Investigation of *N*-Person Games by Agent-Based Modeling

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The recent developments in agent-based modeling of several *N*-person games are reviewed and some features of the author's software tools as well as some applications are described. After explaining the transition from two-person to *N*-person games, the classification and properties of *N*-person games, our software tools, and agent personalities, the following *N*-person games are considered: prisoners' dilemma, chicken dilemma, battle of the sexes game, games with crossing payoff functions, a new game with parabolic payoff functions, transitions between different *N*-person games, and analytical solutions of *N*-person games. Several practical applications are then considered.

1. Introduction

Game theory [1] is a well-developed discipline, but relatively little effort has been devoted to the theoretical investigation of *N*-person games [2–7]. The tragedy of the commons [8] pointed to the practical importance of such games.

Axelrod's famous tournaments [9] have generated an enormous interest in *N*-person games. Unfortunately, however, these tournaments are not really *N*-person games but a series of two-person games among *N* participants that have led to the investigation of numerous two-person game strategies. Nevertheless, the interest is still alive and many papers are devoted to this idea.

The appearance of powerful personal computers has made the modeling and simulation of *N*-person games possible. Several papers have appeared describing simulations of some practical examples [10–12]. Simulations in [13, 14] deserve special attention.

Our simulation tool [15] was developed specifically for the investigation of *N*-person games.

In this paper, we review the recent developments in agent-based modeling of several *N*-person games and describe some features of the author's software tools as well as some applications. After explaining the transition from two-person to *N*-person games, the classification and properties of *N*-person games, our software tools, and agent personalities, we consider the following *N*-person games: prisoners' dilemma, chicken dilemma, battle of the sexes game, games with

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crossing payoff functions, a new game with parabolic payoff functions, transitions between different N-person games, and analytical solutions of N-person games. These games are all played on a lattice.

The following practical applications are then considered in Section 14.

- The tragedy of the commons.
- Segregation.
- Mass transportation as an N-person chicken dilemma.
- Agent-based modeling of a simple market.
- Agent-based modeling of public radio membership campaigns.
- Standing ovation.
- Collective action and the N-person prisoners' dilemma.
- Agent-based modeling of the El Farol Bar problem.

2. Two-Person Games

In simple two-person games, each participant has exactly two available choices of actions (C or D) as indicated by this matrix:

$$\begin{array}{ccc} & C & D \\ C & a_1, a_2 & b_1, b_2 \\ D & c_1, c_2 & d_1, d_2 \end{array}$$

Matrix 1.

The two players are called row and column players, respectively. Four outcomes are possible: CC, CD, DC, and DD (the first choice is that of the row player; the second is that of the column player). Each player receives a payoff (reward or punishment) for each situation as shown in the matrix.

The first of each pair of payoffs are for the row player; the second are for the column player. There are $4\,!= 24$ preference orderings for each player if we require that each payoff is different. Consequently, the matrix represents $24 \times 24 = 576$ preference orderings, but not all of these correspond to distinct games. It turns out that there are exactly 78 distinct two-person games of which only three are constant-sum games [16].

If $a_1 = a_2$, $b_1 = c_2$, $c_1 = b_2$, and $d_1 = d_2$, the game is symmetric. There are 24 symmetric two-person games if each payoff is different. For such a game we can simplify the matrix. Let us call the C choice *cooperation* and the D choice *defection*. R is the reward for mutual cooperation and P is the punishment for mutual defection. T is the

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temptation to defect when the other player cooperates, and S is the sucker's payoff for cooperating when the other player defects. Then:

$$\begin{array}{ccc} & C & D \\ C & R, R & S, T \\ D & T, S & P, P \end{array}$$

Matrix 2.

Note that these names and notations are arbitrary. C and D can be any two actions and the payoffs can have any value.

Among the 24 games, the following are especially well known and important:

- Prisoners' dilemma: $T > R > P > S$ (equilibrium: DD)
- Chicken: $T > R > S > P$ (equilibria: CD and DC)
- Benevolent chicken: $T > S > R > P$ (equilibria: CD and DC)
- Leader: $S > T > R > P$ (equilibria: CD and DC)
- Battle of the sexes: $T > S > P > R$ (equilibria: CD and DC)
- Stag hunt: $R > T > P > S$ (equilibria: CC and DD)
- Deadlock: $T > P > R > S$ (equilibrium: DD)

An equilibrium means that if either of the two players abandon this situation, their position will not improve.

Let us call those games where the temptation's payoff is greater than the reward that is greater than the punishment ($T > R > P$) *dilemmas*. The first four games satisfy this condition. In these games, defection is preferred to cooperation in spite of the fact that cooperation would lead to the most advantageous situation for both players.

Poundstone's definition [17] is different from this one. He calls a game a dilemma if it satisfies the following two conditions: $R > S$ and $T > P$. This is true for six games. From the provided list, four games satisfy both conditions: prisoners' dilemma, chicken, stag hunt, and deadlock. Only the first two are dilemmas by both definitions. Let us look at both of them.

In prisoners' dilemma, no matter what the other player does, both players' best choice is defection. As a result, mutual defection is the equilibrium. It is a dilemma because mutual cooperation would be better for both players.

In the case of chicken, mutual defection is the worst possible outcome that both players try to avoid, but they do not want to be "chickens" either. Therefore, each will do the opposite of their partner's choice.

3. *N*-Person Games

In social and economic games there are more than two participants (the number of participants is N where $N > 2$). These are usually not constant-sum games: every participant can win as in a regulated free-market economy or lose as during market crashes. The participants are almost never rational because even if they are extremely intelligent, it is impossible to calculate the payoffs of all of their choices.

There are additional major differences between two-person and N-person games. In two-person games each participant's choice is clear. If the number of participants is high, defection is anonymous and the harm caused by defections is diffused. Rewards and punishments are dependent on the ratio of cooperators among the participants, that is, on the statistical distribution of the participants' choices.

One of the best examples of N-person games is the tragedy of the commons [8]. There is a pasture open to all herdsmen of a village. They use it to feed their cattle. The pasture can sustain a certain number of animals. The tragedy develops when the herdsmen all try to maximize their gains by adding more animals to their herds. This is a typical N-person prisoners' dilemma. To add more animals always seems to be advantageous (defection is a dominant strategy), but the result is that all animals die because of the finite size of the overcrowded pasture. Freedom in the commons brings ruin to all. Everyone rushes toward ruin pursuing their seemingly best interests.

Fortunately, not all people behave like this. They have different personalities. This may lead to equilibria that are different from total ruin. We can investigate the role of personalities by agent-based modeling.

At a first glance, an N-person game looks like a well-defined problem. However, at least the following questions arise immediately [18]:

1. Who are the players?

   They can be people, groups, organizations, departments, cities, countries, nerve cells, ants, computers, or anything else. They can be very simple or enormously complex. Their common features are that they are autonomous (they have no controller above them) and are interconnected with each other. It is generally accepted to call them *agents*.

2. What is the goal of the game?

   The agents may try to maximize their payoffs, to win a competition, to do better than their neighbors, to behave like the majority, or may have any other goal. This is a critical issue. In real-life situations, different agents have different goals. It is also possible that the agents simply react to their and their neighbors' payoffs without specific goals.

3. What are the agents' options?

   The agents must choose between different available options. In the simplest case there are only two options, for example, cooperate or defect, go to a bar or stay at home, and so forth. In real-life situations, more options are available or the agents even have to choose from a

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continuous spectrum of options, for example, how much to contribute to charity.

4. Do the agents act simultaneously or are the actions distributed in time?

   There is a huge difference between simultaneous actions and actions distributed in time. In the first case, all agents see the same environment at the moment of their simultaneous actions. In most social settings, however, agents act at different and uncorrelated times. Therefore, each agent sees a slightly different world than another agent who acts at a slightly different time [19]. Simulation of this case is a more sophisticated task than that of the previous case.

5. What do they know? Can individual agents see and adapt to the actions of others?

   Even if the agents' actions are distributed in time, they may or may not have information about the actions of others. You may look out of the window and see how many cars are on the road before deciding if you are going to drive your car or take a bus, but you do not know how many children will be born next year before deciding if you are going to have another child.

6. Can they talk to each other? Can they form coalitions?

   Obviously, if they do not know the other participating agents, they cannot form coalitions with them. Even if they know one another, it is not certain that they can communicate, let alone form a coalition. However, coalitions may drastically change the outcome of the game.

7. Is it a one-shot game or an iterated one?

   The one-shot game is less interesting than an iterated one where the agents act repeatedly based on their personalities, their neighbors' situations, and the payoffs received for their previous actions.

8. If it is an iterated game, how are the next actions determined?

   The next choices are determined by updating schemes that are different for different agents.

9. When is an iterated game over?

   There may be a predefined number of iterations or the game ends when an important parameter (e.g., the ratio of cooperators) reaches a constant value or oscillates around such a value. This value is the *solution* of the iterated game.

10. How are the agents distributed in space and time?

    The agents may be distributed in space and time in many different ways.

11. Can the agents move?

    If there are fewer agents than locations in space or if more than one agent may occupy one location, then it is possible that the agents move around in space and their neighborhood constantly changes.

12. Can an agent refuse participation in the game?

    The iterated game may considerably change if an agent may refuse participation in some iteration. It is impossible in chess, but Kutuzov beat Napoleon by avoiding battles.

13. How do the agents react to the reward/penalty received for their actions?

When everything else is fixed, the payoff curves determine the game (see Section 5). There is an infinite variety of payoff curves. In addition, stochastic factors can be specified to represent stochastic responses from the environment. The stochastic factors determine the thickness of the payoff curves. Zero stochastic factors mean a deterministic environment. Even in the almost-trivial case when both payoff curves are straight lines and the stochastic factors are both zero, four parameters specify the environment. Attempts to describe it with a single variable are certainly too simplistic [13, 20]. As shown later, the relative position of the two payoff curves with respect to one another alone does not always determine the outcome of the game. Ordinal preference is not enough to represent the payoff functions: the actual amount of reward and punishment may be as important as the relative situation of the two curves.

