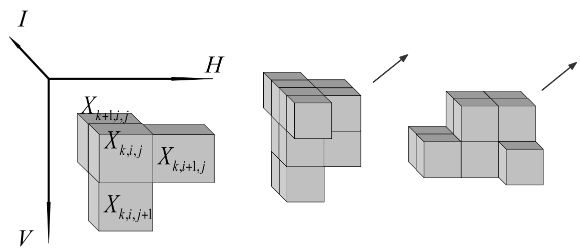

Figure 2. A simple schematic diagram of three coordinate axes and the two periodic patterns of the glider d. The positive directions are prescribed along the H, V, and I axes, and each unit is given a concrete coordinate.

Because of the periodic shift characteristic, each glider is expressed as several word blocks in S^{Z^3}. For instance, the glider d has two periodic configurations, which are distinguished as two $4 \times 3 \times 3$-word blocks. However, the appropriate blocks, especially at larger sizes, should be chosen in order to prevent the adjacent blocks from destroying each other by evolution (add an extra layer of the elements 0 on the exterior of the glider's cube). In particular, the glider d is expressed as two $6 \times 5 \times 5$-word blocks. To some extent, two $6 \times 5 \times 5$-word blocks are mutually independent. Then, by introducing the block transformation $B_{\langle 6\times 5\times 5\rangle}$, each $Y_{k,i,j}$ over $\hat{S} = \{0, 1, \ldots, 2^{150} - 1\}$ stands for a $6 \times 5 \times 5$-word block ${}_{(k,i,j)}X_{6\times 5\times 5} = \{X_{1,5\times 5} / X_{2,5\times 5} / \cdots / X_{6,5\times 5}\}$ over $S = \{0, 1\}$. It is of importance to mention that the elements 316 238 850 and 4 311 252 000 actually refer to the decimal notations of the glider d in two periods, respectively. For instance, when $Y_{k,i,j} = 316\,238\,850$, $X_{1,5\times 5} = X_{6,5\times 5} = \mathbf{0}_{5\times 5}$, which is a zero matrix,

$$X_{2,5\times 5} == X_{5,5\times 5} == \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$X_{3,5\times 5} == X_{4,5\times 5} == \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

When $Y_{k,i,j} = 4\,311\,252\,000$,

$$X_{1,5\times 5} == X_{6,5\times 5} == \mathbf{0}_{5\times 5},\ X_{2,5\times 5} == X_{5,5\times 5} == \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$X_{3,5\times 5} == X_{4,5\times 5} == \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 1. For the glider d, there exists a subset

$$\Lambda_{\mathcal{B}_4} = \left\{Y \in S^{Z^3} \;\middle|\; {}_{(k,i,j)}Y_{3\times3\times3} \in \mathcal{B}_4,\ \forall\, k, i, j \in Z\right\},$$

such that $G^{10}(Y)|_{\Lambda_{\mathcal{B}_4}} = \sigma_{0,1,-1}(Y)|_{\Lambda_{\mathcal{B}_4}}$, $\forall\, Y \in \Lambda_{\mathcal{B}_4}$, where ${}_{(k,i,j)}Y_{3\times3\times3}$ is

$$\left\{\begin{pmatrix} Y_{k,i,j} & Y_{k,i,j+1} & Y_{k,i,j+2} \\ Y_{k,i+1,j} & Y_{k,i+1,j+1} & Y_{k,i+1,j+2} \\ Y_{k,i+2,j} & Y_{k,i+2,j+1} & Y_{k,i+2,j+2} \end{pmatrix} \Big/ \begin{pmatrix} Y_{k+1,i,j} & Y_{k+1,i,j+1} & Y_{k+1,i,j+2} \\ Y_{k+1,i+1,j} & Y_{k+1,i+1,j+1} & Y_{k+1,i+1,j+2} \\ Y_{k+1,i+2,j} & Y_{k+1,i+2,j+1} & Y_{k+1,i+2,j+2} \end{pmatrix} \Big/ \begin{pmatrix} Y_{k+2,i,j} & Y_{k+2,i,j+1} & Y_{k+2,i,j+2} \\ Y_{k+2,i+1,j} & Y_{k+2,i+1,j+1} & Y_{k+2,i+1,j+2} \\ Y_{k+2,i+2,j} & Y_{k+2,i+2,j+1} & Y_{k+2,i+2,j+2} \end{pmatrix}\right\}$$

and $\mathcal{B}_4$ is

$$\left\{\left\{\begin{pmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{pmatrix} \Big/ \begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix}\right\}; \left\{\begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix} \Big/ \begin{pmatrix} k_2 & 0 & k_3 \\ 0 & k_3 & 0 \\ k_3 & 0 & k_4 \end{pmatrix}\right\}\right\},$$

where $k_i = 0, 316\,238\,850, 4\,311\,252\,000$ and $i = 1, 2, 3, 4$. Moreover, $\Lambda_{\mathcal{B}_4}$ is a subshift of finite type of $\left(\hat{S}^{Z^3}, \sigma_{0,1,-1}\right)$.

From the mathematical point of view, directed graph theory provides a powerful tool for studying the subshift of finite type. A fundamental method for constructing finite shifts starts with a finite directed graph and produces collections of bi-infinite walks (i.e., matrices of nodes) on the graph. A directed graph $\mathcal{D}(v, E)$ consists of a finite set v of vertices (or states) together with a finite set E of edges. A finite path $P = v_1 \to v_2 \to \cdots \to v_m$ on a graph $\mathcal{D}(v, E)$ is a finite block of vertices v_i from $\mathcal{D}$. The length of P is $|P| = m$. It is addressed that $\Lambda_{\mathcal{B}_4}$ can be described by a finite directed graph $\mathcal{D}_{\mathcal{B}_4} = \mathcal{D}(\mathcal{B}_4, E)$, where each vertex is a $3\times3\times3$-word block in $\mathcal{B}_4$. Each edge $e \in E$ starts at a block denoted by ${}_{(k,i,j)}Y_{3\times3\times3} \in \mathcal{B}_4$ and terminates at the block $_{(k,\,i,\,j)}Y'_{3\times3\times3}$ if and only if

