A Glimpse of Complex Maps with Memory

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An exploratory study is made on the dynamics of the complex quadratic map endowed with memory (m) of past iterations $z_{T+1} = m_T^2 + c$.

1. Memory in Discrete Maps

Memory can be embedded in conventional finite difference equations (often referred to as *discrete dynamical systems*) with $x_{T+1} = f(x_T)$ by means of $x_{T+1} = f(m_T)$, with m_T being an average value of past states $m_T = m(x_1, \ldots, x_T)$ [1].

Thus, the complex quadratic map ($z, c \in \mathbb{C}$)

$$z_{T+1} = z_T^2 + c \text{ becomes with memory } z_{T+1} = m_T^2 + c. \tag{1}$$

In a general form, at time step T we will consider past states weighted as $m_T = \sum_{t=1}^{T} p_t x_t$, with the weights p fulfilling the probabilistic-like normalization condition $\sum_{t=1}^{T} p_t = 1$, $p_t \geq 0$.

We will consider first the effect of average memory with geometric decay:

$$m_T = \frac{z_T + \sum_{t=1}^{T-1} \alpha^{T-t} z_t}{1 + \sum_{t=1}^{T-1} \alpha^{T-t}} \equiv \frac{\omega(T)}{\Omega(T)} = \frac{z_T + \alpha\omega(T-1)}{\Omega(T)}. \tag{2}$$

The choice of the memory factor α simulates the intensity of the memory effect: the limit case $\alpha = 1$ corresponds to a memory with equally weighted records (*full* memory), whereas $\alpha \ll 1$ intensifies the contribution of the most recent states (short-term memory). The choice $\alpha = 0$ leads to the ahistoric model. This kind of memory implementation will be referred to as α-memory.

Limited trailing memory would keep memory from only the last τ time steps. Thus:

$$m_T = m(z_\top, \ldots, z_T), \text{ with } \top = \max(1, T - \tau + 1).$$

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Limiting the trailing memory would approach the ahistoric model ($\tau = 1$); such an effect is more appreciable when the value of α is high, whereas at low α values (already close to the ahistoric model when memory is not limited), the effect of limiting the trailing memory is not so important.

In the limit case $\tau = 2$, the trailing memory is:

$$m_T = \frac{z_T + \alpha\, z_{T-1}}{1 + \alpha}, \qquad 0 \le \alpha \le 1. \tag{3}$$

This kind of memory implementation will be referred to as τ=2 α-memory.

We will consider also memory of only the last two states implemented in the general form [2]:

$$m_T = (1 - \epsilon)\, z_T + \epsilon\, z_{T-1}, \qquad 0 \le \epsilon \le 1. \tag{4}$$

This kind of memory implementation will be referred to as ϵ-memory. This is a more general model than the $\tau = 2$ α-memory model, because it allows for a higher contribution of the past than of the present state. If $\epsilon \le 1/2$, the models in equations (3) and (4) are interchangeable according to $\epsilon = \alpha/(1 + \alpha)$. The maximum memory charge attainable in the $\tau = 2$ α-memory model is that of $\alpha = 1.0$, which corresponds to $\epsilon = 0.5$, but levels of the ϵ parameter over 0.5 generate dynamics uncovered with $\tau = 2$ α-memory. In the extreme case, if $\epsilon = 1$ it is $m_T = z_{T-1}$, so that every state of the ahistoric evolution is generated twice: $z_1, z_2, z_2, z_3, z_3, \ldots, z_T, z_T, \ldots$.

2. The Mandelbrot Set with Memory

The Mandelbrot set is the set of all c for which the iteration in equation (1), starting from $z = 0$, does not diverge to infinity ([3], Ch. 14). The Mandelbrot set $\mathbf{M}$ is a compact set, contained in the closed disk of radius 2 around the origin. In fact, a point c belongs to the Mandelbrot set if and only if $|z_T| < 2$ for all T.

Figures 1 through 3 show the effect of memory on the Mandelbrot set, represented in base to the escape time, that is, the number of iterations up to reaching a critical *escape* condition or *bailout*. Simulations are run up to $T = 100$. Thus, black color indicates the Mandelbrot set, whereas white/blue colors indicate an odd/even number of iterations up to divergence, precisely up to $|z|$ reaching the breakout value 8.0. The boundary of the contour curves in these figures can be interpreted as equipotential curves. The images shown in Figures 1 through 3 represent computations with memory, which remain symmetrical around the real axis as in the conventional scenario.