The N-person game is a *compound* game (it can be reduced to a series of two-person games) if and only if both payoff functions are linear [4]. Therefore, a dyadic tournament where every agent plays two-person games against each of the $N-1$ other agents [9] represents only a limited subset of N-person games.

14. Are the payoff functions the same for all agents?

    In this case, the game is called *uniform*, but not all games are uniform. This condition is not always guaranteed.

15. What are the personalities of the agents?

    Different people react quite differently to the same situations. The personalities of the agents are one of the most important characteristics of the game. Personalities may represent genetic as well as cultural differences among them.

    The psychological literature on the impact of personalities in social dilemmas is summarized in [21]. It is possible but not easy to quantify personality profiles in the traditional psychological sense. We will use the term "personality" in the sense of decision heuristics (repeated-game strategies) in this paper to represent the fact that different agents react differently to the same stimulus from their environment. This is a rather primitive approach but it is still much better than the unjustified assumption of a uniform response.

    Personalities are usually neglected in the literature. In our research, different agents may have quite different personalities in the same experiment. Personalities may be represented in many different ways (see Section 4).

16. Can the agents change their personalities during the game?

    In the case of adaptive agents, they may change their personalities by learning from their mistakes during the game or because of the influences of other agents.

17. How is the total payoff to all agents related to the number of cooperators?

    The total payoff to all agents is related to the number of cooperators, but the maximum collective payoff is usually not at maximum cooperation.

18. Do the agents interact with everyone else or just with their immediate neighbors?

    The agents may interact with everyone else or just with their neighbors. In the latter case, they behave like cellular automata [22].

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19. How is a neighborhood defined?

    The number of neighborhood layers around each agent and the agent's location determine the number of its neighbors. The depth of agent A's neighborhood is defined as the maximum distance, in three orthogonal directions, that agent B can be from agent A and still be in its neighborhood. An agent at the edge or in the corner of the available space has fewer neighbors than one in the middle. The neighborhood may extend to the entire array of agents.

With so many open questions, it is obviously quite difficult to create a general classification scheme for N-person games and there is a great variety of possible games. They constitute, in fact, a whole family of quite different games. Even in the case of a uniform game with all other parameters fixed, the number of possible variations is infinitely large because of the infinite variety of the payoff curves.

There are an infinite number of different N-person games!

The first serious investigation of an N-person game was Schelling's segregation model [23]. He also introduced the graphical representation of the payoff functions [5]. Hamburger [4] gave a thorough mathematical description of these games. Axelrod's tournaments [9] contributed to the widespread interest in such games.

The effects of alternative strategies on achieving consensus for action were simulated by Feinberg and Johnson [24]. A stochastic learning model was developed by Macy [25] to explain critical states where threshold effects may cause the system to shift from a defective equilibrium to a cooperative one. A computer simulation of temporary gatherings was presented by McPhail et al. [26]. Glance and Huberman [27] used a thermodynamical model to investigate outbreaks of cooperation in a social system. Nowak and May [13] and Lloyd [14] wrote simple computer programs that demonstrated the dynamics of deterministic social behavior based on pair-wise interactions between the participants.

In most of these papers, the simulation is restricted to narrowly defined special cases. For example, in [13] the agents interact with each other only pair-wise, they all have the same predefined personality, and the environment is deterministic and defined through one single parameter.

Epstein and Axtell [28] demonstrated that it is possible to build complex artificial societies based on simple participating agents. This work is still one of the best examples of agent-based modeling.

4. Agent Personalities

We can represent agent personalities in agent-based modeling using different approaches.

1. The model presented in Section 6 includes the following personalities:

   (a) Pavlovian (the probability of choosing the previously chosen action again changes by an amount proportional to the reward/penalty for

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the previous action; the coefficient of proportionality is called the learning rate; of course, the probabilities always remain in the interval between 0 and 1). This personality is based on Pavlov's experiments and Thorndike's law [29]: if an action is followed by a satisfactory state of affairs, then the tendency of the agent to produce that particular action is reinforced. These agents are primitive enough not to know anything about their rational choices, but they have enough "intelligence" to learn a behavior according to Thorndike's law. Kraines and Kraines [30], Macy [31], Flache and Hegselmann [32], and others used such agents for the investigation of iterated two-person games.

(b) Greedy (imitates the neighbor with the highest reward).

(c) Conformist (imitates the action of the majority of its neighbors).

(d) Accountant (acts according to the average payoffs for past actions).

(e) Statistically "predictable" (constant probability p of cooperation):

- Angry agents with short-term rationality always defect ($p = 0$).
- Benevolent agents ignore their short-term interests and always cooperate ($p = 1$).
- Unpredictable agents act randomly ($p = 0.5$).

(f) Any combination of these in the society of agents.

2. The Big Five approach characterizes personalities by the proportions of the following characteristics:

   - Openness
   - Conscientiousness
   - Extroversion
   - Agreeableness
   - Neuroticism

   This approach was used in our study of standing ovation [33].

3. The Markov chain model uses the following propensities:

   - trustworthiness (probability of choosing cooperation after receiving R),
   - forgiveness (probability of choosing cooperation after receiving S),
   - repentance (probability of choosing cooperation after receiving T), and
   - trust (probability of choosing cooperation after receiving P).

4. The most interesting approach is to represent personalities on the basis of 16 mental disorders. Oldham and Morris [34] believe that a normal person is a combination of these disorders (listed in parentheses in the following table). The disorders are just extremes of normal human personality patterns listed in Table 1.

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If every person is a combination of these patterns, it is very easy to represent personalities by just 16 numbers showing the proportions of the individual patterns.

Adventurous (Antisocial)	Self-Confident (Narcissistic)
Aggressive (Sadistic)	Self-Sacrificing (Self-Defeating)
Conscientious (Obsessive-Compulsive)	Sensitive (Avoidant)
Devoted (Dependent)	Solitary (Schizoid)
Dramatic (Histrionic)	Vigilant (Paranoid)
Idiosyncratic (Schizotypal)	Exuberant (Cyclothymiac)
Leisurely (Passive-Aggressive)	Serious (Depressive)
Mercurial (Borderline)	Inventive (Compensatory Narcissistic)

Table 1.

5. Transition from Two-Person to *N*-Person Games

As we have seen, any two-person game can be represented by a two-dimensional 2×2 matrix. Accordingly, a three-person game would require a three-dimensional $3 \times 3 \times 3$ matrix, a four-person game a four-dimensional $4 \times 4 \times 4 \times 4$ matrix, and so on. Obviously, this is not a feasible approach to study multi-person games.

To overcome this difficulty, let us replace the two-dimensional Matrix 2 with Figure 1.

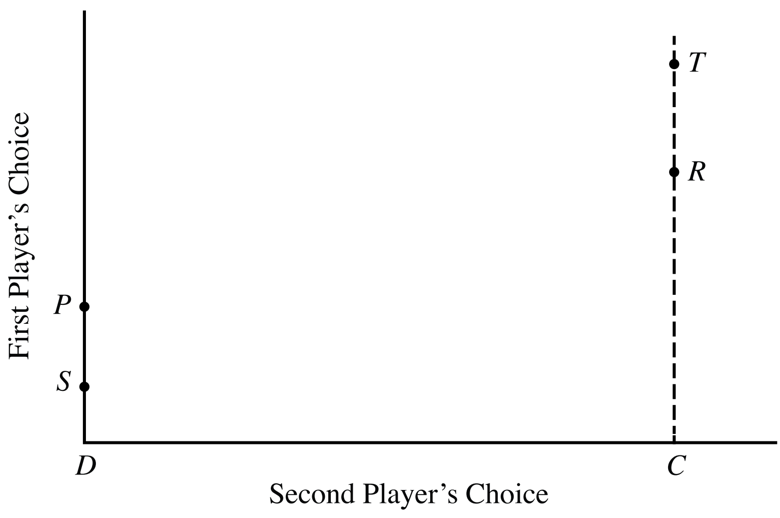

Figure 1.

Now replace the second player choice with the ratio of cooperators among other players x. As there is only one other player, Figure 2 contains exactly the same information as Figure 1. The only difference is that we have added two lines: one connects points P and T; the other connects points S and R. The former is called the defectors' payoff function $D(x)$; the latter is the cooperators' payoff function $C(x)$. These functions do not have to be linear.

Figure 2 can be applied to any number of players. This is the general representation of N-person games. If N is a large number, we can say that x is simply the ratio of cooperators among all players. This is not exactly true, but for large N the error is hardly noticeable.

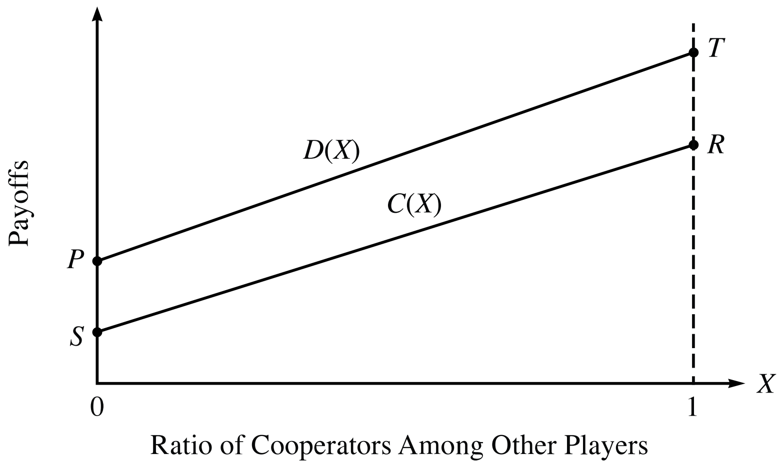

Figure 2.

6. The Model

The first task before building a model is to clearly define the object of the model. In the case of N-person games, we must answer the questions listed in Section 3.

1. The players are agents.

2. The goal of the game is to find out its outcome for a large number of agents.

3. Each of the N agents has a choice between two actions. It is convenient to call these actions cooperation and defection (cooperating with each other for the "common good" or defecting by following their selfish short-term interests), but these names do not restrict the games to this particular example.

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4. The agents act simultaneously.