$$\left(\begin{pmatrix} Y_{k,i,j+1} & Y_{k,i,j+2} \\ Y_{k,i+1,j+1} & Y_{k,i+1,j+2} \\ Y_{k,i+2,j+1} & Y_{k,i+2,j+2} \end{pmatrix} = \begin{pmatrix} Y'_{k,i,j} & Y'_{k,i,j+1} \\ Y'_{k,i+1,j} & Y'_{k,i+1,j+1} \\ Y'_{k,i+2,j} & Y'_{k,i+2,j+1} \end{pmatrix}\right)$$

and

$$\begin{pmatrix} Y_{k+1,i,j+1} & Y_{k+1,i,j+2} \\ Y_{k+1,i+1,j+1} & Y_{k+1,i+1,j+2} \\ Y_{k+1,i+2,j+1} & Y_{k+1,i+2,j+2} \end{pmatrix} = \begin{pmatrix} Y'_{k+1,i,j} & Y'_{k+1,i,j+1} \\ Y'_{k+1,i+1,j} & Y'_{k+1,i+1,j+1} \\ Y'_{k+1,i+2,j} & Y'_{k+1,i+2,j+1} \end{pmatrix}$$

and

$$\begin{pmatrix} Y_{k+2,i,j+1} & Y_{k+2,i,j+2} \\ Y_{k+2,i+1,j+1} & Y_{k+2,i+1,j+2} \\ Y_{k+2,i+2,j+1} & Y_{k+2,i+2,j+2} \end{pmatrix} = \begin{pmatrix} Y'_{k+2,i,j} & Y'_{k+2,i,j+1} \\ Y'_{k+2,i+1,j} & Y'_{k+2,i+1,j+1} \\ Y'_{k+2,i+2,j} & Y'_{k+2,i+2,j+1} \end{pmatrix}.$$

Each element of $\Lambda_{\mathcal{B}_4}$ can be represented as a certain path on the graph $\mathcal{D}_{\mathcal{B}_4}$. All bi-infinite walks on the graph constitute the closed invariant subsystem $\Lambda_{\mathcal{B}_4}$. The finite directed graph of $\Lambda_{\mathcal{B}_4}$ is shown in Figure 3. The $3\times3\times3$-word blocks of vertices are presented in Appendix A in detail.

It can be extrapolated accurately that the finite directed graph $\mathcal{D}_{\mathcal{B}_4}$ only consists of different cycles. A cycle is a path that starts and terminates at the same vertex. When one cycle has repeated vertices, it is called the reducible cycle; otherwise, it is called the irreducible cycle. Any cycle can be compounded by the irreducible cycle. The period with period points of $\mathcal{D}$ on $\Lambda_{\mathcal{B}_4}$ is the length of the cycle. The irreducible cycle can produce the irreducible period point of $\mathcal{D}$. As the length of the irreducible cycle is less than the number of vertices, $\mathcal{D}$ has finite different periods.

For example, the irreducible cycle

$$\begin{aligned} C: v_2 &\to \\ & v_{31} \to v_5 \to v_{41} \to v_{15} \to v_{70} \to v_{17} \to v_{77} \to v_{24} \to v_{98} \to \\ & \qquad v_{18} \to v_{78} \to v_{25} \to v_{99} \to v_{19} \to v_{82} \to v_2 \end{aligned}$$

in $\mathcal{D}_{\mathcal{B}_4}$ can produce a 160-period point Y, which consists of a $16\times16\times16$-word block

$Y_{16\times16\times16} =$
$\{(P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16})/$

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$$(P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1)\,/$$
$$(P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2)\,/$$
$$(P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3)\,/$$
$$(P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4)\,/$$
$$(P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5)\,/$$
$$(P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5, P_6)\,/$$
$$(P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5, P_6, P_7)\,/$$
$$(P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8)\,/$$
$$(P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9)\,/$$
$$(P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10})\,/$$
$$(P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11})\,/$$
$$(P_{13}, P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12})\,/$$
$$(P_{14}, P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13})\,/$$
$$(P_{15}, P_{16}, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14})\,/$$
$$(P_{16}, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15})\}.$$

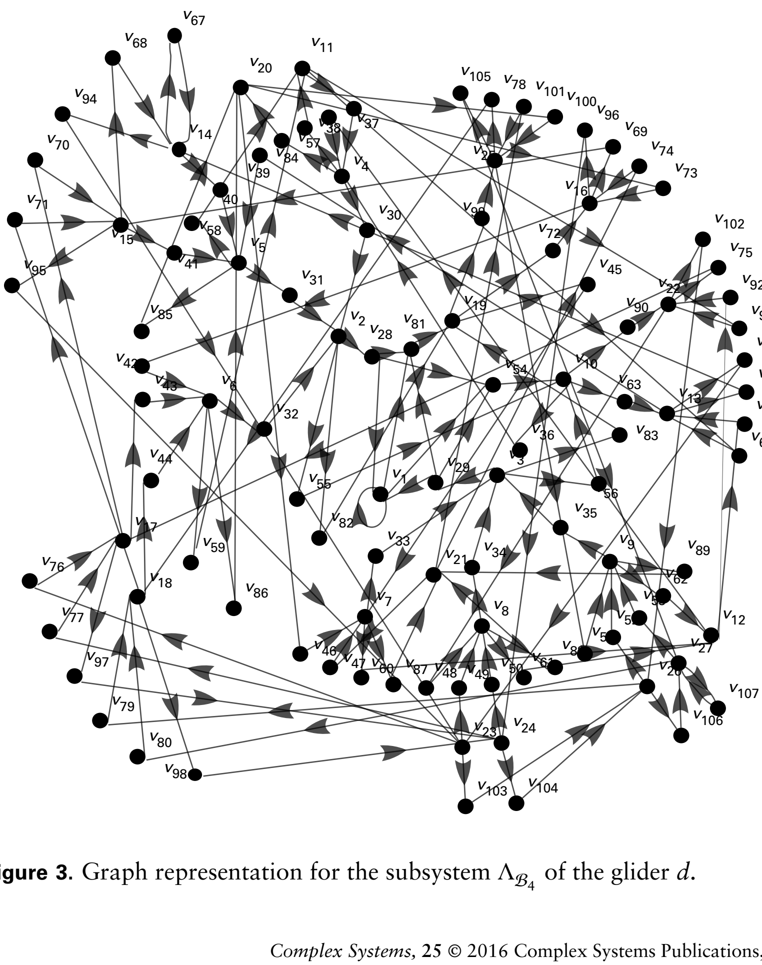

Figure 3. Graph representation for the subsystem $\Lambda_{\mathcal{B}_4}$ of the glider d.