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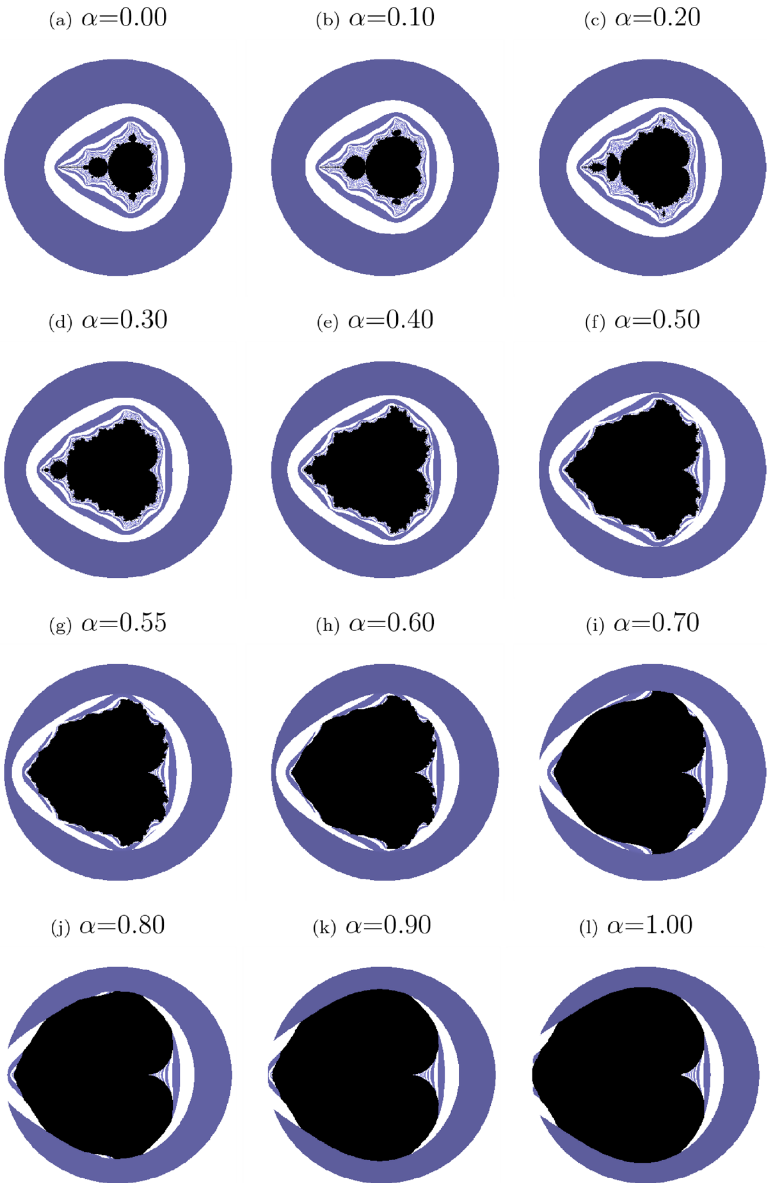

Figure 1. The Mandelbrot set with unlimited α-memory.

Figure 1 shows the effect of α-memory on the Mandelbrot set. The region of the complex plane shown in this figure is $[-4.0, 2.5]\times[-3.3, 3.3]$. As a result induced by the inertial effect that α-memory exerts, **M** grows as α increases, so that it virtually occupies the region bounded by the outer oval-like blue band, which is not very much altered.

At a first glance, the kidney bean-shaped portion of **M** is not dramatically altered by memory, particularly with respect to its far-right part. Thus, for example, the vertex featuring the *elephant valley* (the portion of the Mandelbrot set centered around $0.3 + 0.0\,i$ with size approximately $0.1 + 0.1\,i$ [4]) in the ahistoric context is not displaced too much to the right of its -0.25 conventional value. On the contrary, the adjoining circle (with center at $(-1, 0)$ and radius 0.25 in the ahistoric formulation) becomes increasingly merged with the whole set as memory increases. With full memory, the contour curve of **M** is reminiscent of that of the (reflected) *limaçon* curve $r = 2\cos\theta + b$.

The area of the Mandelbrot set has been estimated to be close to 1.5066 [5]. In simulations up to $T = 1000$ in a lattice with high resolution (1001×1001), the area estimated by pixel counting evolves as: 1.5104, 2.1394, 3.0273, 4.4804, 6.1283, 7.8861, 8.8680, 9.8788, 11.6333, 13.0409, 14.0855, 15.0250 when increasing α as in Figure 1. Thus, the area grows with full memory up to 10 times the area of **M**. The best-known estimate of the real coordinate of the center of gravity is close to 0.2868. In the simulations just described, the estimations by pixel counting of the real coordinate of the center of gravity evolve as: -0.2902, -0.3861, -0.4839, -0.6184, -0.7007, -0.7705, -0.8036, -0.8352, -0.8894, -0.9313, -0.9701, -1.0073.