5. They get information about other agents' choices only through their payoffs.

6. They cannot talk to each other and cannot form coalitions.

7. The game is iterated.

8. The next actions are decided by the updating schemes that are dependent on the agents' personalities.

9. The iterated game is over when a solution is reached (the ratio of cooperators reaches a constant value or oscillates around such a value).

10. The agents are distributed in a two-dimensional array (as cells in a cellular automaton).

11. They cannot move.

12. No agent can refuse participation in the game.

14. The payoff functions are the same for all agents (uniform game).

16. The agents cannot change their personalities during the game.

17. The relationship between the total payoff to the number of cooperators will be discussed later.

We have not answered questions 13, 15, 18, and 19. These will be the user-defined parameters of the game.

13. The user-defined payoff functions determine the outcome of the game. Each agent receives a reward or punishment (payoff) that is dependent on its choice as well as everybody else's (see Figure 2).

15. The personalities of the agents can be chosen according to the first representation of Section 4.

18. The agents can interact with everyone else or with their neighbors of arbitrary depth.

19. The depth of agent A's neighborhood is defined as the maximum distance, in three orthogonal directions, that agent B can be from agent A and still be in its neighborhood.

Thus, to make our task manageable, we assume that the game is uniform and iterated, the agents have no goals, they know nothing about each other, and they cannot refuse participation in any iteration. They are distributed in and fully occupy a finite two-dimensional space and the updates are simultaneous. These restrictions leave the payoff functions, the personalities, and the neighborhood open for investigation. The last two factors are mostly neglected in the literature. In addition, the user has to define the number of agents and their initial states. It should be noted that even if only these factors are considered, there is still an infinite number of different variations of games because of the infinite variety of payoff functions.

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Agent-based computer simulation is the best way to investigate the role of the above-mentioned factors on the outcome of the game. We have developed an agent-based model, *Dilemma*, for the investigation of N-person games with a large number of agents operating in a stochastic environment [15]. Our model has three distinctive features:

1. It is a genuine multi-agent model and not a model for repeated two-person games.
2. It is a general framework for inquiry in which the properties of the environment as well as those of the agents are user-defined parameters and the number of interacting agents is theoretically unlimited.
3. Our agents have various distinct, user-defined personalities.

The participating agents are described as stochastic learning cellular automata, that is, as combinations of cellular automata [22, 35] and stochastic learning automata [36, 37].

A cellular automaton is a discrete dynamic unit whose behavior is specified in a simple way in terms of its local relation to the behavior of its neighbors, that is, the behavior of each unit depends on its own and its neighbors' states. A good example is the Game of Life [38], which shows enormous complexity on the collective level generated by trivial rules for the participating units. The cellular automaton approach is gaining popularity for the simulation of complex nonlinear physical, economic, and social problems as shown by the appearance of popular articles in journals ranging from *Scientific American* [39] to *Forbes Magazine* [40]. Experiments with various cellular automata models confirmed that even trivial, deterministic rules can produce extremely complicated and unforeseeable collective behavior [13]. The cellular automaton format describes the environment in which the agents interact. In our model, this environment is not limited to the agents' immediate neighbors: the agents may interact with a user-defined number of other agents simultaneously.

Stochastic learning rules provide more powerful and realistic results than the deterministic rules usually used in cellular automata. Stochastic learning means that behavior is not determined but only shaped by its consequences, that is, an action of the agent will be more probable but still not certain after a favorable response from the environment. A stochastic learning automaton is a unit characterized by probability distributions for a number of possible actions. The units are connected to the stochastic outside environment. A stochastic reward/penalty is the only input that the units receive from the environment. The asymptotic behavior of collectives of such units can be established by using the theory of Markov processes. The probabilities of the agents' actions are updated by the reward/penalty received from the environment based on their and other agents' behavior. Actions are taken according to these probabilities. The outputs of the stochastic environment are influenced by the actions of all participating units whose existence may not even be known to the other units.

Behavior is *learned* by adjusting the action probabilities to the responses of the environment. The learning capability alters the agents' behavior as they make repeated decisions. The aggregate behavior of the society of agents usually converges to a stable or oscillating state.

We will now briefly explain the model's most important features.

Our simulation environment is a two-dimensional array of the participating agents. The size and shape of the environment are also user-defined variables. Its size is limited only by the computer's virtual memory. The behavior of a few million interacting agents can easily be observed on the computer's screen. A possible special case is a linear environment that consists of at least two participants. (This limiting case is important because it makes the investigation of two-agent games possible.)

Our model is a new approach to nonlinear stochastic dynamic system modeling. The participating agents are combinations of the two types of automata described above. The size of the neighborhood is a user-defined variable. It may be just the immediate Moore or Neumann neighborhood or any number of layers of agents around the given agent. In the limiting case, all agents are considered neighbors, and they collectively form the environment for each participating agent.

The number of neighbors depends on the definition of the neighborhood but also on the location of the agent: an agent in the corner of the array has fewer neighbors than one in the middle. The model would have been made simpler by wrapping the array but we decided against this. A model with edges makes it possible to simulate cities or other confined environments.

The payoff (reward/penalty) functions are given as two curves: they specify the reward/penalty for a cooperator and a defector, respectively, as functions of the number of cooperators related to the total number of neighbors (Figure 2). Each curve may include a constant stochastic factor that is multiplied by a random number to make the response from the environment stochastic. The stochastic factors determine the thickness of the payoff functions (Figure 3). Zero stochastic factor means a deterministic environment.

The payoff (reward/penalty) functions are given as two curves: $C(x)$ for cooperators and $D(x)$ for defectors. The payoff to each agent depends on its choice, on the distribution of other players among cooperators and defectors, and on the properties of the environment. The payoff curves y are functions of the ratio x of cooperators to the total number of neighbors. In the original version of the model, the payoff functions are determined by quadratic functions, but a later version [41] allows any function. The freedom of using arbitrary functions for the determination of the reward/penalty system makes it possible to simulate a wide range of N-person games, including those where the two curves intersect each other [5].

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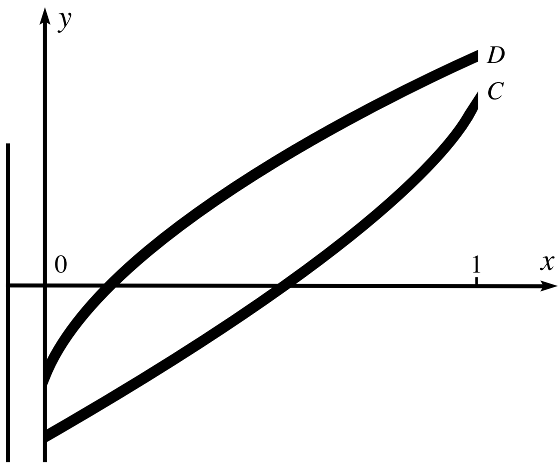

Figure 3.

In an iterative game the aggregate cooperation proportion changes in time, that is, over subsequent iterations. The agents take actions according to probabilities updated on the basis of the reward/penalty received for their previous actions and of their personalities. The user-defined specific probability updating schemes depend on the agents' personalities. They specify the change in the probability of choosing the previously chosen action based on a number of factors such as the reward/penalty received for that action, the history of rewards/penalties received for all actions, the neighbors' actions, and so on.

The updating scheme is different for different agents. Agents with different personalities can interact with each other in the same experiment. Agents with various personalities and various initial states and actions can be placed anywhere in the array. The response of the environment is influenced by the actions of all participating agents.

The updated probabilities lead to new decisions by the agents that are rewarded/penalized by the environment. With each iteration, the software tool draws the array of agents in a window on the computer's screen, with each agent in the array colored according to its most recent action. The experimenter can view and record the evolution of the behavior of any agent and of the society of agents as they change in time.

The outcome of the game depends on the values of the parameters of the game. *Dilemma* reads a text configuration file when it starts. The contents of this file determine all the parameters of the experiment:

- number of rows and columns in the array of agents;

- size and coloration of the agents;
- depth of the neighborhood for each agent;
- payoff curves for cooperators and defectors;
- personalities of individual agents at various locations in the array;
- initial actions of individual agents at various locations in the array; and
- initial states of individual agents at various locations in the array.

Having the experiment defined in a configuration file has significant benefits for the experimenter. This way, experiments can be rerun, shared with other experimenters, or archived and indexed in a natural fashion. This approach is unusual in nonlinear dynamic system simulation.

When *Dilemma* is run, it displays two windows, labeled "graphics output" and "status display". The graphics output window represents all the agents in the simulation and their most recent actions. The user can ask for a four-color display, which shows the current and previous action for each agent, or a black-and-white display, which shows only the current action for each agent. The status display window provides more detailed textual information about individual agents. The experimenter selects the agent to be examined in detail by pointing at it with the mouse. This information includes the agent's coordinates in the array, the agent's two most recent actions, the agent's personality, the last reward or punishment that the agent received, and the agent's current probability of cooperation. As the mouse pointer moves and as iterations are run, both windows are continuously updated. When the experimenter stops the simulation, the history of aggregate cooperation proportions for each iteration is presented as a list of numerical values as well as an automatically generated plot.

If the parameters are selected appropriately, the simulation will exhibit behavior that is close enough to the behavior of real people when they are placed in a similar situation (see, e.g., [42]).

This model was later extended in two directions. The payoffs can now be represented by any function [41] and a continuous model is available [43]. In the latter model, continuous attitude states are used as options for each player instead of just two choices. In addition, degrees of persuasion and propaganda are included as new personality treats.

7. *N*-Person Prisoners' Dilemma

Various aspects of the multi-person prisoners' dilemma have been investigated in the literature [2, 4–6, 13, 21, 44–55]. This game has great practical importance because its study may lead to a better understanding of the factors stimulating or inhibiting cooperative behavior within social systems.

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The dilemma can be formulated by the following two statements [49]:

1. Regardless of what the other agents do, each agent receives a higher payoff for defecting behavior than for cooperating behavior.
2. All agents receive a lower payoff if all defect than if all cooperate.

The condition $T > R > P > S$ satisfies both requirements; therefore, Figure 2 represents this game. The game has $N+1$ distinguishable outcomes: 0, 1, 2, …, $N-1$, or N participants may choose cooperation.