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Let P_j, $1 \leq j \leq 16$ be the different 16-bit column vectors,

$$P_1 = (0, 0, 0, 0, 0, 1, 0, 1, 0, 2, 0, 1, 0, 2, 0, 2)^T,$$
$$P_2 = (0, 0, 0, 0, 1, 0, 1, 0, 2, 0, 1, 0, 2, 0, 2, 0)^T,$$
$$P_3 = (0, 0, 0, 1, 0, 1, 0, 2, 0, 1, 0, 2, 0, 2, 0, 0)^T,$$
$$P_4 = (0, 0, 1, 0, 1, 0, 2, 0, 1, 0, 2, 0, 2, 0, 0, 0)^T,$$
$$P_5 = (0, 1, 0, 1, 0, 2, 0, 1, 0, 2, 0, 2, 0, 0, 0, 0)^T,$$
$$P_6 = (1, 0, 1, 0, 2, 0, 1, 0, 2, 0, 2, 0, 0, 0, 0, 0)^T,$$
$$P_7 = (0, 1, 0, 2, 0, 1, 0, 2, 0, 2, 0, 0, 0, 0, 0, 1)^T,$$
$$P_8 = (1, 0, 2, 0, 1, 0, 2, 0, 2, 0, 0, 0, 0, 0, 1, 0)^T,$$
$$P_9 = (0, 2, 0, 1, 0, 2, 0, 2, 0, 0, 0, 0, 0, 1, 0, 1)^T,$$
$$P_{10} = (2, 0, 1, 0, 2, 0, 2, 0, 0, 0, 0, 0, 1, 0, 1, 0)^T,$$
$$P_{11} = (0, 1, 0, 2, 0, 2, 0, 0, 0, 0, 0, 1, 0, 1, 0, 2)^T,$$
$$P_{12} = (1, 0, 2, 0, 2, 0, 0, 0, 0, 0, 1, 0, 1, 0, 2, 0)^T,$$
$$P_{13} = (0, 2, 0, 2, 0, 0, 0, 0, 0, 1, 0, 1, 0, 2, 0, 1)^T,$$
$$P_{14} = (2, 0, 2, 0, 0, 0, 0, 0, 1, 0, 1, 0, 2, 0, 1, 0)^T,$$
$$P_{15} = (0, 2, 0, 0, 0, 0, 0, 1, 0, 1, 0, 2, 0, 1, 0, 2)^T,$$
$$P_{16} = (2, 0, 0, 0, 0, 0, 1, 0, 1, 0, 2, 0, 1, 0, 2, 0)^T,$$

where 1 stands for 316 238 850, 2 stands for 4 311 252 000, and T refers to the transposed operation.

It is of interest that the irreducible cycles actually define a series of minimal sets of $\Lambda_{\mathcal{B}_4}$. The minimal sets imply the smallest subsystems of $\Lambda_{\mathcal{B}_4}$, which are endowed with simple dynamical properties. As an illustration, the irreducible cycle C defines a minimal set $M_{\mathcal{B}_4} \subset \Lambda_{\mathcal{B}_4}$. Then $G^{10}|_{M_{\mathcal{B}_4}}$ is topologically transitive, yet has zero topological entropy.

Let $\overline{S} = \{R_0, R_1, \ldots, R_{105}, R_{106}\}$ be a new symbolic set, where R_i, $i = 0, \ldots, 106$ stand for elements of $\mathcal{B}_4$, respectively. Then a new symbolic space $\overline{S}^{Z^3}$ can be constructed on $\overline{S}$.

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Denote by

$$\overline{\mathcal{B}_4} = \left\{ \left\{ \begin{pmatrix} R_{k,i,j} & R_{k,i,j+1} \\ R_{k,i+1,j} & \end{pmatrix} \Big/ \begin{pmatrix} R_{k+1,i,j} & \\ & \end{pmatrix} \right\} \middle| R_{k,i,j} = \right.$$

$$\left({}_{(k,i,j)}Y_{3\times3\times3}\right),\ R_{k,i,j+1} = R_{k,i+1,j} = R_{k+1,i,j} =$$

$$\left({}_{(k,i,j)}Y'_{3\times3\times3}\right) \in \overline{S},\ \text{s.t.} \begin{pmatrix} Y_{k,i,j+1} & Y_{k,i,j+2} \\ Y_{k,i+1,j+1} & Y_{k,i+1,j+2} \\ Y_{k,i+2,j+1} & Y_{k,i+2,j+2} \end{pmatrix} =$$

$$\begin{pmatrix} Y'_{k,i,j} & Y'_{k,i,j+1} \\ Y'_{k,i+1,j} & Y'_{k,i+1,j+1} \\ Y'_{k,i+2,j} & Y'_{k,i+2,j+1} \end{pmatrix} \text{and} \begin{pmatrix} Y_{k+1,i,j+1} & Y_{k+1,i,j+2} \\ Y_{k+1,i+1,j+1} & Y_{k+1,i+1,j+2} \\ Y_{k+1,i+2,j+1} & Y_{k+1,i+2,j+2} \end{pmatrix} =$$

$$\begin{pmatrix} Y'_{k+1,i,j} & Y'_{k+1,i,j+1} \\ Y'_{k+1,i+1,j} & Y'_{k+1,i+1,j+1} \\ Y'_{k+1,i+2,j} & Y'_{k+1,i+2,j+1} \end{pmatrix} \text{and} \begin{pmatrix} Y_{k+2,i,j+1} & Y_{k+2,i,j+2} \\ Y_{k+2,i+1,j+1} & Y_{k+2,i+1,j+2} \\ Y_{k+2,i+2,j+1} & Y_{k+2,i+2,j+2} \end{pmatrix} =$$

$$\left. \begin{pmatrix} Y'_{k+2,i,j} & Y'_{k+2,i,j+1} \\ Y'_{k+2,i+1,j} & Y'_{k+2,i+1,j+1} \\ Y'_{k+2,i+2,j} & Y'_{k+2,i+2,j+1} \end{pmatrix} \right\}.$$