Figure 2 shows the effect of $\tau = 2$ α-memory on the Mandelbrot set. The region of the complex plane shown in this figure is the same as in Figure 1. The outer blue band is more altered in Figure 2 compared to the effect with unlimited memory in Figure 1, particularly for high values of the memory factor α, but keeps a space of white pixels between it and the **M** set and its proximity. At variance with what happens with unlimited trailing memory, the adjoining circle in the ahistoric formulation shrinks adopting variable forms, but it does not become merged with the whole **M**-set. The proximity of **M** with $\tau = 2$ α-memory preserves an intricate aspect in a region proportionally similar to that in the conventional ahistoric model. The Mandelbrot set itself adopts with two time-step memory the different aspects shown in Figure 2, but according to the limitation in the length of memory, it grows in a lesser extent compared with the unlimited trailing memory scenario of Figure 1. Thus, the area of the Mandelbrot sets in Figure 2 evolves as: 1.5104, 2.0873, 2.7333, 3.6204, 4.3624, 4.8411, 4.9767, 5.0213, 4.8086, 4.3263, 4.0594, 3.8413. The real coordinate of the center of gravity in Figure 2 evolves as: -0.2902, -0.3870, -0.4880, -0.6508, -0.7656, -0.8338, -0.8551, -0.8657, -0.8441, -0.7662, -0.7390, -0.7236.

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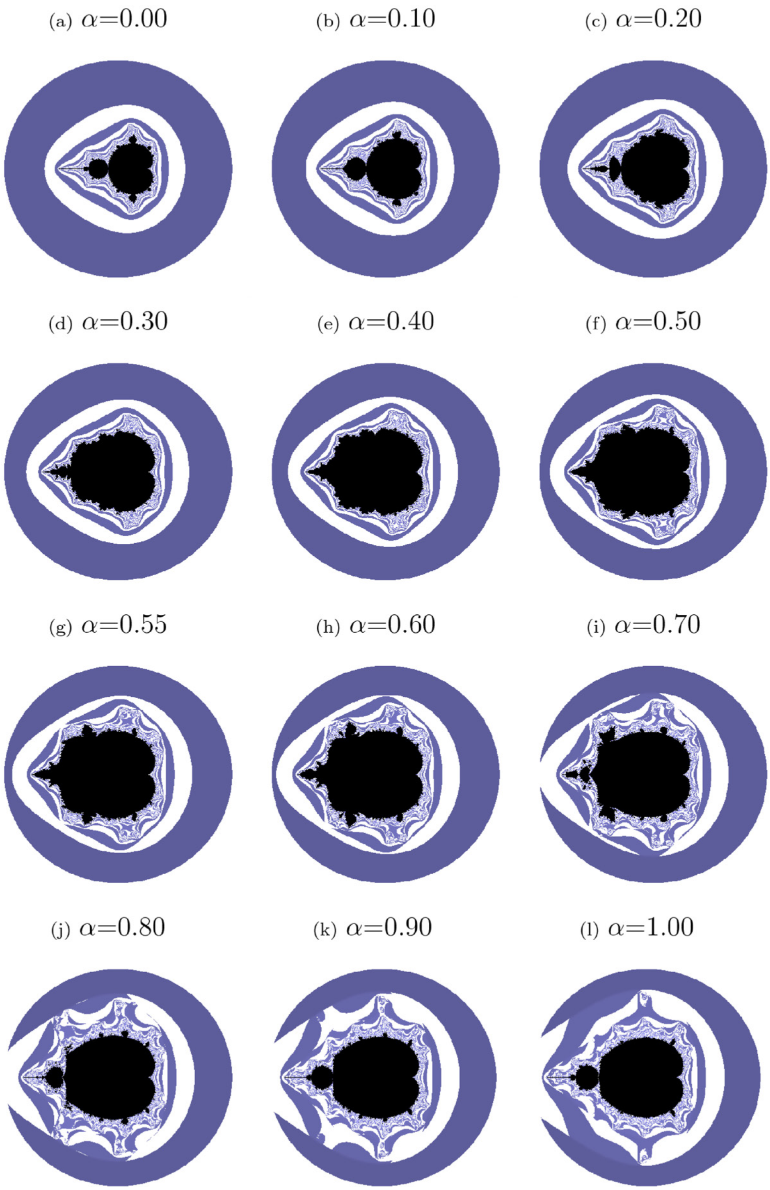

Figure 2. The Mandelbrot set with $\tau = 2$ α-memory.