Each rational player will choose defection and, as a result, everyone loses. Fortunately, human beings are rarely rational. We do not buy stocks for their value but rather because other people buy them or because of a statement by a celebrity. The apparent unpredictability of people is made up of some random behavior plus elements of quite predictable behavior because of a variety of personalities. *Dilemma* makes it possible to get a handle on how to understand the aggregate behaviors in terms of what we know about personality characteristics, the distribution of these characteristics in various populations, and the effects of these characteristics on an agent's desire to compete or cooperate.

The N-person prisoners' dilemma is very important because it represents many real-life applications. The tragedy of the commons [8], for example, clearly shows the dangers of overpopulation.

We have investigated this game for agents with various personalities.

7.1 Pavlovian Agents

Let us assume that the payoff functions are linear and parallel to each other, and their equations are $D(x) = -0.5 + 2x$ and $C(x) = -1 + 2x$ [18]. These functions are shown in Figure 4. The graphics output of Figure 5 represents the initial configuration for the case when the initial actions of all agents are random and their initial probability of cooperation is equal to 0.5. We see an approximately equal number of black (cooperator) and white (defector) spots. Figure 6 shows the graphics output after the 100th iteration with fewer but well above zero cooperators.

The initial state of the system is one of the decisive factors that determine its future state. This can be clearly seen from Figure 7, which shows the evolution of a society for the case when the payoff curves are given by Figure 4 and the neighborhood is the entire society of agents. The graphs show the proportions of cooperating agents for different initial cooperation probabilities as functions of the number of iterations. The initial values of the aggregate cooperation probabilities are shown at the curves.

We can see that there are two distinctly different solutions. When the initial values are in the range between 0 and 0.69, all solutions oscillate around 0.18. When the initial aggregate cooperation probabil-

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ity is above 0.695, the solutions tend to reach well-defined constant values that are dependent on the initial values. These numerical values exactly satisfy equation (1) (see Section 13).

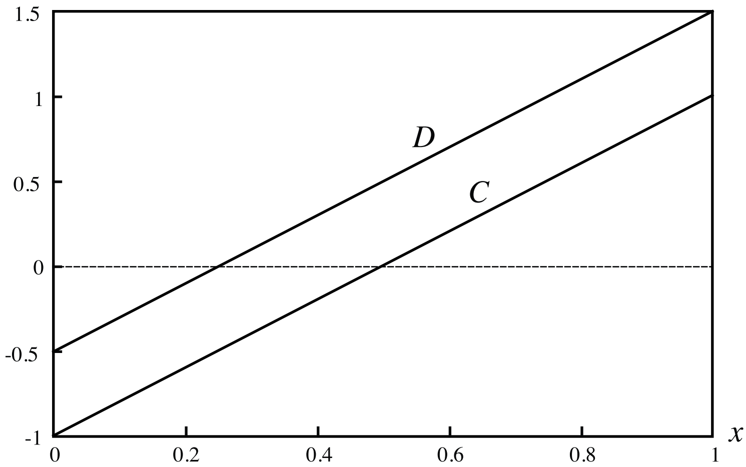

Figure 4.

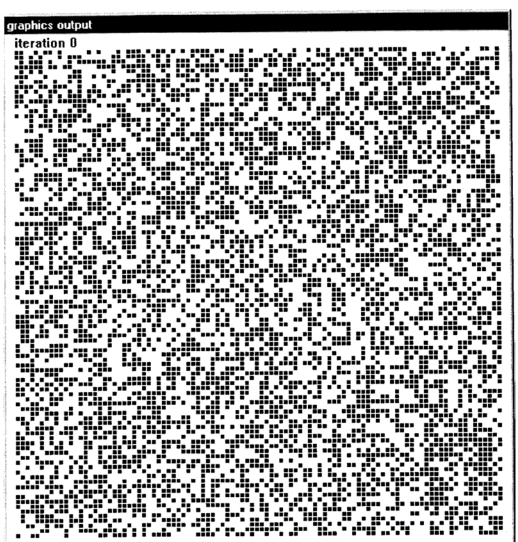

Figure 5.

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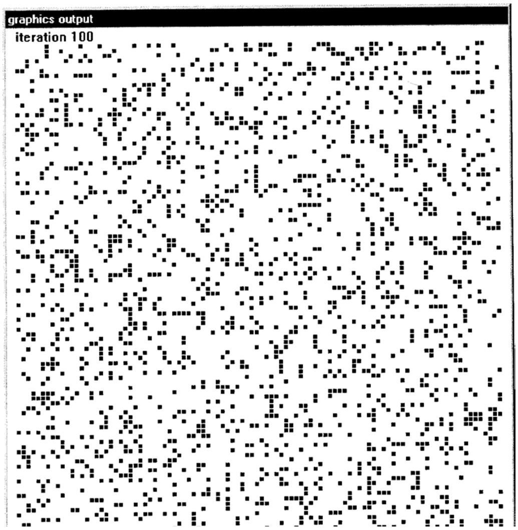

Figure 6.

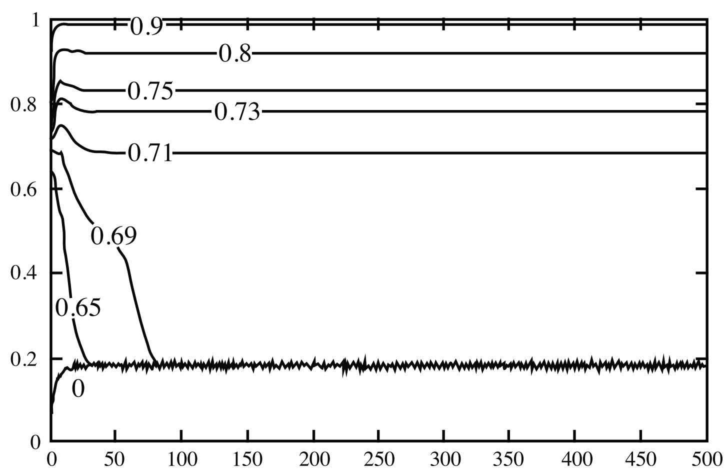

Figure 7.

This result certainly satisfies the definition of chaos as sensitive dependence on initial conditions. It means that a perturbation to the initial state of the system will cause the system to evolve into a different future state within a finite period of time. Thus, a very small difference in the initial cooperation ratio around $x = 0.69$ leads to totally different solutions.

The solutions are similar but somewhat different for the case when the neighborhood is only one layer deep (Figure 8).

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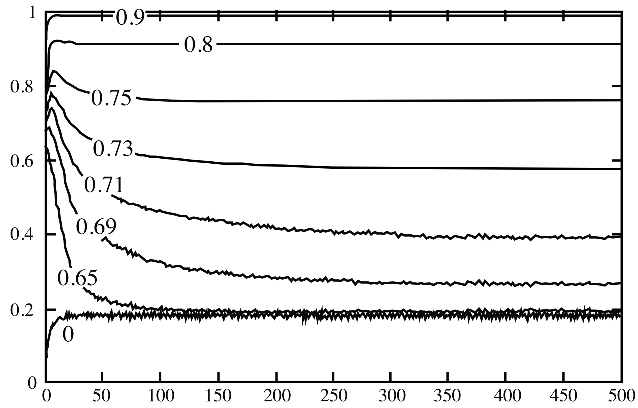

Figure 8.

The following simulations take non-uniform distributions of the agents into account [55]. Figure 9 shows the graphics output of the initial configuration for the case when the initial actions of all agents are random but the society is equally divided into two parts: agents in the upper half initially defect, while those in the lower half initially cooperate.

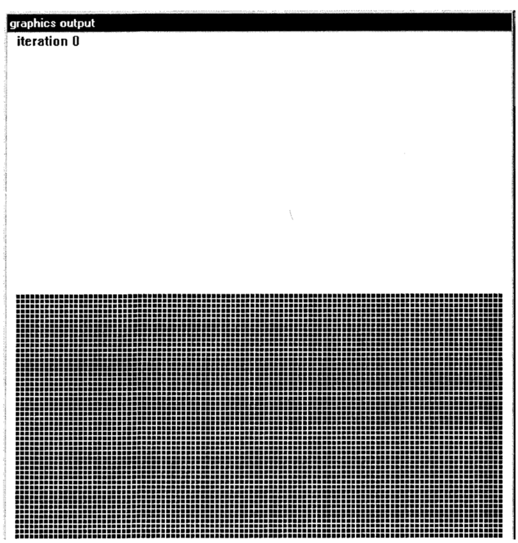

Figure 9.

If the neighborhood is one layer deep, the upper half will be gradually infected with cooperators (Figure 10).

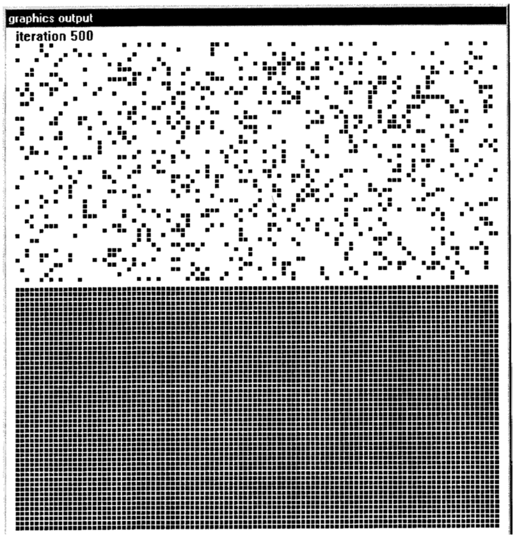

Figure 10.

As the neighborhood depth is increased, a protective layer is formed where no cooperation occurs (Figure 11).