Further, the two-order subshift $\Lambda_{\overline{\mathcal{B}_4}}$ of σ is defined by

$$\Lambda_{\overline{\mathcal{B}_4}} = \left\{ R \in \overline{S}^{Z^3} \middle| R_{k,\,i,\,j} \in \overline{S}, \right.$$

$$\left. \left\{ \begin{pmatrix} R_{k,\,i,\,j} & R_{k,\,i,\,j+1} \\ R_{k,\,i+1,\,j} & \end{pmatrix} \Big/ \begin{pmatrix} R_{k+1,\,i,\,j} & \\ & \end{pmatrix} \right\} \prec \overline{\mathcal{B}_4},\ \forall\ i,\,j \in Z \right\}.$$

Define a map from $\Lambda_{\mathcal{B}_4}$ to $\Lambda_{\overline{\mathcal{B}_4}}$ as follows: $\pi : \Lambda_{\mathcal{B}_4} \to \Lambda_{\overline{\mathcal{B}_4}},\ Y \mapsto R$, where $R_{k,\,i,\,j} =_{(k,\,i,\,j)} (Y_{3\times3\times3}),\ \forall\ k,\,i,\,j \in Z$. Then it follows from the definition of $\Lambda_{\overline{\mathcal{B}_4}}$ that for any $Y \in \Lambda_{\mathcal{B}_4}$, we have $\pi(Y) \in \Lambda_{\overline{\mathcal{B}_4}}$; namely, $\pi\left(\Lambda_{\mathcal{B}_4}\right) \subseteq \Lambda_{\overline{\mathcal{B}_4}}$. It can easily be checked that π is a homeomorphism and $\pi \circ \sigma = \sigma \circ \pi$. Therefore, the topologically conjugate relationship between $\left(\Lambda_{\mathcal{B}_4}, \sigma\right)$ and a two-order subshift of finite type $\left(\Lambda_{\overline{\mathcal{B}_4}}, \sigma\right)$ is established. It is easy to calculate the transition matrix $\mathcal{M}$ of the subshift $\Lambda_{\overline{\mathcal{B}_4}}$. Then $\mathcal{M}$ is positive if all of its entries are non-negative, irreducible if $\forall\ i,\,j$ there exists n such that $\mathcal{M}_{ij}^n > 0$, and aperiodic if there exists N such that $\mathcal{M}_{ij}^n > 0,\ n > N,\ \forall\ i,\,j$.

Proposition 2.

1. The nonwandering set $\Omega\left(G^{10}|_{\Lambda_{\mathcal{B}_4}}\right) = \Lambda_{\mathcal{B}_4}$.
2. $G^{10}|_{\Lambda_{\mathcal{B}_4}}$ is topologically transitive.
3. The topological entropy of $G^{10}|_{\Lambda_{\mathcal{B}_4}}$ is positive.
4. $G^{10}|_{\Lambda_{\mathcal{B}_4}}$ is topologically mixing.

Proof.

1. The elements of $\mathcal{M}^n$ are marked as $\mathcal{M}^n_{i,j}$, $1 \le i, j \le 107$. Here each $\mathcal{M}^n_{i,j}$ shows the number of all the paths from vertex v_i to vertex v_j whose length is n. Thus, $\mathcal{M}^n_{i,i}$ is the number of all the cycles of the i^{th} vertex with length n. As $\mathcal{M}^n_{i,i}$ is positive for $n = 8$, it is easy to verify that each vertex has a particular cycle.
2. Here $\sigma_{0,1,-1}$ is topologically transitive on $\Lambda_{\mathcal{B}_4}$ if the transition matrix $\mathcal{M}$ is irreducible. Further, the irreducibility of $\mathcal{M}$ indicates that $\mathcal{M} + \mathcal{I}$ is aperiodic, where $\mathcal{I}$ is the 107×107 identity matrix. Meanwhile, it is easy to verify that $(\mathcal{M} + \mathcal{I})^n$ is positive for $n \ge 8$. Hence, G^{10} is topologically transitive on $\Lambda_{\mathcal{B}_4}$.
3. Let $\rho(\mathcal{M})$ be the maximum positive real root λ^* of the characteristic equation of $\mathcal{M}$. The characteristic equation is $-\lambda^{101}(\lambda^6 - \lambda^5 - 2\lambda^4 + 2\lambda^3 - 2\lambda^2 + 2\lambda - 2) = 0$. It can be calculated that $\rho(\mathcal{M}) = 1.7654$. Recall that two topologically conjugate systems have the same topological entropy and the topological entropy of $\sigma_{0,1,-1}$ on $\Lambda_{\mathcal{B}_4}$ equals $\log\rho(\mathcal{M})$. Therefore the topological entropy of $G^{10}|_{\Lambda_{\mathcal{B}_4}}$ is $\log\rho(\mathcal{M}) = 0.568376$.
4. A two-order subshift of finite type is topologically mixing if and only if its transition matrix is irreducible and aperiodic. Meanwhile, it is easy to verify that $\mathcal{M}^n$ is positive for $n \ge 12$. This implies that $\mathcal{M}$ is irreducible and aperiodic.

Theorem 2. $G^{10}|_{\Lambda_{\mathcal{B}_4}}$ is chaotic in the sense of both Li–Yorke and Devaney.

Moreover, the three-dimensional shift map can be decomposed into three one-dimensional shift maps σ^I_p, σ^H_s, and σ^V_t. For instance, the decomposition form is expressed as

$$\left[\sigma_{p,s,t}(Y)\right]_{k,i,j} \doteq \left[\sigma^I_p(Y)\right]_{k,i,j} + \left[\sigma^H_s(Y)\right]_{k,i,j} + \left[\sigma^V_t(Y)\right]_{k,i,j}.$$

Thus, a discussion of the symbolic dynamics of the shift configurations in three-dimensional CAs can be implemented respectively to σ^I_p, σ^H_s, and σ^V_t, whose relevant subsystems are found in one-dimensional symbolic string space $S^Z = \cdots S\times S\times S\cdots$. The metric $\hat{d}$ in S^Z is defined as

$$\hat{d}(y, y') = \max_{i \in Z} \left\{ \frac{1}{\max\{|i|\} + 1} \,\middle|\, y_i \neq y_i' \right\},$$

where $y, y' \in S^Z$. In S^Z, an n-word block is a symbolic string $(a_1, a_2, \ldots, a_n)$. For the glider d, in order to understand the symbolic dynamical properties of σ in three-dimensional subsystems, three one-dimensional subsystems are analyzed as follows.

