# Iterations, Wolfram Sequences and Approximate Closed Formulas

**Mojtaba Moniri**
*Department of Mathematics, Western Illinois University*
*Macomb, IL 61455, USA*
*m-moniri@wiu.edu*

Examples of computationally simplifying some sequences of non-negative integers are presented. The reduction might be at the cost of leaving out a set of exceptional inputs of zero or rather small density.

Iterations $(a_m)_{m\in\mathbb{N}}$ of $\sqrt{2+x}$ with specific initial values $x \in [-2, 2]$ are considered. Modulo base-4 normality of $\frac{1}{\pi^2}$, when $x = 0$ and $m$ is outside a set of density about $\frac{1}{12}$, $\left\lfloor \frac{1}{2-a_m} \right\rfloor$ equals $\left\lfloor \frac{4^{m+1}}{\pi^2} \right\rfloor$; plus 1 on the exceptional set. Adding the second term of a series for $\frac{1}{2-a_m} = \frac{1}{4}\csc^2\left(\frac{\pi}{2^{m+2}}\right)$, it is asked whether any exceptions remain.

Next, Wolfram sequences $c$, of iterated $\left\lfloor \frac{3}{2}x \right\rfloor$ starting at 2, $s$ of their base-2 lengths and $r_m = \min\{k \mid s_k \geq m\}$ are discussed. Under some conditions, including $c$ not achieving a power of 2 greater than 4, $r_m = \left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} + \gamma \right\rfloor - 1$ with $\gamma \approx 0.0972\ldots$ expressible via an Odlyzko–Wilf constant. Unconditionally, $\gamma$ can be removed if outside a set of density between 0.9027 and 0.9028, so is $-1$.

## 1. Introduction

For a positive real number $\theta$, the roots $\left(\sqrt[n]{\theta}\right)_{n\in\mathbb{N}^{\geq 1}}$, when the index $n$ goes to infinity, tend to 1. For $\theta > 1$, the sequence $\left(\left\lfloor \frac{1}{\sqrt[n]{\theta}-1} \right\rfloor\right)_{n\in\mathbb{N}^{\geq 1}}$ provides another form of real number representation for $\theta$. It was shown in [1] that for a fixed parameter $\theta$ and given $n$, the $n^{\text{th}}$ term $\left\lfloor \frac{1}{\sqrt[n]{\theta}-1} \right\rfloor$ equals the $n^{\text{th}}$ term of one of the two *inhomogeneous Beatty sequences* $\left(\left\lfloor \frac{n}{\ln\theta} \pm \frac{1}{2} \right\rfloor\right)_{n\in\mathbb{N}}$. Both candidate patterns have *slope* $\frac{1}{\ln(\theta)}$, but their intercepts or *inhomogeneity terms* are $\pm\frac{1}{2}$, and the agreement is dominantly with the lower possibility $\left\lfloor \frac{n}{\ln\theta} - \frac{1}{2} \right\rfloor$, co-finitely so

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when $\ln\theta$ is rational with a bound given in [1]. This path of research was followed in [2], where it was shown that for all $\theta > 1$ the exceptional case (agreement with $\left\lfloor \frac{n}{\ln\theta} + \frac{1}{2} \right\rfloor$) is of density 0, for almost all $\theta > 1$ the counting function is asymptotic to $\frac{\ln\theta}{12} \ln n$, and for certain numbers like $e^{\sqrt{5}-1}$ the exceptional case is empty, while for $e^{2\sqrt{5}}$ it is infinite. So although the sequence $\left(\left\lfloor \frac{1}{e^{(2\sqrt{5})/n}-1} \right\rfloor\right)_{n\in\mathbb{N}^{\geq 1}}$ takes the value $\left\lfloor \frac{n}{2\sqrt{5}} + \frac{1}{2} \right\rfloor$ infinitely many times, outside this set of density 0 it agrees with the expression $\left\lfloor \frac{n}{2\sqrt{5}} - \frac{1}{2} \right\rfloor$, which is simpler than $\left\lfloor \frac{1}{e^{(2\sqrt{5})/n}-1} \right\rfloor$.

We show later in the introduction how these types of functions lead to the optimal modulus of convergence, where the $\epsilon$ are of the form $\frac{1}{n}$ for positive integer inputs $n$. With all this in mind, and further motivated by the recently developed notion of coarse computability [3] where a membership decision procedure for a set of non-negative integers may give an incorrect output on a vanishing fraction of inputs, we present new instances of arithmetically and computationally simplifying some sequences involving integer parts. The reduction might be at the cost of leaving out a set of exceptional inputs of zero or rather small density. A typical simplified output would most likely be the exact intended value, and when it is not, it would just be the predecessor of the actual value. We would have a supplementary standby program of higher complexity to determine whether 1 must be added. In Section 2, we deal with integer parts of some increasing unbounded sequences of reals, and in Section 3, the floor is invoked at each iteration.

Our first group of examples would have their *generalized inverse* (G-inverse, for short) serve as the least modulus of convergence (linear, i.e., when $\varepsilon$ is of the form $\frac{1}{k}$). Broadly speaking, given a function $g$, its G-inverse $h$ is defined by $h(m) = (\mu\, k)(g(k) \geq m)$. Here $\mu$ stands for "the least" over integers or the "inf" over the reals. G-inverses are usually considered for increasing functions $\mathbb{R} \to \mathbb{R}$, see [4], and they are left continuous. The examples will include how nested square roots with addition, obtained by iterating $\sqrt{2+x}$ with special initial values in $[-2, 2]$, converge to 2. The simplified representation candidate would be of the type $\left\lfloor \frac{4^{m+1}}{\pi^2} + \frac{1}{12} \right\rfloor$, where $m$ is the number of iterations.

Countless examples of this sort could be considered; for example, for $f(n) = \left\lfloor \frac{1}{\pi^2 - 6 \sum_{i=1}^{n} 1/i^2} \right\rfloor$ we have $f(25\,000) = 4166$. Here is a much faster sequence $g(k) = \left\lfloor \frac{1}{3 - \underbrace{\sqrt{6 + \sqrt{6 + \cdots + \sqrt{6}}}}_{k \text{ times}}} \right\rfloor$, where the Mathematica code

```
Table[Floor[1/(3-Nest[Function[t,√(t+6)],0,k])],{k,10}]
```

gives the first 10 terms

```
{1,10,64,385,2310,13862,83174,499045,2994272,17965635}.
```

For the latter sequence, given a positive integer $m$, for an integer $k > 0$ to satisfy $3 - \underbrace{\sqrt{6 + \sqrt{6 + \cdots + \sqrt{6}}}}_{k \text{ times}} < \frac{1}{m}$, it is equivalent that $m \le g(k)$. The least $k$ where this holds, that is the $k$ with $g(k-1) < m \le g(k)$, is the G-inverse of $g$ evaluated at $m$. Call that G-inverse function $h$. For $m = 1$, we have $h(m) = 1$; for $m = 2, \ldots, 10$, we have $h(m) = 2$; for $m = 11, \ldots, 64$, we have $h(m) = 3$; for $m = 65, \ldots, 385$ we have $h(m) = 4$; and so on.