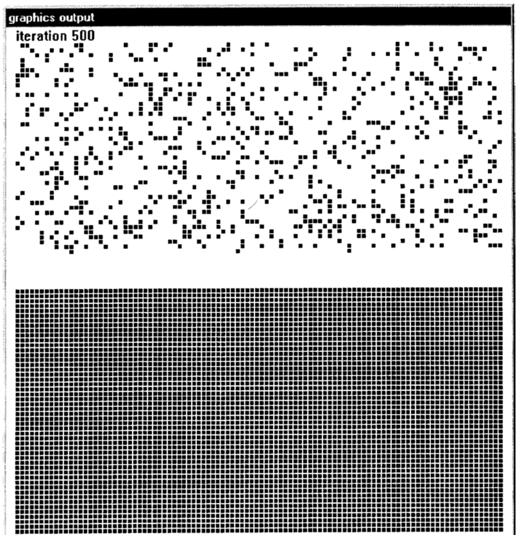

Figure 11.

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The situation is completely different when the neighborhood is the entire society. In this case, change starts in the lower region (Figure 12) and it spreads into the entire society (Figure 13).

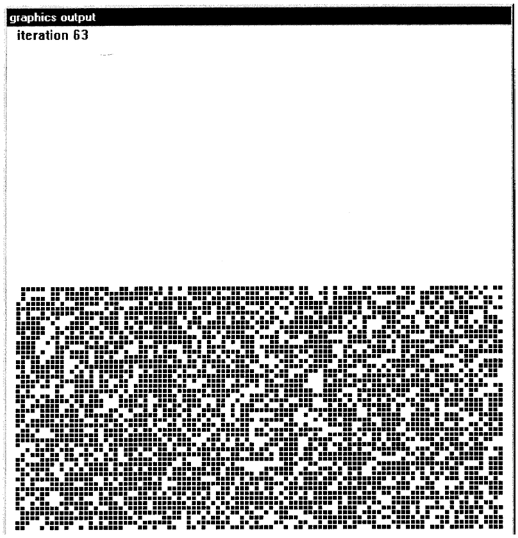

Figure 12.

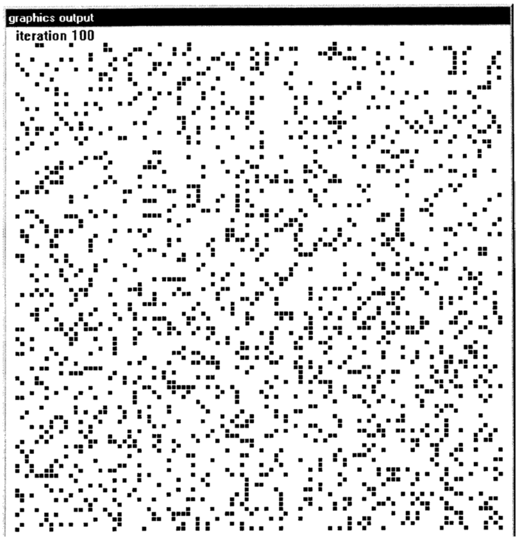

Figure 13.

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7.2 Accountants

The accountant's payoff depends on the average reward for its previous actions. If initially the number of cooperators is approximately equal to the number of defectors for a one-layer-deep neighborhood, the result is universal defection because the defectors' payoff is always higher than that of the cooperators. If, however, the initial distribution is unequal, clusters will form. Agents situated at the borders of cooperative clusters will receive smaller and smaller payoffs. As a result, they will eventually defect; these clusters become smaller and smaller, and after several thousand iterations universal defection takes over.

7.3 Greedy Agents

The greedy agent always imitates the behavior of the neighbor with the highest reward (this is the case investigated for dyadic interactions by [13]). If all agents are greedy and the neighborhood extends to the entire organization, they will all defect immediately at the first iteration because they will all imitate the defectors who received higher rewards than the cooperators for their initial action. The situation is not so hopeless for a one-layer-deep neighborhood: the behavior will stabilize with a relatively small number of cooperators.

If we allow a small number of individual defectors to initially randomly distribute among cooperators, they can produce interesting patterns like those shown in Figure 14 for the case of $C(x) = 5x - 1$ and $D(x) = 5x - 0.5$. The neighborhood is one layer deep. The initial ratio of cooperators is equal to 0.90; the final ratio oscillates between 0.91 and 0.92.

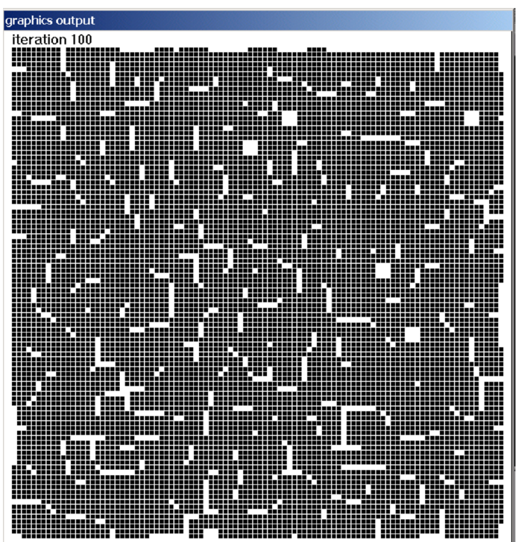

Figure 14.

Beautiful oscillating symmetric fractal patterns may arise when a single defector initially occupies the middle spot in a sea of greedy cooperators with a one-layer-deep neighborhood (Figure 15). It is instructional to investigate the emergence of these patterns. As the $D(x)$ curve is always above the $C(x)$ curve, a layer of defectors will surround the lonely defector after the first iteration. After the second iteration, however, the further development depends on the actual shapes of the payoff curves. Accordingly, the result may be universal defection, a small stable defection pattern around the center, oscillation in the same region, or the symmetric oscillating pattern of Figure 15. Here $D(x) = 1.65\,x$ and $C(x) = x$. The stochastic factor is zero. The initial configuration is a single defector in the middle of 6560 cooperators. The picture shows the situation after 1000 iterations. In Figure 16, we can see the wildly fluctuating proportion of cooperating agents as a function of the number of iterations.

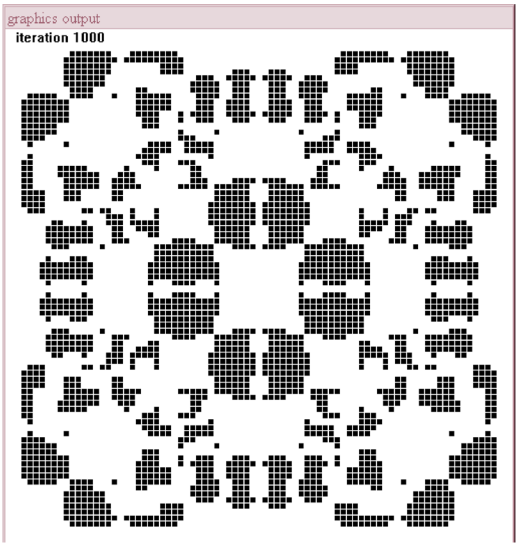

Figure 15.

7.4 Conformists

The conformist agent does not care about the payoff functions. It simply imitates the action of the majority. If all agents are conformists and the neighborhood extends to the entire society of agents, then the outcome depends on the exact relationship between the initial number of cooperators and defectors: every agent will immediately imitate the majority and stay there. The behavior becomes quite interesting for the one-layer-deep neighborhood. In this case, while the proportion of cooperators will not change substantially, their distribution will. Both

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cooperators (black spots) and defectors (white spots) will form mutually intertwined clusters (Figure 17).

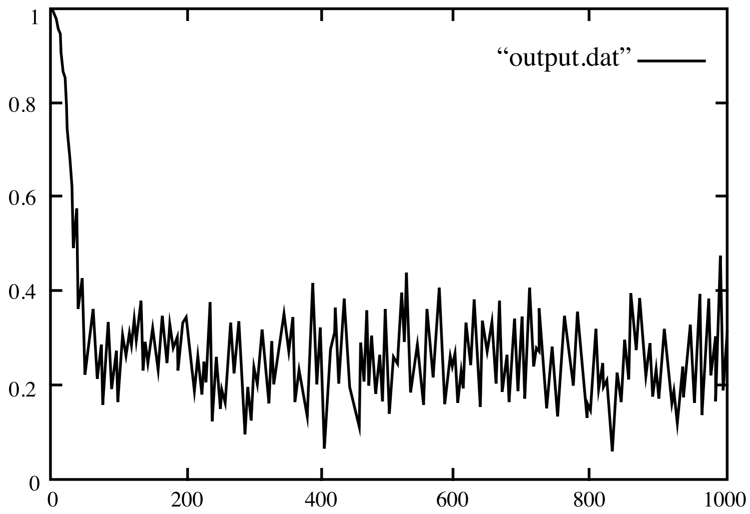

Figure 16.

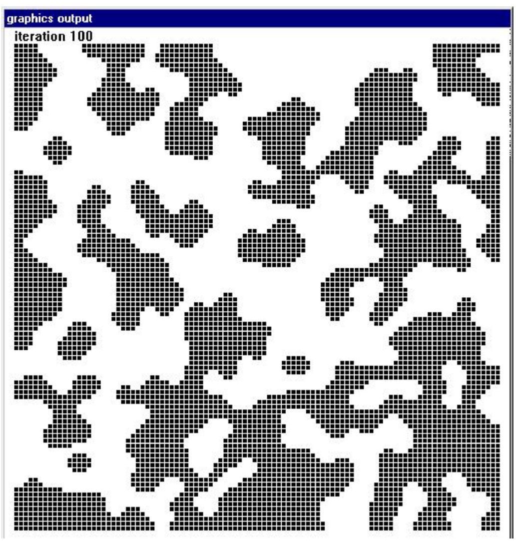

Figure 17.

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7.5 Mixed Personalities

The number of variations is infinitely large. We can change all the parameters simultaneously and mix different personalities in arbitrary ways. Figure 18 shows the evolution of the game for the case when the payoff curves are given by $C(x) = x$ and $D(x) = 1.65\,x$ and the neighborhood is one layer deep. The graphs show the proportions of cooperating agents as functions of the number of iterations. The lower solid curve corresponds to the case when 97% of the agents are greedy and 3% are Pavlovian. For the middle dotted curve 97% of the agents are greedy and 3% are conformists. In the case of the upper solid curve, 45% of the agents are greedy, 45% of them are conformists, and 10% are Pavlovian. The initial cooperation ratio is equal to 0.9.