Corollary 1. The shift of the glider d along the I axis is σ_0^I for each G^{10}. There exists a subset

$$\Lambda_{\mathcal{B}_4^I} = \left\{ y \in S^Z \mid_i y_{[3]} = (y_i, y_{i+1}, y_{i+2}) \in \mathcal{B}_4^I, \forall\, i \in Z \right\},$$

$$\mathcal{B}_4^I = \left\{ (0, k_1, 0), (k_2, 0, k_3) \right\},$$

where $k_i = 0, 316\,238\,850, 4\,311\,252\,000$ and $i = 1, 2, 3$.

Corollary 2. The shift of the glider d along the H axis is σ_1^H for each G^{10}. There exists a subset

$$\Lambda_{\mathcal{B}_4^H} = \left\{ y \in S^Z \mid_i y_{[3]} = (y_i, y_{i+1}, y_{i+2}) \in \mathcal{B}_4^H, \forall\, i \in Z \right\},$$

$$\mathcal{B}_4^H = \left\{ (0, k_1, 0), (k_2, 0, k_3) \right\},$$

where $k_i = 0, 316\,238\,850, 4\,311\,252\,000$ and $i = 1, 2, 3$. $\mathcal{B}_4^H$ is the determinative system of $\Lambda_{\mathcal{B}_4^H}$, which is a configuration set. $\Lambda_{\mathcal{B}_4^H}$ is a subshift of finite type of $\left(\hat{S}^Z, \sigma_s^H\right)$. $\sigma_1^H \mid_{\Lambda_{\mathcal{B}_4^H}}$ is topologically mixing and possesses positive topological entropy.

Corollary 3. The shift of the glider d along the V axis is σ_{-1}^V for each G^{10}. There exists a subset

$$\Lambda_{\mathcal{B}_4^V} = \left\{ y \in S^Z \mid_i y_{[3]} = (y_i, y_{i+1}, y_{i+2}) \in \mathcal{B}_4^V, \forall\, i \in Z \right\},$$

$$\mathcal{B}_4^V = \left\{ (0, k_1, 0), (k_2, 0, k_3) \right\},$$

where $k_i = 0, 316\,238\,850, 4\,311\,252\,000$ and $i = 1, 2, 3$. $\sigma_{-1}^V \mid_{\Lambda_{\mathcal{B}_4^V}}$ is topologically mixing and possesses positive topological entropy.

According to the topological conjugation relation between G and F, the following equation can be easily obtained.

Proposition 3.

$$G^{10} \mid_{\Lambda_{\mathcal{B}_4}} (Y) = \sigma_{0,1,-1}(Y) \iff F^{10} \mid_{\Lambda_{B_4'}} (X) = \sigma_{0,5,-5}(X),$$

where $\Lambda_{B_4'}$ is the corresponding subsystem of S^{Z^3} according to $\Lambda_{\mathcal{B}_4}$.

Consequently, similar to $G^{10}|_{\Lambda_{\mathcal{B}_4}}$, it is relatively trivial to investigate the dynamical properties of $F^{10}|_{\Lambda_{B'_4}}$. Actually, $F^{10}|_{\Lambda_{B'_4}}(X) = \sigma_{0,5,-5}(X)$ can be reduced to $F^2|_{\Lambda_{B'_4}}(X) = \sigma_{0,1,-1}(X)$.

Proposition 4.

1. The nonwandering set $\Omega\left(F^2|_{\Lambda_{B'_4}}\right) = \Lambda_{B'_4}$.
2. $F^2|_{\Lambda_{B'_4}}$ is topologically transitive.
3. The topological entropy of $F^2|_{\Lambda_{B'_4}}$ is positive.
4. $F^2|_{\Lambda_{B'_4}}$ is topologically mixing.

Theorem 3. $F^2|_{\Lambda_{\mathcal{B}'_4}}$ is chaotic in the sense of both Li–Yorke and Devaney.

Then the dynamics of a series of gliders can be analyzed, and their dynamical characteristics are listed in Table 1. It is to be noted that the shift functions have positive topological entropy, as well as being topologically mixing in their subsets $\Lambda_{\mathcal{B}_j}$, $j = 1, \ldots, 6$. Their corresponding decimal code sets of determinative systems $\mathcal{B}_j$ are given in Appendix B.

In addition, there exists a growing number of the other subsystems of the glider d. First, the block transformation $B_{\langle 6\xi\times 5\xi\times 5\xi\rangle}$ is introduced as

$$Y_{k,i,j} = \sum_{k'=1}^{6\xi}\sum_{i'=1}^{5\xi}\sum_{j'=1}^{5\xi} X_{(k-1)6\xi+k',(i-1)5\xi+i',(j-1)5\xi+j'} \cdot 2^{25(k'-1)\xi^2+(i'-1)5\xi+j'-1},$$

where $\xi = 2, 3, 4, \ldots$. Similarly, let $S_{\langle 6\xi\times 5\xi\times 5\xi\rangle} = \{Y_{k,i,j}\}$ be the new symbolic set. After introducing the corresponding extended space $S_{\langle 6\xi\times 5\xi\times 5\xi\rangle}^{Z^3}$ and the particular distance, we can capture an unlimited number of three-dimensional CAs of $2^{150\xi^3}$-states. Let $G_{\langle 6\xi\times 5\xi\times 5\xi\rangle}$ refer to the new evolution function and $\left[G_{\langle 6\xi\times 5\xi\times 5\xi\rangle}(Y)\right]_{k,i,j}$ has $2^{4050\xi^3}$ output results in all. It can be extrapolated accurately that the block transformation $B_{\langle 6\xi\times 5\xi\times 5\xi\rangle}$ is a homeomorphism and the evolution function $G_{\langle 6\xi\times 5\xi\times 5\xi\rangle}$ is topologically conjugate with F.