In Section 2, we raise plausible identities like $\left\lfloor \frac{1}{4} \csc^2 \frac{\pi}{2^{m+2}} \right\rfloor = \left\lfloor \frac{4^{m+1}}{\pi^2} + \frac{1}{12} \right\rfloor$. Any counterexamples, if they exist, would have to make the fractional part of $\frac{4^{m+1}}{\pi^2} + \frac{1}{12}$ close to 1 while making the fractional part of $\frac{1}{4} \csc^2 \left( \frac{\pi}{2^{m+2}} \right)$ small. We entertain the reader by some near integers in a rather similar context presented by Myerson [5]:

$$16 \sin\left(\frac{\pi}{9}\right) \sin\left(\frac{5\pi}{18}\right) \sin\left(\frac{11\pi}{39}\right) \sin\left(\frac{3\pi}{8}\right) = 2.9999999975 \ldots \text{ and}$$

$$16 \sin\left(\frac{2\pi}{45}\right) \sin\left(\frac{4\pi}{25}\right) \sin\left(\frac{20\pi}{49}\right) \sin\left(\frac{17\pi}{40}\right) = 1.0000000054 \ldots.$$

However, for any potential counterexample in our situation, the closeness to integers would have to be much more. For example, at around $m = 20\,000$ the two sides agree for more than 12 000 digits after the decimal point. For 20 000, the fractional parts start with 0.64, and the first difference appears in position 12 042 after the decimal point, namely …546230864945292691182**162**… versus …546230864945292691182**097**…. (The integer parts are also big;

e.g., for $m = 20\,000$ the common value of the two floors is about four pages long in a normal Mathematica output when printed out. They end with … 139 769 790 512 309 585 196.)

Our second group of examples includes the base-2 lengths of approximate multiplications by $\frac{3}{2}$ obtained by iterating $\left\lfloor \frac{3}{2} x \right\rfloor$, initially at 2, and the generalized inverse of that length function. Our interest in this originated from [6, pp. 122, 123], and an early example briefly mentioned in [7, pp. 19, 20]. Quoting the latter, "… embedding of commands, one within another, but taken to another level" and what a fascinating level it turned out to be. The simplified and conditional representation candidate would be the *Sturmian* sequence $\left\lfloor n \log_2\left(\frac{3}{2}\right) + \gamma \right\rfloor + 1$, and for the mentioned G-inverse it would be another Sturmian $\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} + \gamma' \right\rfloor - 1$ with rather small $\gamma \approx 0.113$ and $\gamma' \approx 0.097$, both expressible via an Odlyzko–Wilf constant (see Section 3). With lower complexity, we can calculate the values $\left\lfloor n \log_2\left(\frac{3}{2}\right) \right\rfloor + 1$, respectively $\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 1$. Most likely (for about 90%), that would be the exact intended value, and when it is not, it is just the predecessor of the actual value.

Assuming further powers of 2 do not appear after 2, 3, 4 in the approximate multiplication by $\frac{3}{2}$ starting from 2, we present some discretization of a linear function of slope $\log_2\left(\frac{3}{2}\right)$ by a path in the integer lattice closest to the corresponding line from below, a Christoffel word, see [8, Part I].

## 2. How Iterated $\sqrt{2+x}$ Converges to 2, for $x \in (-2, 2)$

For an increasing and bounded sequence $b_n$ of reals converging to $\ell$, consider the errors $\ell - b_n$. The manner of convergence of these errors to 0 and the reciprocals $1 / (\ell - b_n)$ to infinity can be studied in some aspects via the sequence $\lfloor 1 / (\ell - b_n) \rfloor$. This type of Nathanson–O'Bryant approach and the G-inverse of the resulting sequence motivate this section. Our sequences in this section involve nested square roots of 2, and we use [9, Example 2]. Also see [10, 11] for further related studies.

### 2.1 Starting with 0 or $\sqrt{2}$ with a Modified Number of Iterations

For any positive integer $m$, we have

$$\frac{1}{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{m\text{ times}}} = \frac{1}{4}\csc^2\frac{\pi}{2^{m+2}} \geq \frac{4^{m+1}}{\pi^2}.$$

To see this, just use basic facts like $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, the half-angle formula for cos, and $\lim_{x\to 0^+}\frac{\sin x}{x} = 1$ (this convergence is in an increasing fashion). The latter shows the ratio of the left side of the preceding inequality to its right side (which is greater than or equal to 1) tends to 1, as $m \to \infty$.

For $m = 1, \ldots, 10$, the terms on the left, respectively the rightmost terms, are approximately as follows (shown with eight significant digits):

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 1.7071068 | 6.5685356 | 26.021717 | 103.83627 | 415.09491 |
| 1.6211389 | 6.4845558 | 25.938223 | 103.75289 | 415.01157 |
| 6 | 7 | 8 | 9 | 10 |
| 1660.1296 | 6640.2684 | 26 560.824 | 106 243.04 | 424 971.93 |
| 1660.0463 | 6640.1851 | 26 560.740 | 106 242.96 | 424 971.85 |

We consider whether $\left\lfloor\frac{1}{4}\csc^2\frac{\pi}{2^{m+2}}\right\rfloor = \left\lfloor\frac{4^{m+1}}{\pi^2}\right\rfloor$ or $\left\lfloor\frac{1}{4}\csc^2\frac{\pi}{2^{m+2}}\right\rfloor > \left\lfloor\frac{4^{m+1}}{\pi^2}\right\rfloor$.

Using the expansion $\csc^2 x = \frac{1}{x^2} + \frac{1}{3} + \frac{x^2}{15} + O\left(x^4\right)$ about 0, we see that $\frac{1}{4}\csc^2\frac{\pi}{2^{m+2}} = \frac{4^{m+1}}{\pi^2} + \frac{1}{12} + \frac{\pi^2}{15\times 4^{m+3}} + O\left(4^{-2\,m}\right)$, as $m \to \infty$. Therefore, if the two floors are not the same, then

$$\left\lfloor\frac{1}{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{m\text{ times}}}\right\rfloor = 1 + \left\lfloor\frac{4^{m+1}}{\pi^2}\right\rfloor.$$

Call this "*high*," and let "*regular*" be the one without $+1$.