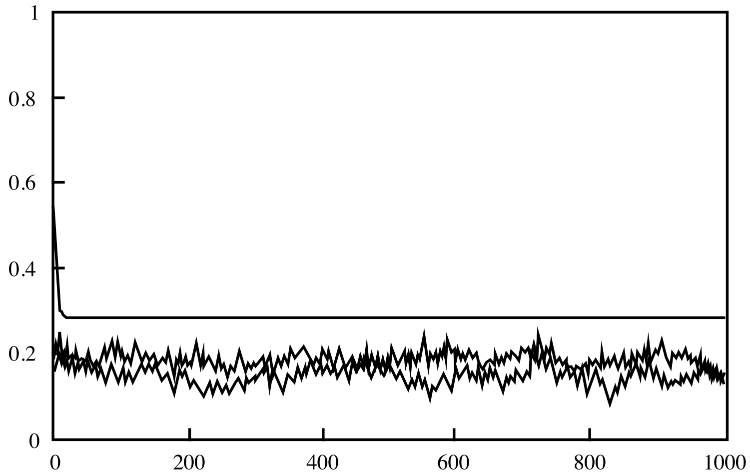

Figure 18.

8. *N*-Person Chicken Dilemma

The chicken game is an interesting social dilemma. When two people agree to drive their cars toward one another, each has two choices: drive straight ahead or swerve. If both swerve (a mutually cooperative behavior), they both receive a certain reward R. If both go ahead (mutual defection), they are both severely punished (P). The dilemma arises from the fact that if one of them swerves (chickens out) but the other does not, then the cooperator receives a sucker's payoff S while the defector gets a high reward for following the temptation T. In this game $P < S < R < T$.

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In the prisoners' dilemma game, defection dominates cooperation: regardless of what the other player does, each player receives a higher payoff for defecting behavior (D) than for cooperating behavior (C), in spite of the fact that both players are better off if they both cooperate than if both defect ($S < P < R < T$). In the chicken game there is no domination. The D choice yields a higher payoff than the C choice if the other person chooses C, but the payoff is less for D than for C if the other player's choice is D. Mutual defection is the worst outcome for both players. Consequently, cooperation is more likely in chicken than in prisoners' dilemma. Evidently, cooperation grows with the severity of punishment for mutual defection. A good example of the chicken dilemma is the Cold War, when mutual defection would have led to a nuclear catastrophe.

The two-person chicken game has received some attention in the literature [17, 21, 56] but the N-person game had been neglected prior to our research [57–59].

The payoff curves are functions of the ratio of cooperators to the total number of agents. Even for linear payoff functions, we have four free parameters that determine the payoff functions that have the following properties: (1) Both payoff functions increase with the increasing number of cooperators. (2) In the region of low cooperation, the cooperators have a higher reward than the defectors. (3) When the cooperation rate is high, there is a higher payoff for defecting behavior than for cooperating behavior. (4) As a consequence, the slope of the D function is greater than that of the C function and the two payoff curves intersect each other. (5) All agents receive a lower payoff if all defect than if all cooperate. Typical payoff functions for the chicken game are shown in Figure 19.

8.1 Pavlovian Agents

Naturally, the results of the simulation strongly depend on the values of the parameters. The four-dimensional problem of linear payoff functions can only be handled by a systematic variation of the parameters. For Pavlovian agents, the actual values of the parameters are important, not only their relative values.

Let us assume that the total number of agents is 10 000 and the initial ratio of cooperators is 50% [57].

We start with the payoff functions with the parameter values as $S = -3$, $P = -13$, $R = 0$, $T = 1$. The result of the simulation in this case is that after a relatively small number of iterations, the ratio of cooperators starts oscillating about the stable value of 76%. The actual number of iterations needed for the oscillation to start depends on the learning rate used in the probability updates.

Let us first vary one parameter at a time. If we increase the value of T, the ratio of cooperators steadily decreases because the defectors receive a higher reward when there are more cooperators.

Increasing the value of R drastically increases the cooperation ratio because a larger and larger part of the C function is above the D func-

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tion. At $R = 0.4$, we already reach a point when no one defects anymore.

Changing the value of S makes a big difference, too. When $S = P = -13$, the result is that the ratio of cooperators oscillates around 48%. This is the limiting case between the chicken dilemma and the prisoners' dilemma. In this case the number of cooperators always decreases with time and can reach the original 50% only in the best case when the two payoff functions are very close to each other. This is in perfect agreement with the analytical study of the multi-agent prisoners' dilemma game (see Section 13).

As we move the value of S up from this limiting case, the ratio of cooperators steadily grows and at $S = 0$ (the unrealistic case of constant payoff for the cooperators) reaches 93%.

If we move point P up, the situation for the defectors steadily improves and the ratio of cooperators decreases to 43% at the limiting case of $S = P = -3$. Compare this result with the previous one when the two coinciding points were much lower.

Let us now keep P and S without change and move both T and R but in such a way that their difference does not change. We start at $R = -1$, $T = 0$ and move both points up. The ratio of cooperators changes from 0.70 to 0.83 when $R = 0.35$ and $T = 1.35$. A further small increase ($R = 0.38$, $T = 1.38$) results in total cooperation.

If we keep R and T without change and move both S and P up so that their difference does not change, the ratio of cooperators increases again and reaches 91% at the limiting case of $S = 0$, $P = -10$.

Finally, if we move P up so that the slope of the D function remains constant (that moves point T up as well), the ratio of cooperators will decrease drastically and reaches 18% in the limiting case of $P = -3$, $T = 11$. Compare this with the result above when the value of T remained 1.

For rational players, the intersection point (x^*) of the two payoff functions is a Nash equilibrium. Indeed, when $x < x^*$, the cooperators get a higher payoff than the defectors; therefore, their number increases. When $x > x^*$, the defectors get a higher payoff and the number of cooperators decreases. This is, however, not true for the more realistic Pavlovian agents. In this case, the relative situation of the two payoff curves with respect to one another does not determine the outcome of the dilemma. It is equally important to know the actual values of the payoffs. For example, in case of payoff functions determined by parameter values of $S = -3$, $P = -13$, $R = 0$, $T = 1$, the solution is 76% cooperation while $x^* = 91\%$ for this case.

If we shift the horizontal axis up and down, then x^* does not change but the solutions do. The following cases are possible:

(a) Both curves are positive for any value of x. In this case, the cooperators and the defectors are all always rewarded for their previous actions; therefore they are likely to repeat those actions. As a result, little change occurs in the cooperation ratio, especially when the rewards are large. The number of cooperators remains approximately equal to that of the defectors.

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(b) The $C(x)$ curve is entirely positive, but $D(x)$ changes sign from negative to positive as the value of x grows. In this case, the cooperation ratio changes from 0.5 to almost 1 as we shift the horizontal axis up, that is, for smaller rewards, the cooperation ratio is larger.

(c) When both $C(x)$ and $D(x)$ change signs, the cooperation ratio changes from near 1 to 0.76 as we shift the horizontal axis up, that is, for smaller rewards, the cooperation ratio is smaller. Maximum cooperation occurs when $S = 0$, that is, the environment is neutral to cooperators during maximum defection.

(d) The $C(x)$ curve is entirely negative but $D(x)$ changes sign from negative to positive as the value of x grows. The solution oscillates around a stable equilibrium that grows with the rewards.

(e) Both $C(x)$ and $D(x)$ are negative for all values of x (punishment for any action). The solution oscillates around a stable equilibrium. As the punishments grow, all actions tend to change at each iteration and the solution approaches 50%.

If we change more than one parameter at the same time, different solutions will appear but the basic trends remain the same.

A perturbation to the initial state may cause the system to evolve into a different future state within a finite period of time or cause no effect at all. In case of the last example above, the result is the same if we start with 10%, 50%, or 90% cooperators.

8.2 Greedy Agents

In this case, only the relative payoffs count; therefore, a systematic investigation is much easier. We assumed that the neighborhood is one layer deep [59].

Our four-dimensional problem is reduced to a two-dimensional one by assuming $D(x) = x$ ($P = 0$ and $T = 1$). The remaining two free parameters R and S satisfy the $P < S < R < T$ condition if and only if $0 \leq R \leq 1$ and $0 \leq S \leq R$ for each R. Writing the equation of the cooperator's payoff function in the form of $C(x) = a + b\,x$, we see that $S = a$ and $R = a + b$.

We have performed a systematic investigation of the game for hundreds of values of these two variables in their entire range. The global ratio $X(t)$ of the total number of cooperators in the entire array as a function of time (iterations) was observed for each pair of the parameter values. (Note that X is different from x, which refers to an agent's immediate neighbors only.) The final ratio of cooperators X_{final} around which $X(t)$ oscillates represents the solution of the game.

Figure 19 shows the case when $D(x) = x$ and $C(x) = 0.6 + 0.14\,x$. The broken lines represent the fact that the values $C(0)$ and $D(1)$ do not exist by definition.

The results show that the solutions have predictable tendencies but they are nontrivial and represent quite irregular emergent behavior. The solutions show drastic changes in the parameter ranges $0.6 \leq R \leq 0.65$ for all values of S and $0 \leq S \leq 0.2$ when $R < 0.6$.

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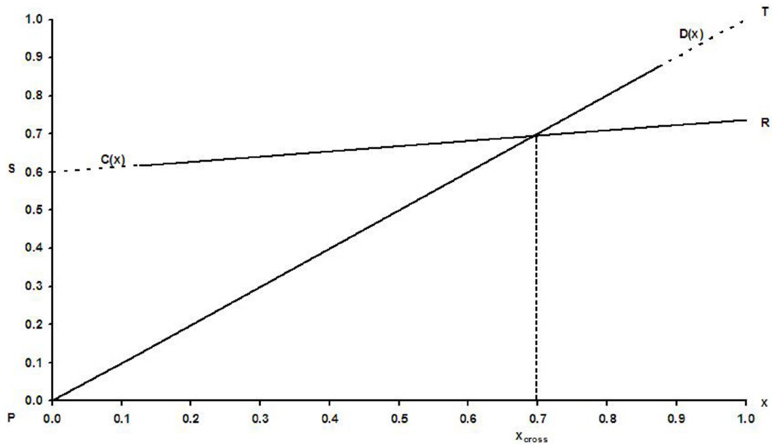

Figure 19.