Type	Rule	Period	Block	Shift	Decomposition
a	5,7/6	4	$5\times5\times4$	$G^{20}\mid_{\Lambda_{\mathcal{B}_1}}(Y)=\sigma_{0,1,-1}(Y)$ $\Updownarrow$ $F^{20}\mid_{\Lambda_{\mathcal{B}'_1}}(X)=\sigma_{0,5,-5}(X)$	σ_0^I Period-1 σ_1^H Bernoulli shift σ_{-1}^V Bernoulli shift
b	2,3/5	4	$6\times6\times6$	$G^{12}\mid_{\Lambda_{\mathcal{B}_2}}(Y)=\sigma_{1,0,0}(Y)$ $\Updownarrow$ $F^{12}\mid_{\Lambda_{\mathcal{B}'_2}}(X)=\sigma_{6,0,0}(X)$	σ_1^I Bernoulli shift σ_0^H Period-1 σ_0^V Period-1
c	3,8/5	2	$7\times7\times6$	$G^{14}\mid_{\Lambda_{\mathcal{B}_3}}(Y)=\sigma_{0,1,-1}(Y)$ $\Updownarrow$ $F^{14}\mid_{\Lambda_{\mathcal{B}'_3}}(X)=\sigma_{0,7,-7}(X)$	σ_0^I Period-1 σ_1^H Bernoulli shift σ_{-1}^V Bernoulli shift
d	8,5	2	$5\times5\times6$	$G^{10}\mid_{\Lambda_{\mathcal{B}_4}}(Y)=\sigma_{0,1,-1}(Y)$ $\Updownarrow$ $F^{10}\mid_{\Lambda_{\mathcal{B}'_4}}(X)=\sigma_{0,5,-5}(X)$	σ_0^I Period-1 σ_1^H Bernoulli shift σ_{-1}^V Bernoulli shift
e	3,7/5	3	$7\times7\times6$	$G^{18}\mid_{\Lambda_{\mathcal{B}_5}}(Y)=\sigma_{1,0,0}(Y)$ $\Updownarrow$ $F^{18}\mid_{\Lambda_{\mathcal{B}'_5}}(X)=\sigma_{6,0,0}(X)$	σ_1^I Bernoulli shift σ_0^H Period-1 σ_0^V Period-1
f	2,5/5	3	$9\times5\times5$	$G^{15}\mid_{\Lambda_{\mathcal{B}_6}}(Y)=\sigma_{1,0,0}(Y)$ $\Updownarrow$ $F^{15}\mid_{\Lambda_{\mathcal{B}'_6}}(X)=\sigma_{5,0,0}(X)$	σ_1^I Bernoulli shift σ_0^H Period-1 σ_0^V Period-1

Table 1. Summary of the quantitative properties of subsystems of the gliders.

In particular, for these three-dimensional CAs of $2^{150\xi^3}$ states, there is a series of subsystems that is similar to the subset in Proposition 1. Their corresponding deterministic systems are

$$\left\{\left\{\begin{pmatrix} 0 & v_1 & 0 \\ v_1 & 0 & v_2 \\ 0 & v_2 & 0 \end{pmatrix} \Big/ \begin{pmatrix} v_1 & 0 & v_2 \\ 0 & v_2 & 0 \\ v_2 & 0 & v_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & v_2 & 0 \\ v_2 & 0 & v_3 \\ 0 & v_3 & 0 \end{pmatrix}\right\};\right.$$
$$\left.\left\{\begin{pmatrix} v_1 & 0 & v_2 \\ 0 & v_2 & 0 \\ v_2 & 0 & v_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & v_2 & 0 \\ v_2 & 0 & v_3 \\ 0 & v_3 & 0 \end{pmatrix} \Big/ \begin{pmatrix} v_2 & 0 & v_3 \\ 0 & v_3 & 0 \\ v_3 & 0 & v_4 \end{pmatrix}\right\}\right\},$$

where $i = 1, 2, 3, 4$, and

$$v_i = \{u_{1,\xi\times\xi} / u_{2,\xi\times\xi} / \cdots / u_{\xi,\xi\times\xi}\} = \left\{ \begin{pmatrix} u_{1,1,1} & \cdots & u_{1,1,\xi} \\ & \cdots & \\ u_{1,\xi,1} & \cdots & u_{1,\xi,\xi} \end{pmatrix} \Bigg/ \begin{pmatrix} u_{2,1,1} & \cdots & u_{2,1,\xi} \\ & \cdots & \\ u_{2,\xi,1} & \cdots & u_{2,\xi,\xi} \end{pmatrix} \Bigg/ \cdots \Bigg/ \begin{pmatrix} u_{\xi,1,1} & \cdots & u_{\xi,1,\xi} \\ & \cdots & \\ u_{\xi,\xi,1} & \cdots & u_{\xi,\xi,\xi} \end{pmatrix} \right\},$$

where each $u_{k,i,j} = 0, 316\,238\,850, 4\,311\,252\,000$, $1 \leq |k| \leq \xi$, $1 \leq |i| \leq \xi$, $1 \leq |j| \leq \xi$.

In this paper, $G_{\langle 6\times 5\times 5\rangle}$ is denoted as G and $S^{Z^3}_{\langle 6\times 5\times 5\rangle}$ refers to $\tilde{S}^{Z^3}$. For clarity, the following diagram commutes:

$$\begin{array}{ccccc} S^{Z^3} & \xrightarrow{B_{\langle 6\times 5\times 5\rangle}} & \tilde{S}^{Z^3} & \xrightarrow{B_{\langle 6\xi\times 5\xi\times 5\xi\rangle}} & S^{Z^3}_{\langle 6\xi\times 5\xi\times 5\xi\rangle} \\ \downarrow F & & \downarrow G & & \downarrow G_{\langle 6\xi\times 5\xi\times 5\xi\rangle} \\ S^{Z^3} & \xrightarrow[B_{\langle 6\times 5\times 5\rangle}]{} & \tilde{S}^{Z^3} & \xrightarrow[B_{\langle 6\xi\times 5\xi\times 5\xi\rangle}]{} & S^{Z^3}_{\langle 6\xi\times 5\xi\times 5\xi\rangle} \end{array}$$

4. Conclusion

In this paper, the chaotic dynamics of the gliders in games of three-dimensional life are explored under the framework of symbolic dynamics. It is shown that the gliders considered here are topologically mixing and possess positive topological entropy on their concrete subsystems. Therefore, it is concluded that these gliders are chaotic in the sense of both Li–Yorke and Devaney. Chaos means deterministic behaviors that are very sensitive to the initial conditions; that is, infinitesimal perturbations of the initial conditions will lead to large variations in dynamical behavior.