The following remark is on the density of the set of high numbers $m$ and relies on the condition that $\left(\frac{4^{m+2}}{\pi^2}\right)_{m\in\mathbb{N}}$ be uniformly distributed mod 1. Recall that this is equivalent to $\frac{1}{\pi^2}$ being normal to the base 4, which is not known to hold.

**Remark 1**. With the regular case being when $\left\lfloor \frac{1}{4}\csc^2\frac{\pi}{2^{m+2}}\right\rfloor = \left\lfloor \frac{4^{m+1}}{\pi^2}\right\rfloor$ and the complementary high case being when $\left\lfloor \frac{1}{4}\csc^2\frac{\pi}{2^{m+2}}\right\rfloor = 1 + \left\lfloor \frac{4^{m+1}}{\pi^2}\right\rfloor$, if $\frac{1}{\pi^2}$ is normal to the base 4, then the high set has density at least $\frac{1}{12}$, and approximately just that.

**Question 1**. Does the representation $\left\lfloor \frac{1}{4}\csc^2\frac{\pi}{2^{m+2}}\right\rfloor = \left\lfloor \frac{4^{m+1}}{\pi^2} + \frac{1}{12}\right\rfloor$ hold for all $m$?

We used Mathematica and found there are no counterexamples for Question 1 for $m \leq 100\,000$.

**Example 1**. These high case numbers $m$ up to 400 where the values equal $1 + \left\lfloor \frac{4^{m+1}}{\pi^2}\right\rfloor$ are as follows: 3, 9, 25, 26, 85, 95, 112, 115, 123, 142, 143, 157, 158, 165, 170, 171, 208, 209, 236, 263, 284, 285, 310, 311, 312, 313, 314, 315, 319, 320, 325, 335, 355, 397.

Looking back at our starting inequality, we multiply the leftmost and rightmost sides by $\pi^2$ and compare the integer parts of the resulting numbers (this time the simplified value is far more so).

**Corollary 1**. For every $m$, we have $\left\lfloor \frac{\pi^2}{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{m\text{ times}}} \right\rfloor = 4^{m+1}$.

*Proof*. The fractional part of the expression in the floor is $\frac{\pi^2}{12} + \frac{\pi^4}{15\times 4^{m+3}} + O\left(4^{-2\,m}\right)$, with the first term being approximately 0.82 and the next starting at approximately 0.025 when $m = 1$. □

## 2.2 Starting with $-\sqrt{2}$, $\sqrt{3}$ or $-\sqrt{3}$

Here we deal with three other sequences where we have varied the initial point from the preceding consideration. For any positive integer $m$, we have

$$1 \Bigg/ \left(2 - \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2-\sqrt{2}}}}}_{(m+1)\text{ times, different innermost sign}}\right) = \frac{1}{4}\csc^2\frac{3\,\pi}{2^{m+3}} \geq \frac{4^{m+2}}{9\,\pi^2},$$

$$1\Bigg/\left(2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{3}}}}}_{(m+1)\text{ times, different innermost number}}\right)=$$

$$\frac{1}{4}\csc^2\frac{\pi}{3\times 2^{m+2}}\geq\frac{9\times 4^{m+1}}{\pi^2}$$

and

$$1\Bigg/\left(2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2-\sqrt{3}}}}}_{(m+1)\text{ times, different innermost sign and number}}\right)=$$

$$\frac{1}{4}\csc^2\frac{5\pi}{12\times 2^{m}}\geq\frac{9\times 4^{m+1}}{25\pi^2}.$$

Again, these can be seen by just using basic facts like $\cos\frac{3\pi}{8}=\frac{\sqrt{2-\sqrt{2}}}{2}$, $\cos\frac{\pi}{3}=\frac{1}{2}$, $\cos\frac{\pi}{12}=\frac{\sqrt{2+\sqrt{3}}}{2}$, the half-angle formula for cos, and $\lim_{x\to 0^+}\frac{\sin x}{x}=1$, in an increasing fashion. The latter shows the ratio of the left side of any of the three preceding inequalities to its right side (which is greater than or equal to 1) tends to 1, as $m\to\infty$.

For $m=1,\ldots,10$, the terms on the left, respectively the rightmost terms, are approximately as follows (shown with eight significant digits):

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 0.80995720 | 2.9668240 | 11.611795 | 46.195820 | 184.53294 |
| 0.72050619 | 2.8820248 | 11.528099 | 46.112396 | 184.44959 |
| 6 | 7 | 8 | 9 | 10 |
| 737.88168 | 2951.2767 | 11 804.857 | 47 219.177 | 188 876.46 |
| 737.79834 | 2951.1934 | 11 804.773 | 47 219.094 | 188 876.38 |

and

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 14.673870 | 58.444407 | 233.52736 | 933.85937 | 3735.1874 |
| 14.590250 | 58.361002 | 233.44401 | 933.77603 | 3735.1041 |
| 6 | 7 | 8 | 9 | 10 |
| 14 940.500 | 59 761.749 | 239 046.75 | 956 186.74 | 3 824 746.7 |
| 14 940.416 | 59 761.666 | 239 046.66 | 956 186.65 | 3 824 746.6 |

and

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 0.67459909 | 2.4195891 | 9.4215417 | 37.434486 | 149.48753 |
| 0.58361002 | 2.3344401 | 9.3377603 | 37.351041 | 149.40416 |
| 6 | 7 | 8 | 9 | 10 |
| 597.70000 | 2390.5500 | 9561.9499 | 38 247.549 | 152 989.95 |
| 597.61666 | 2390.4666 | 9561.8665 | 38 247.466 | 152 989.86 |

We would have to continue a bit to get the first index where the corresponding floors differ (by 1 again); those indices are 15, 38 and 16, respectively (see Example 2).