Let us fix the value of $x_{\text{cross}} = 0.7$ (it means that the values of R and S must change simultaneously). In this case, $0.7 \leq R \leq 1$ and $0 \leq S \leq 0.7$ while a higher value of R means a lower value of S (Figure 20). The solution of the game X_{final} as a function of S is given in Figure 21. In this case, we expect a decreasing dependence on S, which is indeed the case, but for this value of x_{cross} the cooperation ratio is relatively high ($0.78 \leq X_{\text{final}} \leq 0.92$) for the entire range. The function is quite irregular.

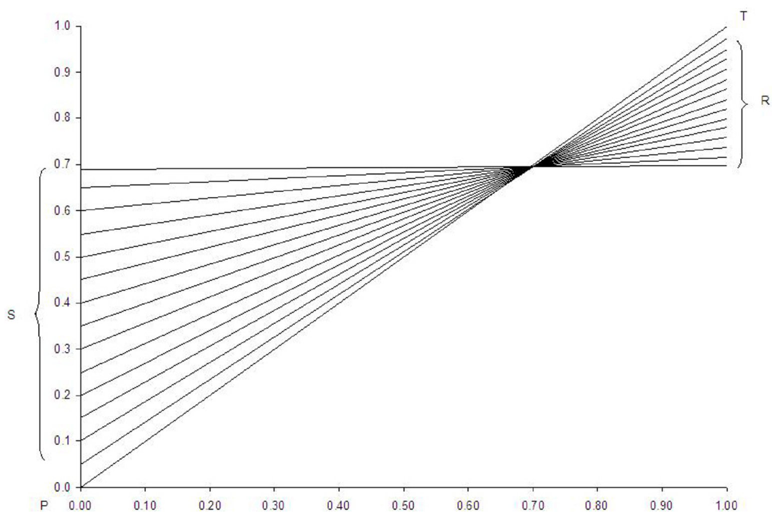

Figure 20.

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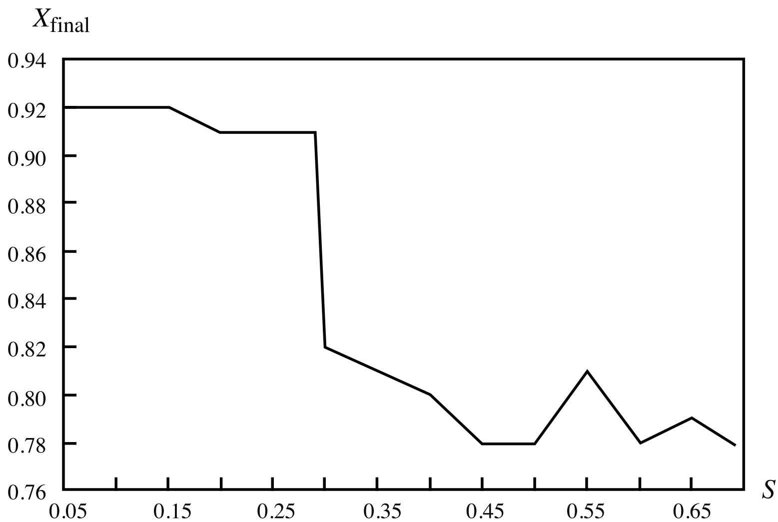

Figure 21.

For rational players the intersection point x_{cross} of the two payoff functions is a Nash equilibrium (see above). This is not true for the greedy simpletons, either, because x refers to the immediate neighbors only while the final ratio of cooperators X_{final} represents the ratio of the total number of cooperators in the entire array. In the case of the payoff functions presented in Figure 19, for example, the solution is $X_{\text{final}} = 0.775$ while $x_{\text{cross}} = 0.7$ in this case.

9. *N*-Person Battle of the Sexes Game

In this game $T > S > P > R$; therefore, the payoff functions are crossing each other again. Even in the two-person case, there are 16 decision combinations. The N-person game is not a simple extension of the two-person game in this case [60]. Defection and cooperation have deeper meanings here than in any other game. We cannot say that the preferences are conflicting for all pairs of players. It causes additional difficulties that were resolved by agent-based simulation [61].

10. *N*-Person Games with Crossing Payoff Functions

There are 12 different orderings of the values of P, R, S, and T that lead to crossing payoff lines. Each of them represents a different type of game. We have systematically investigated uniform N-person

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games with crossing payoff functions for the case when the agents are all greedy [62].

The results show nontrivial and in some cases quite irregular emergent solutions. They show drastic changes in the case of the leader game in the narrow parameter range of $1.72 < P < 1.75$. This behavior is similar to that observed for the N-person chicken game. Emergent solutions were found also for the reversed stag hunt game.

For example, let us investigate the role of the relative angle between the two payoff lines. We fixed the $C(x)$ function as $C(x) = 2.9 - x$, the intersection point of the two lines at $x_{\text{cross}} = 0.7$, $y_{\text{cross}} = 2.2$, and rotated the $D(x)$ function around this point. As we rotate $D(x)$, its end points change: $-10.0 < P < 2.9$, $R = 1.9$, $S = 2.9$, and $T = P + (2.2 - P) / 0.7$ $(7.43 > T > 1.9)$.

The upper limit of P is chosen as S to maintain the condition that at $x < x_{\text{cross}}$, the cooperators get a higher payoff than the defectors and at $x > x_{\text{cross}}$, the defectors get a higher payoff than the cooperators. Under these conditions for $-10.0 < P < 0.56$ we have the benevolent chicken game, for $0.56 < P < 1.9$ it is the leader game, for $1.9 < P < 2.2$ we have the reversed benevolent chicken game, and for $2.2 < P < 2.9$ it is the reversed stag hunt game. $P = 0.56$, $P = 1.9$, and $P = 2.2$ are borderline cases in between two games. The transitions between games are quite smooth.

We present the result of the simulation X_{final} as a function of P in Figure 22. For the region $-10.0 < P < -0.6$, the result is nearly constant around the value of $X_{\text{final}} = 0.82$. Therefore, we only show the result for $-1.0 < P < 2.9$. Several remarkable features can be immediately noticed. First, neither the benevolent chicken nor the reversed benevolent chicken games are sensitive to the rotation of the $D(x)$ function. However, the leader and the reversed stag hunt games behave quite strangely.

The behavior of the reversed stag hunt game is quite remarkable. At $P = 2.275$, X_{final} suddenly jumps from 0.66 to 0.80, then rises to 0.83, then at $P = 2.394$ has a dip down to 0.76, rises again to 0.81, then at $P = 2.4$ jumps down to 0.65, rises again, and finally at $P = 2.525$ suddenly changes from 0.63 to 0.23; at $P = 2.65$ it reaches its final value of 0.11. The character of the $X(t)$ function also drastically changes at these points. The $X(t)$ function for $P = 2.24$ wildly fluctuates in between 0.40 and 0.91 so that $X_{\text{final}} = 0.65$. In the case of $P = 2.41$, $X_{\text{final}} = 0.65$ again, but there are practically no fluctuations. The graphics outputs are also different.

The leader game behaves even more strangely. There are several wild fluctuations in between $X_{\text{final}} = 0.64$ and $X_{\text{final}} = 0.90$ in the narrow region of $1.72 < P < 1.75$.

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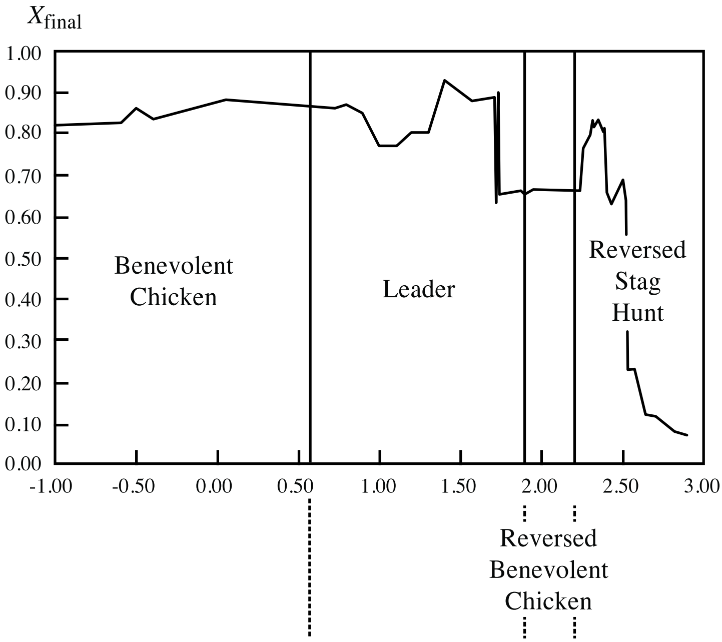

Figure 22.

11. *N*-Person Game of Life

This new game is an extension of Conway's Game of Life in which having a small or a large number of neighbors is equally disadvantageous. Accordingly, in this game the payoff functions have the same tendency for both small and large number of cooperators [63]. The *C* curve of Figure 23 corresponds to the Game of Life: this choice is punished when too few or too many agents choose it and is rewarded for a mediocre behavior. The *D* curve is the mirror image of the *C* curve and it represents the daylight saving problem [5]. If everyone is on daylight saving or everyone is on standard time, there is no problem. However, if some prefer the former and others the latter, the result will be chaos.

There are six possible cases of the mutual positions of parabolic payoff functions crossing each other at two points: $x = 0.3$ and 0.7. (The role of parabolic payoff functions was investigated in [64].) We simulated all of them for Pavlovian, greedy, and conformist agents. The solutions have predictable tendencies only when the neighborhood is the entire array of greedy or conformist agents. In all other cases, unexpected properties emerge.

