High-Probability Trajectories in the Phase Space and the System Complexity

Dan C. Marinescu

Computer Science Division
Department of Electrical Engineering and Computer Science
University of Central Florida, Orlando, FL 32816, USA
dcm@cs.ucf.edu
http://www.cs.ucf.edu/~dcm

The dynamic behavior of a system can be modeled as the trajectory of the system in the phase space. A phase space is an abstraction where each possible state of the system is represented by a unique point; each dimension of the phase space represents a degree of freedom of the system. Individual trajectories have different probabilities, with some of them more likely than others. For a complex system, it is conjectured that the highly probable trajectories in the phase space are dominant. Random walks are analyzed for fully connected finite state machines. We show that the cardinality of the set of highly probable trajectories is very large; its lower bound is exponential in the number of states traversed by the random walk and in an expression of the entropy of a system.

1. Introduction and Motivation

When we think about complex systems, the human brain comes immediately to mind; the number of neurons in the human brain is estimated to be between 80 and 120 billion. Systems with a very large number of components such as the space shuttle, a modern microprocessor, the internet, and computer clouds are examples of complex manmade systems.

Indeed, "the main elements of the space shuttle … are assembled from more than 2.5 million parts, 230 miles of wire, 1040 valves, and 1440 circuit breakers" [1]. The microprocessors vintage 2011 have several million transistors: the 10-core Xeon Westmere-EX produced by Intel has 2.5 million transistors, the Tahiti graphics processing unit (GPU) by Advanced Micro Devices (AMD) has 4.3 million, and the Virtex-7 FPGA by Xilinx has 6.8 million.

In January 2010, the internet connected some 800 million hosts. Cloud computing reflects the realization that information processing can be done more efficiently on large farms of computing and storage

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systems accessible via the internet than on local systems [2]. A March 16, 2012 posting on ZDNet reveals that Amazon's Elastic Compute Cloud (EC2) was made up of 454 600 servers; if the number of servers supporting other Amazon Web Services (AWS) are added, then the total number of Amazon systems dedicated to cloud computing is much larger. EC2 is one of the services provided by AWS; it provides a resizable compute capacity. An unofficial estimation puts the number of servers used by Google in January 2012 at close to 1.8 million; this number was expected to be close to 2.4 million by early 2013 [3].

Computing and communications systems are increasingly more complex and, at the same time, play a more vital role for our society. A modern server is typically built with several multicore processors, multiple servers can be linked together to form a cluster, and, finally, a cloud computing infrastructure could be built from many such clusters interconnected by high-speed networks. This is possible because the software acts as a "glue" and pushes further and further the limits of composability of such systems. In this example, each server is controlled by a software stack consisting of a hypervisor, multiple virtual machines running possibly different operating systems, and applications; additional control and communication software allows the servers in a cluster and the clusters to communicate and interact with one another.

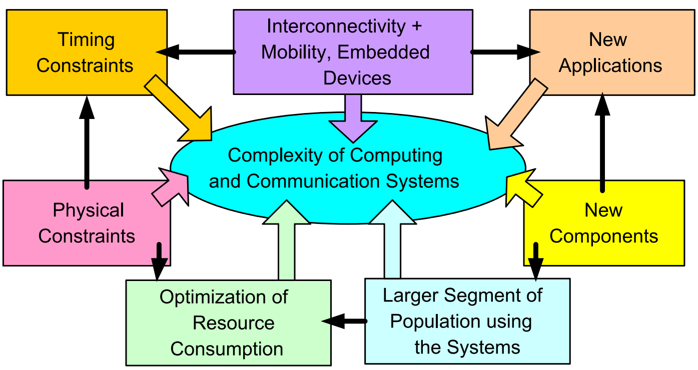

Figure 1. Factors contributing to the complexity of computing and communication systems. Causality relations between individual factors are indicated by slim arrows; for example, physical constraints demand optimization of resource consumption.

Computing and computing systems are at the heart of the critical infrastructure of society. Thus, it seems reasonable to ask what factors are contributing to their complexity, try to quantify their com-

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plexity, and attempt to use this knowledge to test and verify their dynamic behavior.

Some of the factors contributing to the complexity of computing and communication systems are illustrated in Figure 1 [2]:

- The rapid pace of technological developments and the availability of relatively cheap and efficient new system components, such as multicore processors, sensors, and high-density storage devices.
- The development of new applications, which take advantage of the new technological developments.
- The ubiquitous use of the systems in virtually every area of human endeavor, which, in turn, demands a faster pace for hardware and software development.
- The need for interconnectivity and the support for mobility.
- The need to optimize the resource consumption.
- Last but not least, constraints imposed by the laws of physics, such as heat dissipation and finite speed of light.