Using the expansion of $\csc^2 x$ about 0 as before, we see that

$$\frac{1}{4}\csc^2\frac{3\pi}{2^{m+3}} = \frac{4^{m+2}}{9\pi^2} + \frac{1}{12} + \frac{3\pi^2}{5\times 4^{m+3}} + O\left(4^{-2m}\right),$$

$$\frac{1}{4}\csc^2\frac{\pi}{3\times 2^{m+2}} = \frac{9\times 4^{m+1}}{\pi^2} + \frac{1}{12} + \frac{\pi^2}{135\times 4^{m+3}} + O\left(4^{-2m}\right),$$

$$\frac{1}{4}\csc^2\frac{5\pi}{12\times 2^{m}} = \frac{9\times 4^{m+1}}{25\pi^2} + \frac{1}{12} + \frac{5\pi^2}{27\times 4^{m+3}} + O\left(4^{-2m}\right),\ m\to\infty.$$

Consider the new regular and high complementary cases for each of the three examples:

$$\left\lfloor\frac{1}{4}\csc^2\frac{3\pi}{2^{m+3}}\right\rfloor = \left\lfloor\frac{4^{m+2}}{9\pi^2}\right\rfloor \quad \text{vs.} \quad \left\lfloor\frac{1}{4}\csc^2\frac{3\pi}{2^{m+3}}\right\rfloor = 1 + \left\lfloor\frac{4^{m+2}}{9\pi^2}\right\rfloor,$$

$$\left\lfloor\frac{1}{4}\csc^2\frac{\pi}{3\times 2^{m+2}}\right\rfloor = \left\lfloor\frac{9\times 4^{m+1}}{\pi^2}\right\rfloor \quad \text{vs.} \quad \text{LHS} = 1 + \left\lfloor\frac{9\times 4^{m+1}}{\pi^2}\right\rfloor \text{ and}$$

$$\left\lfloor\frac{1}{4}\csc^2\frac{5\pi}{12\times 2^{m}}\right\rfloor = \left\lfloor\frac{9\times 4^{m+1}}{25\pi^2}\right\rfloor \quad \text{vs.} \quad \text{LHS} = 1 + \left\lfloor\frac{9\times 4^{m+1}}{25\pi^2}\right\rfloor.$$

The following remark is on the density of the set of high numbers in each case and relies on the condition that (a fixed rational multiple in each case of) $\left(\frac{4^{m+2}}{\pi^2}\right)_{m\in\mathbb{N}}$ be uniformly distributed mod 1. Recall that this is equivalent to $\frac{1}{\pi^2}$ being normal to the base 4, which is not known to hold.

**Remark 2**. If $\frac{1}{\pi^2}$ is normal to the base 4, then the high set for each of the three cases has density at least $\frac{1}{12}$, and approximately just that.

**Question 2**. Do we have the representation

$$\left\lfloor \frac{1}{4}\csc^2 \frac{3\pi}{2^{m+3}} \right\rfloor = \left\lfloor \frac{4^{m+2}}{9\pi^2} + \frac{1}{12} \right\rfloor,$$

$$\left\lfloor \frac{1}{4}\csc^2 \frac{\pi}{3\times 2^{m+2}} \right\rfloor = \left\lfloor \frac{9\times 4^{m+1}}{\pi^2} + \frac{1}{12} \right\rfloor \text{ or}$$

$$\left\lfloor \frac{1}{4}\csc^2 \frac{5\pi}{12\times 2^{m}} \right\rfloor = \left\lfloor \frac{9\times 4^{m+1}}{25\pi^2} + \frac{1}{12} \right\rfloor$$

for all $m$?

We used Mathematica and found there are no counterexamples for any of these for $m \le 20\,000$.

**Example 2**. Starting with $-\sqrt{2}$, the high case numbers $m$ up to 400 where the values equal $1 + \left\lfloor \frac{4^{m+2}}{9\pi^2} \right\rfloor$ are as follows: 15, 47, 61, 62, 134, 136, 152, 160, 161, 178, 179, 192, 195, 203, 207, 208, 209, 210, 217, 227, 234, 248, 254, 264, 265, 269, 297, 327, 331, 334, 335, 336, 363, 371, 384, 388, 391, 394.

Starting with $\sqrt{3}$, the high case numbers $m$ up to 400 where the values equal $1 + \left\lfloor \frac{9\times 4^{m+1}}{\pi^2} \right\rfloor$ are as follows: 38, 41, 49, 56, 59, 60, 61, 62, 68, 73, 74, 92, 139, 140, 145, 146, 148, 149, 157, 170, 197, 204, 205, 217, 218, 219, 251, 252, 266, 284, 295, 296, 302, 307, 310, 311, 312, 313, 314, 319, 329, 383, 384.

Starting with $-\sqrt{3}$, the high case numbers $m$ up to 400 where the values equal $1 + \left\lfloor \frac{9\times 4^{m+1}}{25\pi^2} \right\rfloor$ are as follows: 16, 17, 18, 49, 50, 51, 56, 81, 82, 87, 99, 129, 130, 140, 165, 195, 196, 238, 288, 290, 296, 317, 318, 338, 339, 381, 383.

Looking back at our three starting inequalities, we multiply the leftmost and rightmost sides by $\pi^2$ and compare the integer parts of the resulting numbers (and the presented equal values are far simpler).

**Corollary 2**. If $m \equiv 1 \pmod 3$, then

$$\left\lfloor \frac{\pi^2}{2 - \underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 - \sqrt{2}}}}}_{(m+1)\text{ times, different innermost sign}}} \right\rfloor = \frac{4^{m+2} - 1}{9}.$$

If $m \equiv 0$, respectively 2 (mod 3), then the corresponding value is $\frac{4^{m+2}+2}{9}$, respectively $\frac{4^{m+2}+5}{9}$.

*Proof*. We have

$$\left\lfloor \frac{\pi^2}{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2-\sqrt{2}}}}}_{(m+1)\text{ times, different innermost sign}}} \right\rfloor = \left\lfloor \frac{4^{m+2}}{9} + \frac{\pi^2}{12} + \frac{3\,\pi^4}{5\times 4^{m+3}} + O\left(4^{-2\,m}\right) \right\rfloor,$$

with the term $\frac{\pi^2}{12}$ being approximately 0.82, and the next term starting at approximately 0.23 when $m = 1$ and phasing out. We observe that all $m \equiv 1$ (mod 3) with the exception of $m = 1$ follow the formula $\left\lfloor \frac{4^{m+2}}{9} \right\rfloor$, while all $m \equiv 0, 2$ (mod 3) follow $1 + \left\lfloor \frac{4^{m+2}}{9} \right\rfloor$. □

**Corollary 3**. For all $m \in \mathbb{N}$, we have

$$\left\lfloor \frac{\pi^2}{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{3}}}}}_{(m+1)\text{ times, different innermost number}}} \right\rfloor = 9\times 4^{m+1}.$$

*Proof*. The fractional part of the expression in the floor is $\frac{\pi^2}{12} + \frac{\pi^4}{135\times 4^{m+3}} + O\left(4^{-2\,m}\right)$, with $\frac{\pi^2}{12} \approx 0.82$ and the next term starting at less than 0.003 when $m = 1$. □

**Corollary 4**. If $m \equiv 2$, respectively 3 (mod 10), then

$$\left\lfloor \frac{\pi^2}{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2-\sqrt{3}}}}}_{(m+1)\text{ times, different innermost sign and number}}} \right\rfloor = \frac{9\times 4^{m+1}-1}{25},$$

respectively $\frac{9\times 4^{m+1}-4}{25}$.