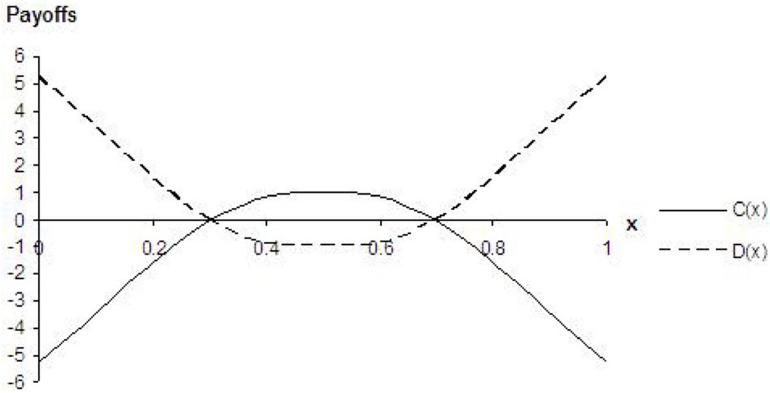

Figure 23.

12. Transitions between Different *N*-Person Games

The classification of games according to the mutual relationships between the R, S, T, and P parameters ignores the cases when some of these parameters are equal to each other. With the exception of the simulations reported in Section 10, these cases are not considered in the previous sections. We have investigated these transitional cases as well. The results are very interesting but their explanation would require much more space than what is available here. The reader is referred to the original publication [65].

13. Analytical Solutions of *N*-Person Games

It is possible to replace simulation with an algorithm that accurately predicts the final aggregate outcome for any combination of Pavlovian agents and any payoff functions [54]. The predictions are exact for an infinite number of agents, but the experimental results of simulations approximate these predictions very closely even for a few hundred agents. The algorithm was further improved in [41]. However, it is very desirable to find analytical solutions for N-person games.

In many cases, it is possible for Pavlovian agents to use an analytical formula for the prediction of the solutions. Let us assume that in a society of N Pavlovian agents, the neighborhood is the entire collective of agents, the ratio of cooperators is x, and the ratio of defectors is $(1-x)$ at a certain time. We have shown [54, 66] that when the cooperators receive the same total payoff as the defectors, that is,

$$x\,C(x) = (1-x)\,D(x), \tag{1}$$

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an equilibrium occurs. This may happen if $C(x)$ and $D(x)$ are either both negative or both positive. In the first case, a stable equilibrium was observed. In the second case, an unstable equilibrium occurs.

In the case of linear payoff functions, the equilibrium equation is quadratic. If its solutions are real, they are x_1 (stable attractor) and x_2 (unstable repulsor). When the initial cooperation ratio is below x_2, the solution of the game converges toward x_1 as an oscillation while it stabilizes exactly when the initial cooperation ratio is above x_2. The latter case does not result in the aggregate cooperation proportion converging to 1, as would be expected. This is because, for an individual agent that started as a defector, there is always some likelihood that the agent will continue to defect. This probability is initially small but continues to increase if the agent is always rewarded for defecting. If the number of agents is sufficiently large, then there will be some agents that continue to defect until their cooperation probability reaches zero due to the successive rewards they have received, and these agents will defect forever. In the case of complex solutions, equation (1) does not give any information about the game.

We have performed numerous experiments with our simulation tool for N-person games. When the agents have Pavlovian personalities, the following cases are possible for the application of equation (1):

(a) Both payoff curves are positive for any value of x. In this case, only the unstable equilibrium exists and the solution of the game depends on the value of this equilibrium and on the initial ratio of cooperators. When the initial cooperation ratio is below x_2, the solution of the game stabilizes at a lower value between 0 and x_2. When the initial cooperation ratio is above x_2, the final stable ratio has a higher value between x_2 and 1.

(b) Both $C(x)$ and $D(x)$ are negative for all values of x. In this case, only the stable equilibrium exists and the solution of the game always converges to x_1.

(c) The $C(x)$ curve is entirely positive but $D(x)$ changes sign from negative to positive as the value of x grows or the $D(x)$ curve is entirely positive and $C(x)$ changes sign. The situation is similar to case (a). The only difference is that in this case the region where both $C(x)$ and $D(x)$ are positive may be too narrow to produce a solution according to equation (1).

(d) The $C(x)$ curve is entirely negative but $D(x)$ changes sign from negative to positive as the value of x grows or the $D(x)$ curve is entirely negative but $C(x)$ changes sign. The situation is similar to case (b). However, the region where both $C(x)$ and $D(x)$ are negative may be too narrow to produce a solution according to equation (1).

(e) The most interesting case is when both $C(x)$ and $D(x)$ change sign. In this case, both equilibria exist and equation (1) always works.

An investigation of equation (1) by linear programming showed that for linear payoff functions in the N-person prisoners' dilemma game, if the initial cooperation rate is small, a maximum 50% cooperation rate can be achieved. This limit can only be increased if the initial cooperation rate is above 50% [67].

14. Practical Applications

There are numerous practical applications of N-person games in various fields. A few of them include: the evolution of cooperation in societies, the behavior of economic systems, various problems of social psychology, the development of social norms, the fight against terrorism, artificial life and societies, social experiments without the involvement of people, and public policy.

We have investigated the following problems.

14.1 The Tragedy of the Commons

As was mentioned in Section 7, the famous tragedy of the commons [8] is an N-person prisoners' dilemma. Accordingly, any agent-based model of this game is also a model for the tragedy of the commons.

14.2 Segregation

Section 7.4 deals with the simulation of conformist agents. This is basically Schelling's segregation simulation [23] for the case when all agents follow the behavior of their neighbors. A typical result is shown in Figure 17.

14.3 Mass Transportation

Today's economy is based on the automobile. Every day, hundreds of millions of people use their cars to visit a remote place or just to go to a supermarket nearby. Trillions of dollars have been spent on building highways. This is a tremendous waste, let alone air pollution and dependence on foreign oil. In most cars one person uses hundreds of horsepower. Can we do better?

The answer to this question is public transportation. If there were no cars but reliable trains and buses, everyone could get anywhere quickly and without traffic jams. The individual agents in this example may cooperate with each other for the collective interest by using public transportation or may defect, that is, pursue their selfish interests by driving their cars.

Let us consider the payoffs for both choices in two extreme situations. If everyone is using a car, the roads will be clogged and no one can move. If some agents choose the bus, they are in a slightly better situation because in many large cities there are special lanes for buses, but in most cases these lanes are also filled with cars. Both the car drivers and the bus riders are punished for the behavior of the majority. On the other hand, if everyone uses a bus, the buses will be crowded, but they can freely move on the empty streets. All agents can get relatively quickly to their destinations, which corresponds to a reward. If some agents yield to the temptation and choose to use their cars anyway, they get an even larger reward because they can move even faster and avoid the crowd in the bus. This is a typical N-person chicken dilemma (see Figure 19).

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Our simulation results show that it is quite possible to achieve a situation when the majority of people would prefer using mass transportation to driving cars [58]. The best solutions appear when the $D(x)$ function changes sign with increasing x, that is, when the car drivers are rewarded when there are few of them and punished when there are many of them.

14.4 A Simple Market

We have investigated an agent-based model of a simple market based on trade and price satisfaction functions for both buyers and sellers. A rather long paper [68] describes the simulation, which is also based on the chicken dilemma. The price satisfaction dominates trade satisfaction in most cases.

14.5 Public Radio Membership Campaigns

We assumed that each possible contributor is a mixture of eight motivation types: benefitting from the radio programs, reaching short-term goals, feeling guilty if no contribution is made, valuing reputation, feeling good for participating, matching contributions from others, fulfilling promises, and giving gifts to contributors. Our simulations show how campaigns depend on these motivations, on manipulations of the organizers, on the wealth of the community, and on other model parameters [69].

14.6 Standing Ovation

It was mentioned in Section 4 that we used the Big Five personality representation in our study of standing ovation [33]. This is a complex phenomenon and we proved that different personality types participate in it in different ways.

14.7 Collective Action

We have shown that there is a direct quantitative relationship between the production function of collective action and the payoff functions of the N-person prisoners' dilemma game [70].

14.8 The El Farol Bar Problem

The famous El Farol Bar problem is an excellent demonstration of the self-referential expectation formation. As such, it serves also as a simple model of financial markets.

Arthur [71] has shown by using computer simulation based on a sophisticated reasoning that the bar attendance fluctuates rather wildly around the capacity level L of the bar. This result was arrived at by many other papers as well [72–74].

The El Farol problem has been extended to the so-called minority game [75]. The players must choose between two options and those in the minority side win. Many variants of this game have been devel-

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oped, a large number of papers have been published [76–79], and even books have been written about it [80–82].

We found a much simpler approach [42]. Considering the problem as an *N*-person battle of the sexes or leader game and using equation (1), we showed that the solutions x_{final} of our games will always converge to $x_{\text{final}} = L$, that is, they indeed fluctuate around the capacity level of the bar.

Our computer simulations confirmed these solutions and showed that the amplitude of fluctuations around this value is inversely proportional to the number of participating agents.

15. Additional Literature

This paper reviewed only games that are played on a lattice. Other aspects of agent-based modeling are reviewed in [83, 84].

As we mentioned in Section 1, there is a huge amount of literature about series of two-person games among *N* participants. A review of these papers can be found in [85]. Even a NetLogo program [86] is available to play such games. We, however, believe that the participants of a genuine *N*-person game should play with all the other players at the same time. As Anatol Rapoport noted [87], the "tournaments demonstrated neither evolution nor learning because nothing evolved and nothing was learned."

16. Conclusion

This review clearly demonstrates the usefulness of agent-based modeling in the investigation of *N*-person games. Agent-based modeling also advances the state of the art in simulating nonlinear dynamic systems in general.

It is now possible to:

- develop collective behavioral models for finding the best sets of parameters and rules that accurately predict group behavior of a large number of participants in realistic situations;
- simulate real-life social phenomena, especially collective action and social dilemmas involving choices of conflict versus cooperation, as well as situations of long-term social change;
- investigate how decision-making actions of the participants can be influenced;
- investigate what personality characteristics govern the propagation of cooperative and competitive behavior; and
- demonstrate potential applications to human decision support.

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In future work, the simulation of the following problems is especially promising:

- Propagation of cooperation from a small group of originators.
- The dependence of crowd reactions on the location of instigators and the density of social ties.
- The role of group size in the evolution of cooperation.
- The emergence of social norms.

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