Though it is not feasible to quantify the effects on system complexity of all factors presented in Figure 1, we can draw qualitative conclusions regarding testing and verifications of these systems. For example, the emergence of new applications and the ever-increasing segment of the population with access to them lead to innovative, unorthodox, and often unpredictable use of the systems. In addition to the prescribed, and thus well-tested execution modes, such systems often end up in an undesirable state affecting the entire community of users.

There is also the issue of malicious use, which is increasingly more threatening to society. New technologies have provided deeply troubling means to attack such systems. For example, a hardware Trojan horse (HTH) is a malicious modification of an integrated circuit [4]. Critical infrastructure and defense systems of a country often use embedded hardware devices that could be produced by companies other than those that designed them; such companies can re-engineer the devices to make them vulnerable to attacks. Verification of embedded hardware devices affected by HTH is very hard [5].

It seems prudent that random walks through their state space should be part of the testing and verification of such systems, in addition to the most likely execution paths. But how much confidence should we have as a result of verifications including such random walks?

In this paper, we are concerned with random walks on fully connected finite state machines (FSM) and show that the cardinality of the set of highly probable trajectories is very large; its lower bound is exponential in the number of states traversed by the random walk

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and in an expression of the entropy of the system. This result reflects our intuition that among all possible system evolutions, a random walk in the state space exposes a more complex system behavior, and this behavior is amplified when the states are fully connected so that the system could reach all states from any state. This makes verification of such systems an elusive task.

2. Dynamic Behavior of Physical Systems

There is little doubt that the dynamic behavior of a system must be considered when we attempt to quantify the complexity of a system. The concept of "state" is used to describe the current condition of a system. In general, the state represents an ensemble of information; for example, the state of a computing system represents the ensemble of information necessary to restart an interrupted process or thread. An *event* is a change of state; the information describing the condition of the system prior to the event is replaced by the information describing the condition of the system after the event. Identification and characterization of the state of the system is critical for understanding the dynamic behavior of systems or entities in diverse areas, including biology and psychology, chemistry, computing, control theory, physics, and linguistics.

Dynamic behavior can be seen as the trajectory the system follows in a phase space [6]; a phase space is an abstraction where each possible state of the system is represented by a unique point. In a phase space, every degree of freedom of the system is represented as an axis of this multidimensional space. A trajectory in this space links these points and describes the evolution in time of the system; individual trajectories have different probabilities, with some of them more likely than others.

Thus, it seems reasonable to conjecture that as the ratio of high-probability trajectories in the phase space to the low-probability ones increases, so does the system complexity. When there are only a few highly probable trajectories, it is easier to debug the system and to optimize its performance. *Optimize for the common case* is a well-established design principle exploited to reduce latency by caching in the case of domain names service (DNS), web browsers, and other systems. When all or nearly all cases are "common," the design and system verification becomes very challenging.

Proving this conjecture for an irregular topology of the state transition diagram is likely to be very challenging, so we choose to focus our discussion on a fully connected FSM. An FSM is a model of a system that can be in a finite number of states; this abstraction is used in

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the design of a wide range of computing and communication systems, including hardware components, software systems, and communication protocols. For example, in computer architecture the control unit of a processor is described as an FSM. Many complex applications are based on a finite state model. In the internet, the border gateway protocol (BGP) is used for routing decisions among autonomous systems. A BGP peer maintains a table of IP networks and uses an FSM with six states; for each peer-to-peer session, a state variable tracks the state the session is in.

The complexity of an FSM depends on the number of states and the connectivity among these states. A fully connected FSM allows transitions from any state to any other state. For example, the system for minimally invasive surgery described in [7] is modeled by a fully connected FSM.

A random walk is an abstraction of a trajectory when successive steps are taken at random; random walks model stochastic activities in a wide range of disciplines, including physics, biology, computer science, and chemistry. Random walks can be conducted on graphs, lines, planes, or higher-dimensional objects.

3. Quantifying System Complexity

Abstract questions about systems consisting of an ensemble of components have preoccupied the minds of humans for millennia. For example, Aristotle stated that "...the whole is something over and above its parts, and not just the sum of them all." In *The Republic*, Plato introduces the concept of "level of knowledge," ranging from total ignorance to total knowledge. "True knowledge" exists only if a foundation of axioms or a priori knowledge exists [8], and this cannot be the case for complex systems.

In modern times, in addition to the philosophical aspects of composability and complexity, we are also concerned with practical questions including quantitative characterization of complexity. To understand and facilitate the rapid adoption of increasingly more complex computing and communication systems, the metrics that allow us to assess the complexity of a system are investigated; we also attempt to gain insight into how complex systems behave in nature.