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Appendix

A. The Vertices of a Finite Directed Graph

The vertices of a finite directed graph of $\Lambda_{\mathcal{B}_4}$ are presented as follows, where 1 stands for 316 238 850, and 2 stands for 4 311 252 000.

$$v_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad v_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

$$v_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\}, \quad v_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$v_5 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}, \quad v_6 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\},$$

$$v_7 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad v_8 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

$$v_9 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\}, \quad v_{10} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{11} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}, \quad v_{12} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\},$$

$$v_{13} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad v_{14} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

$$v_{15} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\}, \quad v_{16} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{17} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}, \quad v_{18} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\},$$

$$v_{19} = \left\{ \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad v_{20} = \left\{ \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

$$v_{21} = \left\{ \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\}, \quad v_{22} = \left\{ \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{23} = \left\{ \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}, \quad v_{24} = \left\{ \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\},$$

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$$v_{25} = \left\{ \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad v_{26} = \left\{ \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

$$v_{27} = \left\{ \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \right\}, \quad v_{28} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$v_{29} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\}, \quad v_{30} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{31} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}, \quad v_{32} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \right\},$$

$$v_{33} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \right\}, \quad v_{34} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \right\},$$

$$v_{35} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \right\}, \quad v_{36} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{37} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad v_{38} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\},$$

$$v_{39} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}, \quad v_{40} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\},$$

$$v_{41} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \right\}, \quad v_{42} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{43} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \right\}, \quad v_{44} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \right\},$$

$$v_{45} = \left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad v_{46} = \left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$v_{47} = \left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\}, \quad v_{48} = \left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{49} = \left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}, \quad v_{50} = \left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \right\},$$

$$v_{51} = \left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \right\}, \quad v_{52} = \left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \right\},$$

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$$v_{53} = \left\{ \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \right\}, \quad v_{54} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{55} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad v_{56} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\},$$

$$v_{57} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}, \quad v_{58} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\},$$

$$v_{59} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \right\}, \quad v_{60} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{61} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \right\}, \quad v_{62} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \right\},$$

$$v_{63} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad v_{64} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$v_{65} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\}, \quad v_{66} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{67} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}, \quad v_{68} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \right\},$$

$$v_{69} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \right\}, \quad v_{70} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \right\},$$

$$v_{71} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \right\}, \quad v_{72} = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{73} = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad v_{74} = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\},$$

$$v_{75} = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}, \quad v_{76} = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\},$$

$$v_{77} = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \right\}, \quad v_{78} = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \right\},$$

$$v_{79} = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \right\}, \quad v_{80} = \left\{ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \right\},$$

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$$v_{81} = \left\{\begin{pmatrix}2&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}, \quad v_{82} = \left\{\begin{pmatrix}2&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\},$$

$$v_{83} = \left\{\begin{pmatrix}2&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&2\end{pmatrix}\right\}, \quad v_{84} = \left\{\begin{pmatrix}2&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\right\},$$

$$v_{85} = \left\{\begin{pmatrix}2&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}\right\}, \quad v_{86} = \left\{\begin{pmatrix}2&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&2\end{pmatrix}\right\},$$

$$v_{87} = \left\{\begin{pmatrix}2&0&0\\0&0&0\\0&0&2\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&2\\0&2&0\end{pmatrix}\begin{pmatrix}0&0&2\\0&2&0\\2&0&0\end{pmatrix}\right\}, \quad v_{88} = \left\{\begin{pmatrix}2&0&0\\0&0&0\\0&0&2\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&2\\0&2&0\end{pmatrix}\begin{pmatrix}0&0&2\\0&2&0\\2&0&1\end{pmatrix}\right\},$$

$$v_{89} = \left\{\begin{pmatrix}2&0&0\\0&0&0\\0&0&2\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&2\\0&2&0\end{pmatrix}\begin{pmatrix}0&0&2\\0&2&0\\2&0&2\end{pmatrix}\right\}, \quad v_{90} = \left\{\begin{pmatrix}2&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\},$$

$$v_{91} = \left\{\begin{pmatrix}2&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}, \quad v_{92} = \left\{\begin{pmatrix}2&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&2\end{pmatrix}\right\},$$

$$v_{93} = \left\{\begin{pmatrix}2&0&1\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\1&0&0\end{pmatrix}\right\}, \quad v_{94} = \left\{\begin{pmatrix}2&0&1\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}\right\},$$

$$v_{95} = \left\{\begin{pmatrix}2&0&1\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\1&0&2\end{pmatrix}\right\}, \quad v_{96} = \left\{\begin{pmatrix}2&0&1\\0&1&0\\1&0&2\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&2\\0&2&0\end{pmatrix}\begin{pmatrix}1&0&2\\0&2&0\\2&0&0\end{pmatrix}\right\},$$

$$v_{97} = \left\{\begin{pmatrix}2&0&1\\0&1&0\\1&0&2\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&2\\0&2&0\end{pmatrix}\begin{pmatrix}1&0&2\\0&2&0\\2&0&1\end{pmatrix}\right\}, \quad v_{98} = \left\{\begin{pmatrix}2&0&1\\0&1&0\\1&0&2\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&2\\0&2&0\end{pmatrix}\begin{pmatrix}1&0&2\\0&2&0\\2&0&2\end{pmatrix}\right\},$$

$$v_{99} = \left\{\begin{pmatrix}2&0&2\\0&2&0\\2&0&0\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}2&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}, \quad v_{100} = \left\{\begin{pmatrix}2&0&2\\0&2&0\\2&0&0\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}2&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\},$$

$$v_{101} = \left\{\begin{pmatrix}2&0&2\\0&2&0\\2&0&0\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}2&0&0\\0&0&0\\0&0&2\end{pmatrix}\right\}, \quad v_{102} = \left\{\begin{pmatrix}2&0&2\\0&2&0\\2&0&1\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}2&0&1\\0&1&0\\1&0&0\end{pmatrix}\right\},$$

$$v_{103} = \left\{\begin{pmatrix}2&0&2\\0&2&0\\2&0&1\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}2&0&1\\0&1&0\\1&0&1\end{pmatrix}\right\}, \quad v_{104} = \left\{\begin{pmatrix}2&0&2\\0&2&0\\2&0&1\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}2&0&1\\0&1&0\\1&0&2\end{pmatrix}\right\},$$

$$v_{105} = \left\{\begin{pmatrix}2&0&2\\&2&0\\2&0&2\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&2\\0&2&0\end{pmatrix}\begin{pmatrix}2&0&2\\0&2&0\\2&0&0\end{pmatrix}\right\}, \quad v_{106} = \left\{\begin{pmatrix}2&0&2\\&2&0\\2&0&2\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&2\\0&2&0\end{pmatrix}\begin{pmatrix}2&0&2\\0&2&0\\2&0&1\end{pmatrix}\right\},$$

$$v_{107} = \left\{\begin{pmatrix}2&0&2\\&2&0\\2&0&2\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&2\\0&2&0\end{pmatrix}\begin{pmatrix}2&0&2\\0&2&0\\2&0&2\end{pmatrix}\right\},$$

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B. The Determinative Systems of Gliders

The corresponding decimal code sets of determinative systems $\mathcal{B}_j$, $j = 1, \ldots, 6$ are presented.