Figure 3 shows the effect of ϵ-memory on the Mandelbrot set, again in the same region of the complex plane as in Figure 1. The outer blue band is highly altered in Figure 3, tending to vanish as the memory factor increases. The Mandelbrot set is appreciably altered by ϵ-memory up to $\epsilon = 0.70$, but tends to recover the original configuration for higher values of ϵ. Recall that the scenario of $\epsilon = 0.5$ coincides with that of $\tau = 2$ $\alpha = 1.0$-memory, as can be checked here in Figures 2 and 3. In parallel to the effect of ϵ-memory on **M** itself, the sophisticated aspect of the proximity of **M** is fairly preserved up to $\epsilon = 0.70$, but shrinks with higher values of ϵ, and finally disappears with $\epsilon = 1.0$. The area of the Mandelbrot sets in Figure 3 evolves as: 1.5104, 3.8413, 3.4739, 2.8589, 2.4480, 2.2230, 1.7775, 1.4932, 1.5136. The real coordinate of the center of gravity in Figure 3 evolves as: −0.2902, −0.7236, −0.6938, −0.5400, −0.4343, −0.4071, −0.3178, −0.2575, −0.2907.

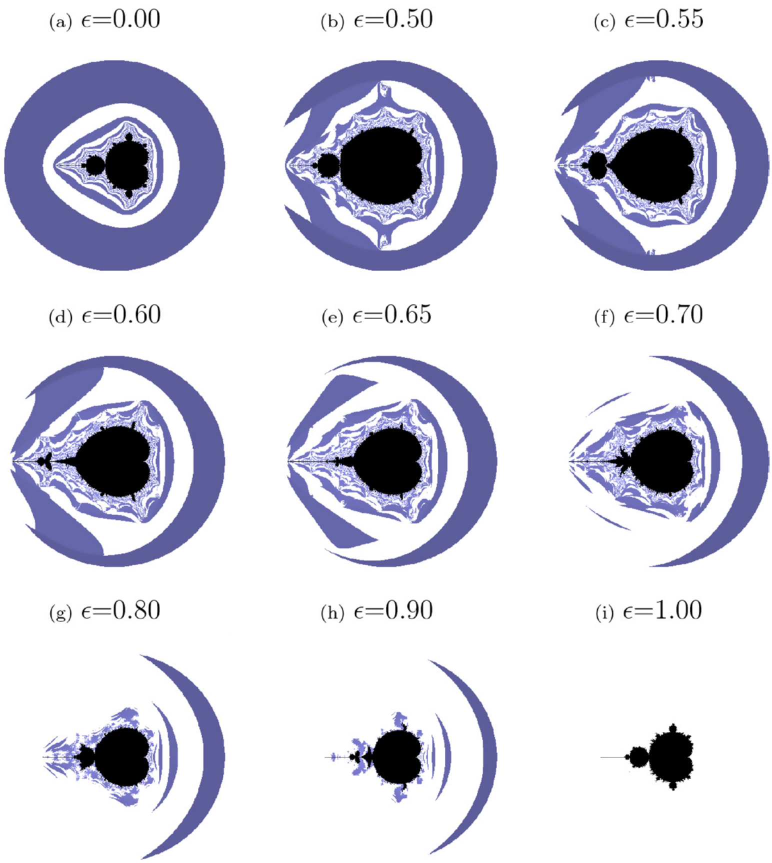

Figure 3. The Mandelbrot set with ϵ-memory.

2.1 Programming

The computing in this paper has been made by means of a Fortran code in a GNU environment (gfortran). The code generates portable

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pixmap format (.ppm) files after converting into PNG format. Nevertheless, to facilitate the divulgation, a MATLAB® code adapted from that provided in [6], implementing α-memory in the Mandelbrot set, is given in Table 1. The MATLAB language recognizes the symbol `i` in formulas as the imaginary unit ($\sqrt{-1}$), so that `C=X+i*Y` and `Z= X+i*Y` directly refer to complex numbers, with no need of type declaration as is customary in other programming languages, for example, Fortran.

```
col = 30; m = 100; cx = -.6; cy = 0; l = 1.5;
for ii = 1 : 10; alpha = 0.1 * (ii - 1);
    if (alpha = = 0.9) alpha = 1.0; end
    x = linspace (cx - l, cx + l, m);
    y = linspace (cy - l, cy + l, m);
    [X, Y] = meshgrid (x, y); Z = zeros (m);
    C = X + i * Y; Z = X + i * Y; ZZ = Z; ww = 0;
    for k = 1 : col;
        ZZ = alpha * ZZ + Z;  ww = alpha * ww + 1;
        Z = ZZ.^ 2 + C;  W = exp (-abs (Z));
    end
    subplot (2, 5, ii); colormap jet (256);
    pcolor (W); shading flat; axis ('off');
end
```

Table 1. A MATLAB code implementing α-memory in the Mandelbrot set.