While we have an intuitive notion of what complexity means, no rigorous definition allowing us to quantify and measure the complexity of a system is universally accepted. Certainly, the scale of a system, the topology of the interconnect linking the individual components, the unpredictability of the next state, and the length of time a system has been in existence may affect its complexity, but none of these elements by itself allows us to conclude that a system is complex or not.

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All systems are thus physical; the concept of state reflects an intrinsic property of a physical system. The transition from one state to another has informational as well as thermodynamic consequences; in this process, the information about the current state is erased and the one about the new state is engraved in the memory of the system.

Thus, there is an intrinsic relationship between matter, system dynamics, and information. This relationship allows us to explain the apparent paradox known as James Maxwell's demon. Maxwell imagined a gedanken experiment involving the famous demon as a challenge to the second law of thermodynamics. Landauer's law [9] states that erasing a single bit of information generates an amount of heat equal to $k_B T \ln 2$, with $k_B = 1.3806488 \times 10^{-23}$ the Boltzmann constant.

According to Bennett, this law implies that "any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment" [10].

The concept of entropy will be used in our derivation in Section 4, and next we shall discuss several flavors of it. Shannon entropy, thermodynamic entropy, and von Neumann entropy are related to the number of states of a system, thus they reflect to some extent the system complexity [11].

Shannon's entropy is a measure of the uncertainty of a single random variable X with the probability density function $p_X(x)$ before it is observed, or the average uncertainty removed by observing it:

$$H(X) = -\sum_x p_X(x) \log p_X(x). \tag{1}$$

The thermodynamic entropy of a microscopic system, for example, N molecules of gas, is

$$S = k_B \ln \Omega, \tag{2}$$

with k_B the Boltzmann's constant and Ω the number of microstates of the system. If the N molecules are grouped together in m macro states depending on their energy, then the number of bits required to label the individual microstates is

$$Q = H(p_1, p_2, \ldots, p_m), \tag{3}$$

with $H(p_1, p_2, \ldots, p_m)$ the Shannon entropy of a system with m states. If n_i is the number of molecules in state i, then $p_i = n_i / N$ is the probability of the system being in state i [12].

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In turn, the von Neumann entropy of a quantum system with the density matrix ρ

$$S(\rho) = -\operatorname{tr}[\rho \log \rho] \tag{4}$$

is equal to the Shannon entropy if the system is prepared in a *maximally mixed state*, a state where all pure states are equally likely.

The *relative predictive efficiency*, $e = E / C$ with E the excess entropy and C the statistical complexity [13], is also a measure of complexity. The *excess entropy* measures the complexity of the stochastic process and can be regarded as the fraction of historical information about the process that allows us to predict the future behavior of the process. The *statistical complexity* reflects the size of the model of the system at a certain level of abstraction. The scale of organization considered by an external observer plays a critical role in assessing the relative predictive efficiency. For example, at the microscopic level, the calculation of e for a volume of gas requires very complex molecular dynamics computations in order to accurately predict the excess entropy; both E and C are very high and the predictive efficiency is low. On the other hand, at the macroscopic level, the relationship between the pressure P, the volume V, and the temperature T is very simple: $P\,V = n\,R\,T$, with n the number of moles of gas and R the universal gas constant. In this case E maintains a high value, but now C is low and the predictive efficiency E / C is large.

Physical systems in equilibrium display their most complex behavior at *critical points*. In thermodynamics, a critical point specifies the conditions of temperature and pressure at which a phase boundary, for example, between liquid and gas, ceases to exist. The time to reach equilibrium becomes very high at critical points, a phenomena called *critical slowing*.

Wolpert and Macready [14] argue that *self-similarity* can be used to quantify complexity; the patterns exhibited by complex systems at different scales are very different, while the patterns exhibited by simple systems such as gases and crystals do not vary significantly from one scale to another.

We could use the complexity of a program that simulates the system as a measure of complexity of the system; this will reflect not only the number of states but also the pattern of transitions among states. This idea has its own limitations, as we generally simulate approximate models of a system, rather than exact ones.

This measure of complexity is consistent with the concept of *depth* defined as the number of computational steps needed to simulate a state of a system; the author of [15] argues that the emergence of complexity requires a long history, but a measure stricter than physical time is needed to reflect this history. The depth reflects not how long the system remains in equilibrium, but how many steps are necessary

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to reach equilibrium following some efficient process. The rate of change of the system state and the communication time do not reflect the complexity of a system. Indeed, two rotating structures involving very different physical processes—a hurricane and a spiral galaxy—are at the limit of today's realistic computer simulation, both of similar depth and, consequently, of similar complexity. Yet, galaxy formation occurs at a scale of millions of light years and is bounded by communication at the speed of light, while the time for hurricane formation is measured in days, the atmospheric disturbances propagate more slowly, and the scale of hurricane formation is only hundreds of kilometers.