If $m \equiv 0, 1, 4, 5, 6, 7, 8, 9 \pmod{10}$, then the corresponding value is $\frac{9\times 4^{m+1}+t}{25}$ with $t = 14, 6, 9, 11, 19, 1, 4, 16$, respectively.

*Proof*. We have

$$\left\lfloor \frac{\pi^2}{2 - \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2-\sqrt{3}}}}}_{(m+1)\text{ times, different innermost sign and number}}} \right\rfloor =$$

$$\left\lfloor \frac{9\times 4^{m+1}}{25} + \frac{\pi^2}{12} + \frac{5\,\pi^4}{27\times 4^{m+3}} + O\left(4^{-2\,m}\right) \right\rfloor,$$

where again $\frac{\pi^2}{12} \approx 0.82$, and the next term starts at roughly 0.07 and phases out. We observe that all $m \equiv 2, 3 \pmod{10}$ follow the formula $\left\lfloor \frac{9\times 4^{m+1}}{25} \right\rfloor$ and all of the other $m$ follow $1 + \left\lfloor \frac{9\times 4^{m+1}}{25} \right\rfloor$. □

## 3. Dilating by $\frac{3}{2}$ and Rounding Down, Starting from 2

The G-inverses we dealt with in the previous section were to the floors of multiplicative inverses of the error in approximating limits by sequence terms. In Section 3.2, we will consider the G-inverse of an increasing function whose defining equation here is not of that sort. We begin with this sequence itself.

### 3.1 Base-2 Length of Iterated $\left\lfloor \frac{3}{2}\,x \right\rfloor$, Initially 2

Let $c_1 = 2$, and for all $n \in \mathbb{N}$, $c_{n+1} = \left\lfloor \frac{3}{2}\,c_n \right\rfloor$. Also define $s_n = \lfloor \log_2 c_n \rfloor + 1$.

The first 20 terms of the sequence $c$ are

2, 3, 4, 6, 9, 13, 19, 28, 42, 63, 94, 141,
211, 316, 474, 711, 1066, 1599, 2398, 3597;

and the sequence $s$ starts with

2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 8, 8, 9, 9, 10, 11, 11, 12, 12.

The predecessor sequence to $c$ is generated via $b_1 = 1$, $b_{n+1} = \left\lceil \frac{3}{2}\,b_n \right\rceil$ (a simple induction shows that $b_n = c_n - 1$ for all $n$). Similarly, the sequence $a_1 = 1$ and $a_{n+1} = 3\left\lceil \frac{a_n}{2} \right\rceil$ gives the sequence $b$ when all the terms except the first are divided by 3. The sequences

$c$, $b$, $a$ are A061418, A061419 and A070885 on OEIS [12], respectively. Some related studies (besides [6]) are [13, Section 5] and [14].

Another easy induction shows that $\left(\frac{3}{2}\right)^n \leq b_n$ and so $\left(\frac{3}{2}\right)^n < c_n$. It is also clear that $c_n \leq \frac{4}{3}\left(\frac{3}{2}\right)^n$ (since dropping all floors cannot make it bigger than what is inside). Therefore $n \log_2\left(\frac{3}{2}\right) < \log_2 c_n < n \log_2\left(\frac{3}{2}\right) + 0.42$. This already shows that $\lfloor \log_2 c_n \rfloor$ equals either $\left\lfloor n \log_2\left(\frac{3}{2}\right)\right\rfloor$ or $\left\lfloor n \log_2\left(\frac{3}{2}\right)\right\rfloor + 1$ (and so $s_n$ is the successor of that in each case). We would like to have estimates on the density of the types of $n$; this is achieved by improving the bounds, where we also explore the possibility that our sequence is an inhomogeneous Beatty sequence (i.e., a Sturmian sequence). The sequence $c$ is also $\left\lceil K(3)\left(\frac{3}{2}\right)^{n-1}\right\rceil$, where $K(3)$ is a number related to the *Josephus problem*; see [14], where the approximate value

$$K(3) \approx 1.6222705028847673159569509828993241 \ldots$$

is presented. Therefore

$$\left(\frac{3}{2}\right)^n \times \frac{2\,K(3)}{3} \leq c_n < \left(\frac{3}{2}\right)^n \times \left(\frac{2\,K(3)}{3} + \left(\frac{2}{3}\right)^n\right)$$

and so

$$\left\lfloor n \log_2\left(\frac{3}{2}\right) + \log_2\left(\frac{2\,K(3)}{3}\right)\right\rfloor \leq$$
$$\lfloor \log_2 c_n \rfloor \leq \left\lfloor n \log_2\left(\frac{3}{2}\right) + \log_2\left(\frac{2\,K(3)}{3} + \left(\frac{2}{3}\right)^n\right)\right\rfloor.$$

We used Mathematica to find that the leftmost and rightmost sides agree for $m \leq 200\,000$. The inhomogeneity term $\log_2\left(\frac{2\,K(3)}{3} + \left(\frac{2}{3}\right)^n\right)$ computed to its first 26 terms decreases from about 0.8058 to about 0.1130, and the latter agrees to four decimals with its limit $\log_2\left(\frac{2\,K(3)}{3}\right)$. Hence we have the following:

**Proposition 1**. On a set of $n$ with density from 0.8869 to 0.8870, we have $s_n = \left\lfloor n \log_2\left(\frac{3}{2}\right)\right\rfloor + 1$, and on an exception set of density from 0.1130 to 0.1131, we have $s_n = \left\lfloor n \log_2\left(\frac{3}{2}\right)\right\rfloor + 2$. If $c_n$ for $n > 3$ is never a power of 2, then $s_n = \left\lfloor n \log_2\left(\frac{3}{2}\right) + \log_2\left(\frac{2\,K(3)}{3}\right)\right\rfloor + 1$ for all $n \neq 1, 3$.

**Example 3**. Here is a list of the exceptional numbers up to 400 found with Mathematica (similar to the code we will see in the next few pages for the G-inverse): 1, 3, 5, 17, 29, 34, 41, 46, 58, 70, 82, 87, 94, 99, 111, 123, 135, 140, 147, 152, 164, 176, 188, 193, 200, 205, 217, 229, 241, 246, 253, 258, 270, 282, 294, 299, 306, 311, 323, 335, 340, 347, 352, 364, 376, 388, 393, 400.