2.2 Generalization

Two generalizations of the Mandelbrot set can be achieved by replacing z^2 in equation (1), either by z^k or by $\overline{z}^2$ (Mandelbar sets, [7]). Memory may then be endowed in these maps as $z_{T+1} = m_T^k + c$ and $z_{T+1} = \overline{m}_T^k + c$, respectively.

Figures 4 through 6 show the effect of memory on the cubic Mandelbrot set, that is,

$$z_{T+1} = m_T^3 + c. \tag{5}$$

Figure 4 shows the effect of α-memory on the cubic Mandelbrot set in the $[-1.2, 1.2]\times[-1.9, 1.9]$ region of the complex plane. Figure 5 shows the effect of $\tau = 2$ α-memory on the cubic Mandelbrot set in the $[-1.1, 1.1]\times[-2.4, 2.4]$ region of the complex plane. Figure 6 shows the effect of ϵ-memory on the cubic Mandelbrot set, in the same region of the complex plane as in Figure 5. The threshold $|z| = 8.0$ is also applied in the cubic figures here.

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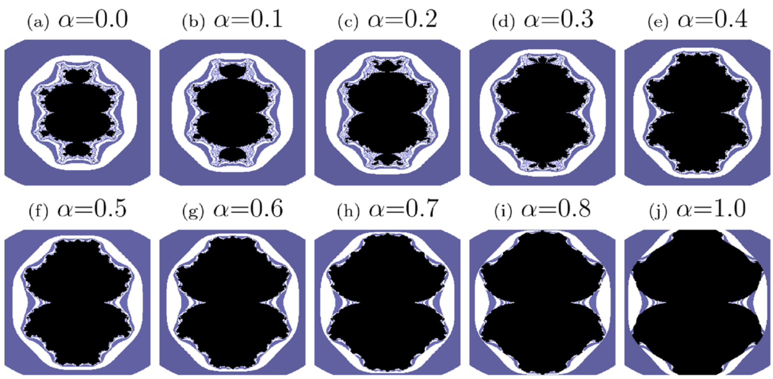

Figure 4. The cubic Mandelbrot set with unlimited α-memory.

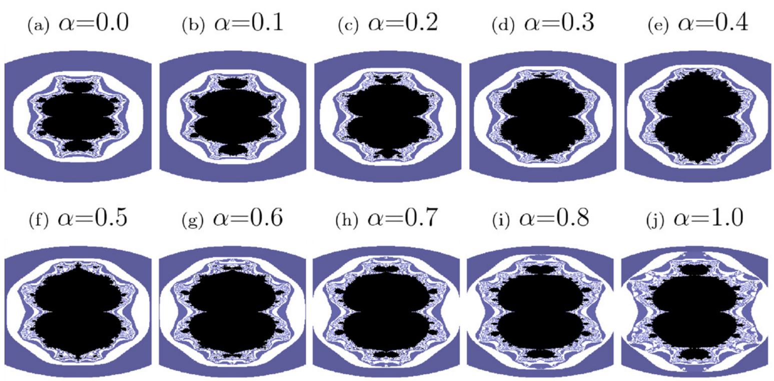

Figure 5. The cubic Mandelbrot set with $\tau = 2$ α-memory.

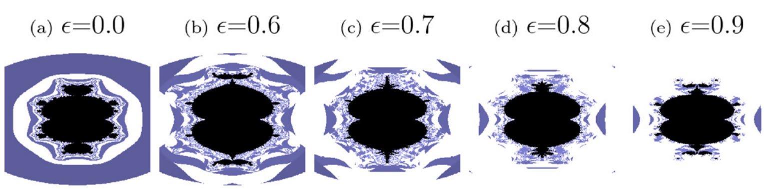

Figure 6. The cubic Mandelbrot set with ϵ-memory.