Complexity could be related to the description of a system and may consist of structural, functional, and possibly other important properties of the system. The question of how to measure the descriptive complexity of an object was addressed by Kolmogorov [16] and independently by Solomonoff [17] and Chaitin [18].

The Kolmogorov complexity $K_{\mathcal{V}}(s)$ of the string s with respect to the universal computer $\mathcal{V}$ is defined as the minimal length over all programs $\text{Prog}_{\mathcal{V}}$ that print s and halt:

$$K_{\mathcal{V}}(s) = \min[\text{Length}(s)] \text{ over all Prog} : \mathcal{V}\left(\text{Prog}_{\mathcal{V}}\right) = s. \tag{5}$$

This approach is intuitive and has been known for centuries. "Nunquam ponenda est pluritas sine necesitate," the famous principle formulated by William of Ockham (c. 1290–1349), states that an explanation should not be extended beyond what is necessary [19]. Bertrand Russell translates this as, "It is vain to do with more what can be done with fewer." An application of Kolmogorov complexity to the characterization of scheduling on a computational grid is discussed in [20].

The concept of *computational irreducibility* was introduced by Stephen Wolfram in [21]; it captures the intuition of inability to shortcut a program, or to describe the behavior of a system in a simple way.

It was then observed that some computationally irreducible elementary cellular automata have properties that are predictable, and so these properties are computationally reducible. In the experiments reported in [22], several cells of an automaton were replaced, fused into a single cell; a new rule for the low-resolution automaton was then devised that would lead to the same long-term behavior as the original automaton. An analogy for this experiment is to make a low-resolution digital image from a higher-resolution version.

The conclusion of [22] is that for some rules, the low-resolution version behaves simply and predictably, even when the high-resolution version is computationally irreducible and therefore unpre-

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dictable. In other words, the complexity can only be reflected in the unimportant details and when only approximate results are acceptable, such results are predictable in spite of the system complexity.

As pointed out by Wolfram, the real question is "under what circumstances will large-scale rules emerge that allow a simple, predictable description of a complex phenomenon" [23]. The results discussed in Section 4 cast some doubts that such circumstances are realistic for many systems of interest; the expectation that computational irreducibility can be avoided seems rather utopic.

4. Random Walks in Fully Connected Finite State Machines

Now we return to our conjecture, namely, that the cardinality of the set of highly probable trajectories is very large for a system we suspect to exhibit complex dynamics. We study random walks in a fully connected FSM and we consider $\mathcal{F}$ an FSM initially in state S_i. We assume that:

1. The system can transition from the initial state to a finite number of states $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ and $|\mathcal{S}| = m$;
2. The FSM is fully connected and p_i is the probability of an input that causes a transition from the current state to state S_i, $1 \leq i \leq m$;

$$\sum_{i=1}^{m} p_i = 1. \tag{6}$$

Call σ^n a random walk of length n and assume that in any random walk the next state is selected independently. Then the probability of the random walk σ^n traversing the states $(S_{i_1}), (S_{i_2}), \ldots, (S_{i_n})$ in this order is

$$P(\sigma^n) = p(S_{i_1})\, p(S_{i_2}) \ldots p(S_{i_n}). \tag{7}$$

We define the entropy of the source controlling the behavior of the FSM $\mathcal{F}$ as

$$H(\mathcal{S}) = -\sum_{i=1}^{m} p_i \log p_i. \tag{8}$$

The *weak law of large numbers* states that $\overline{x} = \sum_i x_i\, p_{x_i}$; the mean of a large number of independent, identically distributed random variables x_i approaches $1/n \sum_i x_i$, the average, with a high probability when n is large:

$$\text{Prob}\left[\left|\frac{1}{n}\sum_{i=1}^{n} x_i - \overline{x}\right| > \delta\right] < \epsilon, \tag{9}$$

with δ and ϵ two arbitrarily small positive real numbers.