## 3.2 G-Inverse of Base-2 Length of Iterated $\left\lfloor \frac{3}{2} x \right\rfloor$, Initially 2

Let us now turn to the G-inverse of $s$, namely define $r_m = (\mu\, n)\left[s_n \geq m\right]$. For any $m$, the predicate $s_r \geq m$ is successively equivalent to the following: $\log_2 c_r \geq m-1$, $c_r \geq 2^{m-1}$ (if $c_n$ for $n \geq 4$ is never a power of 2, then $c_r > 2^{m-1}$ $(*)$; this is referred to in what follows in the proof),

$$\left\lceil K(3)\left(\frac{3}{2}\right)^{r-1}\right\rceil \geq 2^{m-1},\ K(3)\left(\frac{3}{2}\right)^{r-1} > 2^{m-1}-1,$$
$$\left(\frac{3}{2}\right)^{r-1} > \frac{2^{m-1}-1}{K(3)},\ (r-1)\log_2\left(\frac{3}{2}\right) > \log_2\left(\frac{2^{m-1}-1}{K(3)}\right),$$
$$r-1 > \frac{\log_2\left(\frac{2^{m-1}-1}{K(3)}\right)}{\log_2\left(\frac{3}{2}\right)},\ r > 1 + \frac{\log_2\left(\frac{2^{m-1}-1}{K(3)}\right)}{\log_2\left(\frac{3}{2}\right)}.$$

If the latter compound fraction is not an integer, then

$$r_m = 1 + \left\lceil \frac{\log_2\left(\frac{2^{m-1}-1}{K(3)}\right)}{\log_2\left(\frac{3}{2}\right)} \right\rceil = 2 +$$
$$\left\lfloor \frac{\log_2\left(\frac{2^{m-1}-1}{K(3)}\right)}{\log_2\left(\frac{3}{2}\right)} \right\rfloor = 2 + \left\lfloor \frac{\log_2\left(\frac{2^{m-1}\left(1-\frac{1}{2^{m-1}}\right)}{K(3)}\right)}{\log_2\left(\frac{3}{2}\right)} \right\rfloor =$$
$$\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} + 3 + \frac{\log_2\left(1-\frac{1}{2^{m-1}}\right) - \log_2 K(3) - 1}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 1$$

(under $(*)$ this would be $\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} + 3 + \frac{-\log_2 K(3)-1}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 1$, and we used Mathematica to find that the two agree for $m \leq 200\,000$). In any case, the inhomogeneity term $3 + \frac{\log_2\left(1-\frac{1}{2^{m-1}}\right)-\log_2 K(3)-1}{\log_2\left(\frac{3}{2}\right)}$ computed for $6 \leq m \leq 18$ increases from about 0.0189 to about 0.0972, and the latter agrees to four decimals with its limit $3 + \frac{-\log_2 K(3)-1}{\log_2\left(\frac{3}{2}\right)}$. Hence we have:

**Proposition 2**. On a set of $n$ with density between 0.9027 to 0.9028 we have $r_m = \left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 1$, and on an exception set of density between 0.0972 to 0.0973 we have $r_m = \left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor$. If $c_n$ for $n > 3$ is never a power of 2, and $\frac{2^{m-1}-1}{K(3)}$ is never a power of $\frac{3}{2}$, then $r_m = \left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} + 3 + \frac{-\log_2 K(3)-1}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 1$ for all $m \geq 4$.

The inhomogeneity terms $3 + \frac{\log_2\left(1-\frac{1}{2^{m-1}}\right)-\log_2 K(3)-1}{\log_2\left(\frac{3}{2}\right)}$ for $m$ among 2, 3, 4, 5 are about $-1.6$, $-0.6$, $-0.2$ and $-0.06$ (and then become positive). The first two of these do cause an isolated pair of exceptional values where $r_m = \left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 2$ for $m = 2, 3$, but for $m = 4, 5$, the majority formula $r_m = \left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 1$ prevails.

The approximation $\frac{m}{\log_2\left(\frac{3}{2}\right)}$ for $r_m$ turned out to be a good one, approximating it from the right by less than two units. We can produce $c_n$ up to $n$ being the mentioned approximation and present the last two components, where the intended number $m$ appears as the first or just the second. In the former case, the right answer would be $\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 1$ and in the latter case, $\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor$. We observe that most would have $\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 1$, while some have $\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor$. The very first two terms form an exceptional group of their own for which the G-inverse $r_m$ follows $\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 2$ and the values are 1 and 3.

**Example 4**. We can proceed through the common value formula, which most often would be $\left\lfloor \frac{m}{\log_2\left(\frac{3}{2}\right)} \right\rfloor - 1$, and see whether that is short by 1:

```
Table[m-Floor[Log[2,Last[RecurrenceTable[{c[n+1]== Floor[3c[n]/2], c[1]==
2},c,{n,Floor[m/Log[2,3/2]] -1}]]]+1],{m,4,60}]
```

```
{0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0
,0,0,0,0,0,0,0,0,0}.
```

Here are the exceptional inputs.

```
Table[If[m-Floor[Log[2,Last[RecurrenceTable[{c[n+1]==
Floor[3c[n]/2], c[1]==2},c, {n,Floor[m/Log[2,3/2]] -1}]]]+1]==1,m,""],{m,4,60}]
```

{,,,7,,,,,,,14,,,,,,,,,,,,,,,,,31,,,,,,,38,,,,,,,45,,,,,,,,,,,,,,,}.

Regarding the high numbers in Example 3 (for $s$) and in Example 4 (for $r$), see Figure 1.