The general comments made on the effect of memory on the quadratic Mandelbrot set are also applicable regarding the cubic Mandelbrot set in Figures 4 through 6. Thus, for example, the inertial ef-

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fect of unlimited trailing α-memory induces a notable growth of **M**, maintaining the fringe just outside the **M**-set with rich structure for not-too-high levels of the memory factor α. This region is particularly rich with $\tau = 2$ α-memory for every value of α, and with ϵ-memory it is preserved up to $\epsilon = 0.60$, but shrinks with higher values of ϵ, and finally disappears with $\epsilon = 1.0$, in which case (not shown in Figure 6) the ahistoric cubic **M**-set is recovered.

Memory preserves the symmetry of the cubic **M**-set around the real axis as already observed in the quadratic map.

Figure 7 shows the effect of α-memory on the quadratic Mandelbar set, that is, $z_{T+1} = \overline{m}_T^2 + c$, in the $[-2.4, 1.5]\times[-2.2, 2.4]$ region of the complex plane. The characteristic tricorn turns out somehow elongated with memory.

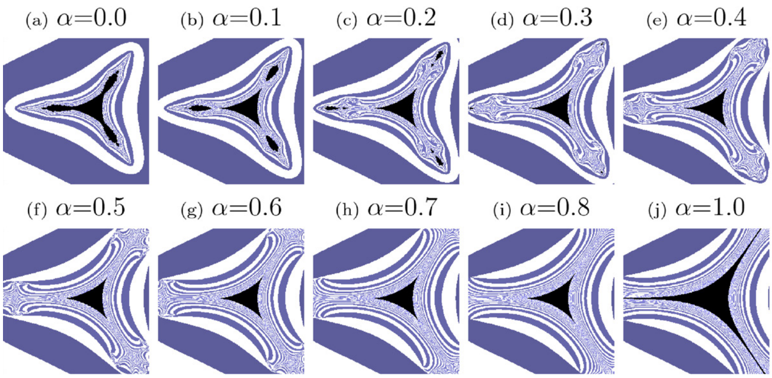

Figure 7. The Mandelbar set with unlimited α-memory.

3. Julia Sets with Memory

For a complex number c, the filled-in Julia set of c is the set of all z for which equation (1) does not diverge to infinity ([3], Ch. 13). The Julia set is the boundary of the filled-in Julia set. For almost all c, these sets are fractals. Julia sets are either connected (one piece) or a dust of infinitely many points.

Figures 8 and 10 show an example of the effect of memory on a Julia set based on small positive values for Re(c) and Img(c), that is, $c = 0.70 + 0.4\,i$. A sketch-like atlas of Julia set shapes may be found in [8]. The threshold $|z| = 8.0$ also applies in this context. Regarding the color scheme in these figures, for the values of (x, y) that do not break out, no red color is present, and green and blue colors are

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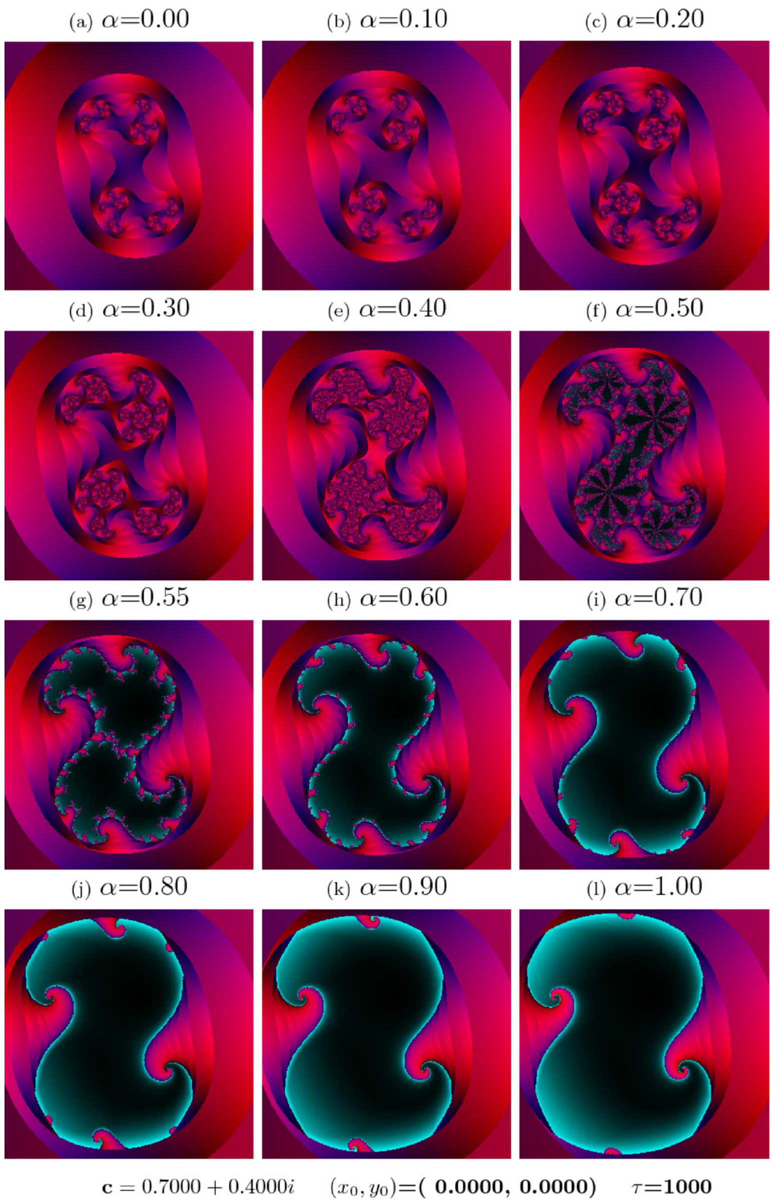