1. For the glider a, the determinative system is

$$\mathcal{B}_1 == \left\{\left\{\begin{pmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{pmatrix} \Big/ \begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix}\right\};\right.$$
$$\left.\left\{\begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix} \Big/ \begin{pmatrix} k_2 & 0 & k_3 \\ 0 & k_3 & 0 \\ k_3 & 0 & k_4 \end{pmatrix}\right\}\right\},$$

$k_i = 0, n_1, n_2, n_3, n_4$ and $i = 1, 2, 3, 4$, where

$n_1 = 157\,301\,732\,472\,753\,487\,872,$

$n_2 = 382\,914\,066\,129\,277\,157\,376,$

$n_3 = 307\,109\,474\,942\,224\,171\,008,$

$n_4 = 87\,838\,209\,750\,016\,720\,896.$

2. For the glider b, the determinative system is

$$\mathcal{B}_2 == \left\{\left\{\begin{pmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{pmatrix} \Big/ \begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix}\right\};\right.$$
$$\left.\left\{\begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix} \Big/ \begin{pmatrix} k_2 & 0 & k_3 \\ 0 & k_3 & 0 \\ k_3 & 0 & k_4 \end{pmatrix}\right\}\right\},$$

$k_i = 0, n_1, n_2, n_3, n_4$ and $i = 1, 2, 3, 4$, where

$n_1 = 1\,426\,112\,365\,437\,110\,779\,069\,669\,048\,320,$

$n_2 = 1\,441\,199\,759\,666\,120\,901\,651\,836\,436\,480,$

$n_3 = 71\,248\,204\,819\,881\,517\,263\,855\,747\,960\,355\,907\,987\,296\,380 \ddots$
$518\,400,$

$n_4 = 71\,248\,204\,820\,918\,315\,100\,583\,069\,735\,449\,887\,616\,897\,216 \ddots$
$675\,840.$

3. For the glider c, the determinative system is

$$\mathcal{B}_3 == \left\{\left\{\begin{pmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{pmatrix} \middle/ \begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \middle/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix}\right\}; \right.$$
$$\left. \left\{\begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \middle/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix} \middle/ \begin{pmatrix} k_2 & 0 & k_3 \\ 0 & k_3 & 0 \\ k_3 & 0 & k_4 \end{pmatrix}\right\}\right\},$$

$k_i = 0, n_1, n_2$ and $i = 1, 2, 3, 4$, where

$$n_1 = 1\,737\,652\,812\,531\,576\,869\,185\,173\,538\,735\,313\,197\,061\,298 \ddots$$
$$377\,740\,063\,979\,195\,810\,709\,504,$$
$$n_2 = 107\,839\,992\,356\,672\,231\,205\,423\,157\,951\,804\,061\,011\,470\,513 \ddots$$
$$723\,684\,758\,003\,267\,634\,462\,720.$$

4. For the glider d, the determinative system is

$$\mathcal{B}_4 == \left\{\left\{\begin{pmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{pmatrix} \middle/ \begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \middle/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix}\right\}; \right.$$
$$\left. \left\{\begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \middle/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix} \middle/ \begin{pmatrix} k_2 & 0 & k_3 \\ 0 & k_3 & 0 \\ k_3 & 0 & k_4 \end{pmatrix}\right\}\right\},$$

$k_i = 0, n_1, n_2$ and $i = 1, 2, 3, 4$, where

$$n_1 = 162\,264\,707\,324\,156\,913\,662\,044\,271\,542\,272,$$
$$n_2 = 10\,384\,594\,273\,175\,548\,853\,037\,316\,624\,613\,376.$$

5. For the glider e, the determinative system is

$$\mathcal{B}_5 == \left\{\left\{\begin{pmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{pmatrix} \middle/ \begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \middle/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix}\right\}; \right.$$
$$\left. \left\{\begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \middle/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix} \middle/ \begin{pmatrix} k_2 & 0 & k_3 \\ 0 & k_3 & 0 \\ k_3 & 0 & k_4 \end{pmatrix}\right\}\right\},$$

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$k_i = 0, n_1, n_2, n_3$ and $i = 1, 2, 3, 4$, where

$$n_1 = 89\,530\,297\,364\,923\,605\,508\,146\,705\,664\,818\,118\,918\,144,$$
$$n_2 = 539\,231\,843\,434\,284\,405\,901\,976\,779\,735\,939\,138\,261\,719\,529 \ddots 625\,828\,884\,104\,490\,338\,222\,080,$$
$$n_3 = 539\,231\,843\,434\,159\,220\,014\,183\,401\,125\,745\,276\,163\,384\,563 \ddots 987\,138\,183\,581\,413\,285\,036\,032.$$

6. For the glider f, the determinative system is

$$\mathcal{B}_6 == \left\{\left\{\begin{pmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{pmatrix} \Big/ \begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix}\right\};\right.$$
$$\left.\left\{\begin{pmatrix} k_1 & 0 & k_2 \\ 0 & k_2 & 0 \\ k_2 & 0 & k_3 \end{pmatrix} \Big/ \begin{pmatrix} 0 & k_2 & 0 \\ k_2 & 0 & k_3 \\ 0 & k_3 & 0 \end{pmatrix} \Big/ \begin{pmatrix} k_2 & 0 & k_3 \\ 0 & k_3 & 0 \\ k_3 & 0 & k_4 \end{pmatrix}\right\}\right\},$$

$k_i = 0, n_1, n_2, n_3$ and $i = 1, 2, 3, 4$, where

$$n_1 = 42\,187\,411\,976\,088\,540\,494\,546\,617\,854\,066\,688,$$
$$n_2 = 1\,098\,980\,723\,400\,879\,481\,039\,162\,633\,706\,099\,850\,677\,091\,041 \ddots 280,$$
$$n_3 = 1\,673\,237\,945\,072\,905\,942\,430\,424\,223\,935\,627\,264.$$

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