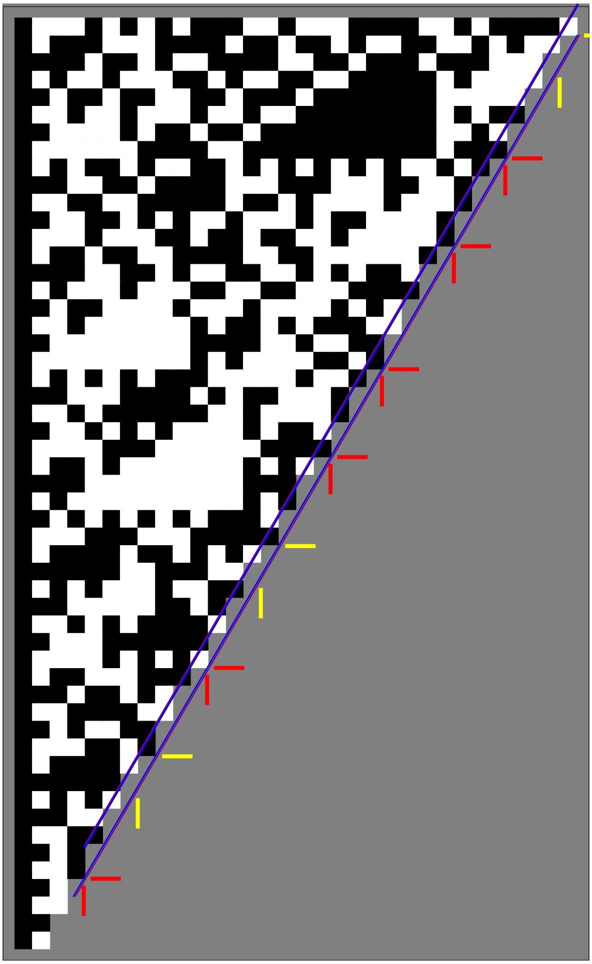

**Figure 1.** Beginning of evolution of $s$ (step 1 up through step 53) and $r$ (input 2 through 31), easy/approximate/majority bounds shown with two lines in blue, refined (conditional/plausible) bounds shown with two lines in purple, the six $s$-high cases $3 < h < 53$ (namely, 5, 17, 29, 34, 41, 46) at the red arrowheads (their outputs lie between the two bounds), the three $r$-high cases $3 < h \leq 31$ (namely, 7, 14, 31) at the yellow arrowheads (for these the blue line crosses a rightmost vertical edge not precisely on top of another).

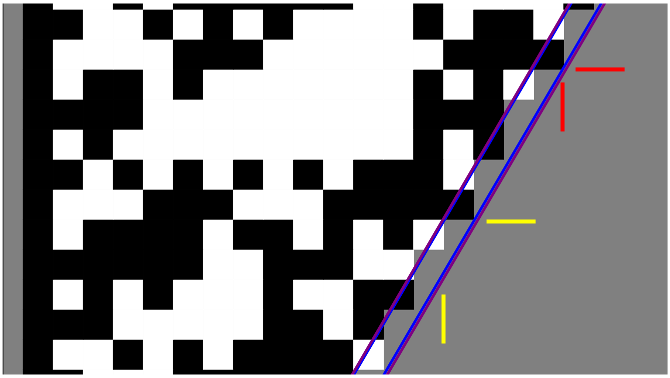

**Figure 2.** We zoom in on Figure 1 and look at a region around $h = 29$ for $s$ and $h = 14$ for $r$.

**Example 5.** The last exceptional input below 1000 is 995, as the code:

```
Table[list={2}; c=2; Do[c=Floor[c3/2];
AppendTo[list,c],{n,Floor[m/Log[2,3/2]] -2}];
If[m-Last[Floor[Log[2,list]]+1]==1,m,""],{m,980,1000}]
```

returns

```
{, , , , , , , ,988, , , , , , ,995, , , , , }.
```

Let us try the input 994 in that range not equal to either of these outputs; the code

```
Floor[994/Log[2,3/2]] -1
```

outputs {1698}.
Also the code

```
list={2}; c=2; Do[c=Floor[c3/2]; AppendTo[list,c],
{n,Floor[994/Log[2,3/2]] -2}]; 994-Last[Floor[ Log[2,list]] +1]
```

outputs 0. So the value is indeed 1698.

For the bigger of the two exceptional values, we first get the at-worst predecessor:

```
Floor[995/Log[2,3/2]] -1
```

gives 1699. The standby algorithm can then be run and will output 1:

```
list={2}; c=2; Do[c=Floor[c3/2]; AppendTo[list,c],{n,Floor[995/Log[2,3/2]] -2}]; 995-Last[Floor[
Log[2,list]]+1].
```

So 1 should be added, and the actual value is 1700.

Putting everything together, we double-check our findings (inputs 1697–1700 give outputs 993, 994, 994, 995):

```
list={2}; c=2; Do[c=Floor[c3/2]; AppendTo[list,c],{n,1699}];
Take[Floor[Log[2,list]]+1,-4]
```

```
{993,994,994,995}.
```

**Example 6**. The rate of exceptions through 6000, in groups of 100, can be found with the code:

```
Table[{k, Count[Table[list={2}; c=2; Do[c=Floor[c3/2] ;
AppendTo[list,c],{n,Floor[m/Log[2,3/2]] -2}];
m-Last[Floor[Log[2,list]]+1], {m,100 k+1,100 k+100}],x_/;x≠0] /100.}, {k,0,59}].
```

This shows the following three classes (roughly 0.0978 exception rate through 6000):

$$0, 3, 15, 27, 38, 50 : 0.11;$$

$$\begin{aligned}&4, 6, 7, 9, 10, 11, 12, 16, 18, 19, 20, 21, 23,\\&24, 26, 30, 31, 32, 33, 35, 36, 39, 41, 42, 44,\\&45, 46, 47, 51, 53, 54, 56, 57, 58, 59 : 0.1;\end{aligned}$$

$$\begin{aligned}&1, 2, 5, 8, 13, 14, 17, 22, 25, 28, 29,\\&34, 37, 40, 43, 48, 49, 52, 55 : 0.09.\end{aligned}$$

**Example 7**. Let us find any exceptional inputs from 59981 to 60000:

```
Table[list={2}; c=2; Do[c=Floor[c3/2]; AppendTo[list,c],
{n,Floor[m/Log[2,3/2]] -2}]; If[m-Last[Floor[Log[2,list]]+1]== 1,m,""],{m,59981,60000}].
```

We get

```
{,59982, , , , , , , , , , , , , , , ,59999,}.
```

Let us try the input 59981 in that range not equal to either of these two outputs. The code

```
Floor[59981/Log[2,3/2]]-1
```

returns {102537}. We would expect the next output to be 0, and that is the case:

```
list={2}; c=2; Do[c=Floor[c3/2]; AppendTo[list,c],
{n,Floor[59981/Log[2,3/2]]-1}]; 59981-Last[Floor[Log[2,list]]+1].
```

The output is {0}, and so the original output is confirmed to be 102537.

For the smaller exceptional number we found there, the code

```
Floor[59982/Log[2,3/2]] -1
```

returns {102538}.

Next we would expect an output 1:

```
list={2}; c=2; Do[c=Floor[c3/2]; AppendTo[list,c],
{n,Floor[59982/Log[2,3/2]]-2}]; 59982-Last[Floor[Log[2,list]]+1]
```

outputs {1}, and so the actual output for the original would be 102539.