Figure 8. Julia sets with unlimited α-memory.

mixed proportionally to the maximum value of $|z|$ achieved during the iteration. For values of (x, y) that break out, no green color is present, and red and blue are mixed according to the last value of (x, y) before the breakout. The original Fortran code implementing the coloring adopted here may be found at http://oldit.itmetr.net/rgraphics/julia/julia.f90.

Figure 8 shows the effect of unlimited α-memory in such a c-Julia set. The inertial effect that α-memory exerts makes the trespassing of the established threshold more difficult, and consequently with high memory charges, that is, beyond $\alpha = 0.50$, the aspect of the snapshots in Figure 8 is notably uniformly obscure. But below $\alpha = 0.50$, the aspect of the snapshots remains fairly as rich as in the ahistoric model, and even becomes fairly enriched in the $\alpha = 0.50$ simulation.

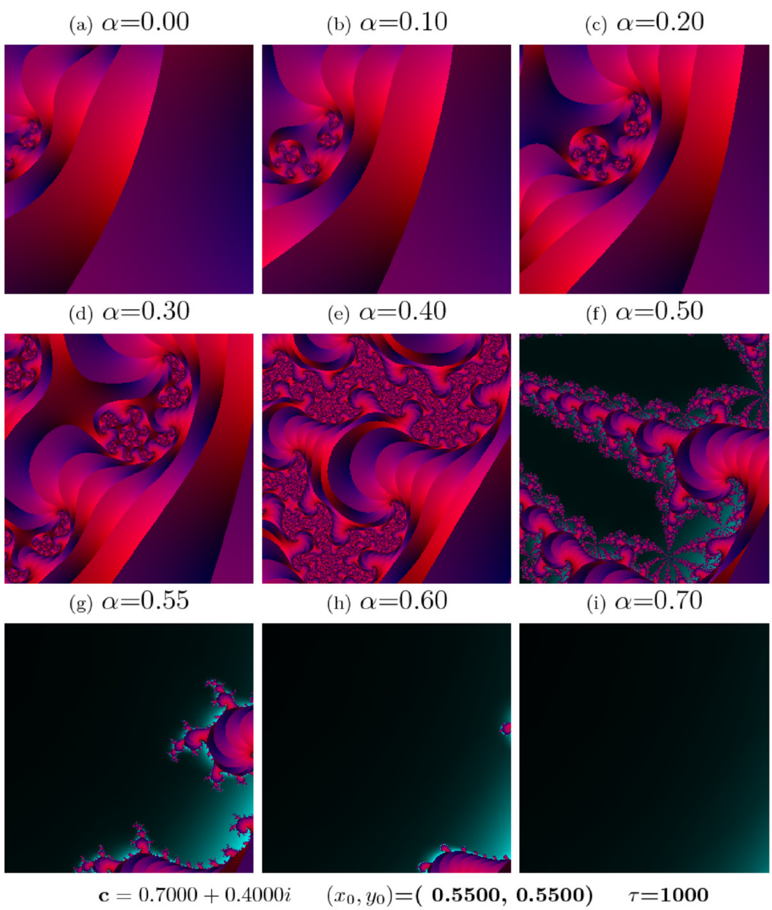

Figure 9. Close-up of Figure 8 in (0.55, 0.55).

Figure 9 shows a close-up of Figure 8 in $(0.55, 0.55)$. With $\alpha > 0.7$, the snapshots present a blackout-like aspect, so that they are not shown in Figure 9. Some snapshots, such as that for $\alpha = 0.5$ in

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Figure 9, are reminiscent of the intricate patterns that can be found when zooming in on regions near the border of the conventional **M** set (see, e.g., [9] or [10], p. 193).

Unlike in Figure 8, $\tau = 2$ α-memory (whose patterns are not shown here) somehow preserves the main features of the conventional Julia set, albeit expanding the picture.

The case of $\tau = 2$ $\alpha = 1.0$-memory is shown under the $\epsilon = 0.5$ label in Figure 10, which shows the effect of ϵ-memory in the Julia set considered in Figure 8. Far beyond $\epsilon = 0.5$, some essentials of the original snapshot are somehow recovered, and in the extreme scenario $\epsilon = 1.0$ the ahistoric picture reappears.

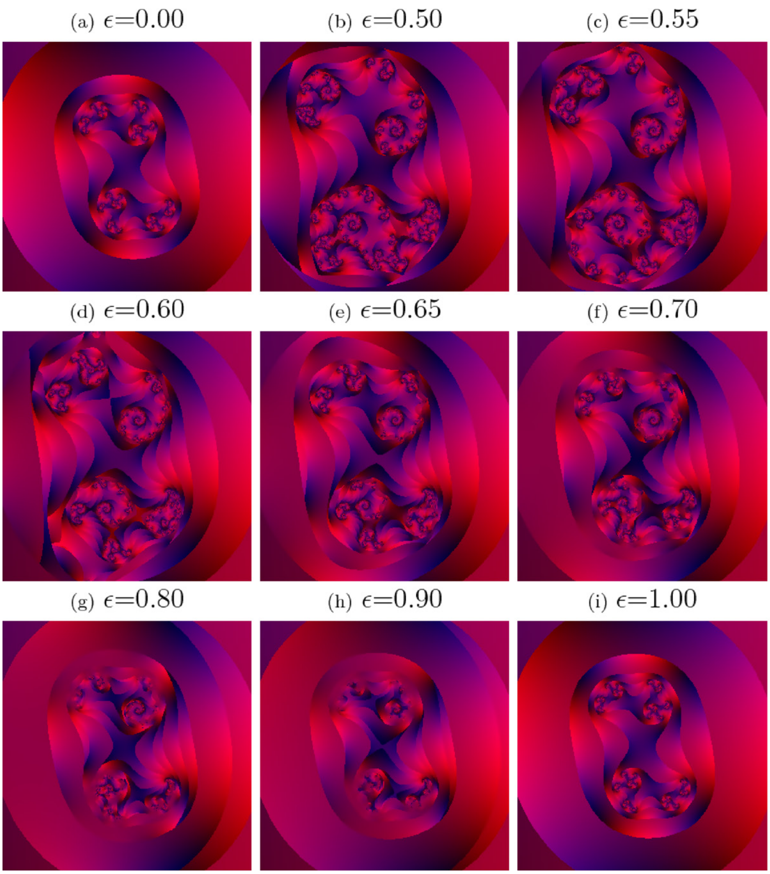

Figure 10. Julia sets with ϵ-memory.

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4. Future Work

Further study of the effect of memory on complex maps is needed, not only regarding a more detailed scrutiny of geometry, but also with respect to its general mathematical foundation. Fractional calculus has been applied in the analysis of discrete maps with memory [11, 12]. Maybe it can help in this respect.

Let us conclude with a note addressed to the reader unfamiliar with the literature concerning the study of memory in discrete dynamical systems. We have considered here a kind of embedded memory implementation:

$$z_{T+1} = f\left(\sum_{t=1}^{T} p_t\, z_t\right). \tag{6}$$

This approach differs from others in which the original mapping function f is somehow extended to g, in order to consider the influence of previous states $z_{T+1} = g\,(z_T, z_{T-1}, \ldots, z_1)$, often in the form

$$z_{T+1} = \sum_{t=1}^{T} p_t\, f(z_t). \tag{7}$$

The approach in equation (7) is often referred to as dynamic systems with delay. (Often, but not always.) To give just one example, the study developed in [13] follows the approach in equation (7) (unlike our study [14]), but uses the word "memory" instead of "delay." (Incidentally, [13] is cited in Chapter 18 of [12], "Fractional Dynamics and Discrete Maps with Memory." It seems that fractional calculus proved to be suitable to deal with the equation (7)-memory approach. Thus, its application to the equation (6)-memory approach appears particularly challenging.)

Acknowledgments

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