Putting it all together, things match. That is, inputs 102536–102539 give outputs {59980, 59981, 59981, 59982}, as the code below shows:

```
Floor[Log[2, RecurrenceTable[{c[n+1]==Floor[3c[n]/2], c[1]=2}, c,
{n,102536,102539}]]]+1.
```

## 4. Conclusion

### 4.1 On the Iterated $\sqrt{2+x}$, and a Likely $\lfloor \Sigma_{k=0}^{\infty} a_k \rfloor = \lfloor \Sigma_{k=0}^{1} a_k \rfloor$

Section 2 included some results, along with some plausible Diophantine-analytic identities, as $\frac{\sin x}{x}$ converges to 1 along bisections of $\pi$. In more detail, using a work by Kowalenko [15, p. 73, equ. (217)], we can see that

$$\csc^2(x) = \sum_{k=0}^{\infty} \frac{(4k-2)\,\zeta(2k)}{\pi^{2k}} x^{2k-2}$$

and so

$$\frac{1}{4}\csc^2\left(\frac{\pi}{2^{m+2}}\right) = \frac{4^{m+1}}{\pi^2} \sum_{k=0}^{\infty} \frac{(4k-2)\,\zeta(2k)}{4^{(m+2)k}}.$$

We talked about the rather large density of $m$ where the initial term of the series already determines the integer part, and that when it does not, we just need to add 1. We left it open whether for all $m \in \mathbb{N}$, the floor of the series is determined by just the first two terms. We note that for each positive integer $m$, the number $\frac{1}{4}\csc^2\left(\frac{\pi}{2^{m+2}}\right)$ is an algebraic number of degree $2^m$, and its reciprocal $4\sin^2\left(\frac{\pi}{2^{m+2}}\right)$ is an algebraic integer. This could be helpful in answering the question.

### 4.2 On the Iterated $\left\lfloor \frac{3}{2} x \right\rfloor$, and the Apparently Balanced Words

In Section 3, we dealt with the process of "start with 2, then repeat: multiply by 1.5 and round down," which after two loops experiences another power of 2. If it never later lands at another $2^k$ (and we checked that up to 5000 there are no more), our "exact" Sturmian representation formula for the base-2 length of generated iterations holds. But the inhomogeneity term would involve the curious number $K(3)$, so using our mentioned approximate Beatty representation makes sense.

An indirect way to check whether a fixed initial segment of the sequence $f(m)$ can be extended to a Sturmian sequence is whether it is *balanced*, that is, whether

$$(\forall\, x, y, z)\big[(f(x+y)-f(x))-(f(z+y)-f(z))\big] \in \{-1, 0, 1\},$$

see [16]. We used this for several blocks in the sequence with Mathematica and did not find a counterexample to being Sturmian. For example, for all values of $n$ that we checked, when $n$ is increased by 900, $s(n)$ increases by either 526 or 527.

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## References

[1] M. B. Nathanson, "On the Fractional Parts of Roots of Positive Real Numbers," *The American Mathematical Monthly*, **120**(5), 2013 pp. 409–429. doi:10.4169/amer.math.monthly.120.05.409.

[2] K. O'Bryant, "The Sequence of Fractional Parts of Roots," *Acta Arithmetica*, **169**(4), 2015 pp. 357–371. doi:10.4064/aa169-4-4.

[3] C. G. Jockusch Jr. and P. E. Schupp, "Asymptotic Density and the Theory of Computability: A Partial Survey," in *Computability and Complexity, Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday* (A. Day, M. Fellows, N. Greenberg, B. Khoussainov, A. Melnikov and F. Rosamond, eds.), Cham, Switzerland: Springer International Publishing, 2017 pp. 501–520.
doi:10.1007/978-3-319-50062-1_ 30.

[4] P. Embrechts and M. Hofert, "A Note on Generalized Inverses," *Mathematical Methods of Operations Research*, **77**(3), 2013 pp. 423–432. doi:10.1007/s00186-013-0436-7.

[5] G. Myerson, "Rational Products of Sines of Rational Angles," *Aequationes Mathematicae*, **45**(1), 1993 pp. 70–82. doi:10.1007/BF01844426.

[6] S. Wolfram, *A New Kind of Science*, Champaign, IL: Wolfram Media, Inc., 2002.

[7] B. F. Torrence and E. A. Torrence, *The Student's Introduction to Mathematica*, 2nd ed., New York: Cambridge University Press, 2009.

[8] J. Berstel, A. Lauve, C. Reutenauer and F. V. Saliola, *Combinatorics on Words, Christoffel Words and Repetitions in Words*, Providence, RI: American Mathematical Society, 2009.

[9] L. D. Servi, "Nested Square Roots of 2," *The American Mathematical Monthly*, **110**(4), 2003 pp. 326–330. doi:10.2307/3647881.

[10] M. A. Nyblom, "More Nested Square Roots of 2," *The American Mathematical Monthly*, **112**(9), 2005 pp. 822–825.
doi:10.2307/30037604.

[11] S. G. Moreno and E. M. García-Caballero, "Chebyshev Polynomials and Nested Square Roots," *Journal of Mathematical Analysis and Applications*, **394**(1), 2012 pp. 61–73. doi:10.1016/j.jmaa.2012.04.065.

[12] N. J. A. Sloane, ed., "The On-Line Encyclopedia of Integer Sequences." (Apr 25, 2017) oeis.org.

[13] J. C. Lagarias and N. J. A. Sloane, "Approximate Squaring," *Experimental Mathematics*, **13**(1), 2004 pp. 113–128.
doi:10.1080/10586458.2004.10504526.

[14] A. Odlyzko and H. S. Wilf, "Functional Iteration and the Josephus Problem," *Glasgow Mathematical Journal*, **33**(2), 1991 pp. 235–240. doi:10.1017/s0017089500008272.

[15] V. Kowalenko, "Applications of the Cosecant and Related Numbers," *Acta Applicandae Mathematicae*, **114**(1), 2011 pp. 15–134. doi:10.1007/s10440-011-9604-z.

[16] M. Boshernitzan and A. S. Fraenkel, "Nonhomogeneous Spectra of Numbers," *Discrete Mathematics* **34**(3), 1981 pp. 325–327. doi:10.1016/0012-365x(81)90013-3.

[17] M. Moniri, "Arithmetic Complexity of Reals," in *Proceedings of Third Conference on Frontiers in Mathematical Sciences*, Tehran, Iran (E. Eftekhary and A. Kamalinejad, eds.), Tehran, Iran: IPM, 2016 pp. 127–154.