If we define a random variable $x_i = -\log p(S_i)$, then we can establish the following correspondence with the quantities in the expression of the weak law of large numbers:

$$\sum_{i=1}^{n} x_i = -\sum_{j=1}^{n} \log p\left(S_{i_j}\right) = -\log P(\sigma^n), \tag{10}$$

$$\overline{x} = \sum_{i=1}^{m} x_i\, p(x_i) = -\sum_{i=1}^{m} p(S_i) \log p(S_i) = H(\mathcal{S}). \tag{11}$$

It follows that given two arbitrarily small real numbers $(\delta, \epsilon) \geq 0$, then for sufficiently large n, the following inequality holds:

$$\text{Prob}\left[\left|-\frac{1}{n} \log P(\sigma^n) - H(\mathcal{S})\right| > \delta\right] < \epsilon. \tag{12}$$

This inequality partitions Λ, the set of random walks of length n, into two subsets:

(a) The subset Λ_{hpt} of high-probability trajectories (hpt) (see Figure 2) is defined by

$$\Lambda_{\text{hpt}} = \left\{\sigma_{\text{hpt}} : \left|-\frac{1}{n} \log P\left(\sigma_{\text{hpt}}\right) - H(\mathcal{S})\right| \leq \delta\right\}. \tag{13}$$

These random walks occur with probability $\text{Prob}\left(\sigma_{\text{hpt}}\right) \geq (1 - \epsilon)$.

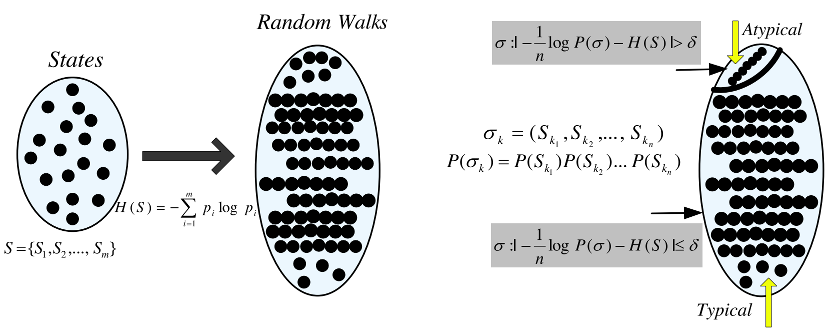

Figure 2. Random walks in a fully connected FSM. Typical high-probability random walks are dominant.

(b) The subset of low-probability trajectories (lpt) is defined by

$$\text{Prob}\left[\left|-\frac{1}{n} \log P(\sigma) - H(\mathcal{S})\right| > \delta\right] < \epsilon. \tag{14}$$

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These random walks occur with a vanishing probability, $\text{Prob}(\sigma_{\text{lpt}}) < \epsilon$. The two subsets are disjoint and complementary:

$$\Lambda_{\text{hpt}} \bigcap \Lambda_{\text{lpt}} = \emptyset \qquad \text{and} \qquad \Lambda = \Lambda_{\text{hpt}} \bigcup \Lambda_{\text{lpt}}. \tag{15}$$

We concentrate on high-probability random walks, as low-probability walks occur with a vanishing probability and can be ignored. Now we shall determine $|\Lambda_{\text{hpt}}|$, the cardinality of the set of high-probability random walks, $\sigma \in \Lambda_{\text{hpt}}$. The inequality defining the random walks in this subset can be rewritten as

$$\begin{aligned} &-\delta \leq -\frac{1}{n} \log P(\sigma_{\text{hpt}}) - H(\mathcal{S}) \leq +\delta \\ &\Longrightarrow 2^{-n(H(\mathcal{S})+\delta)} \leq P(\sigma_{\text{hpt}}) \leq 2^{-n(H(\mathcal{S})-\delta)}. \end{aligned} \tag{16}$$

The first inequality can be expressed in terms of the cardinality set:

$$\sum_{\sigma \in \Lambda_{\text{hpt}}} P(\sigma_{\text{hpt}}) \geq \sum_{\sigma \in \Lambda_{\text{hpt}}} 2^{-n(H(\mathcal{S})+\delta)} = |\Lambda_{\text{hpt}}| \, 2^{-n(H(\mathcal{S})+\delta)}. \tag{17}$$

It follows that

$$|\Lambda_{\text{hpt}}| \leq 2^{n(H(\mathcal{S})+\delta)}. \tag{18}$$

Similarly, the second inequality can be expressed as

$$\sum_{\sigma \in \Lambda_{\text{hpt}}} P(\sigma_{\text{hpt}}) \leq \sum_{\sigma \in \Lambda_{\text{hpt}}} 2^{-n(H(\mathcal{S})-\delta)} = |\Lambda_{\text{hpt}}| \, 2^{-n(H(\mathcal{S})-\delta)}. \tag{19}$$

This implies that

$$|\Lambda_{\text{hpt}}| \geq 2^{n(H(\mathcal{S})-\delta)}. \tag{20}$$

It follows that

$$\delta \mapsto 0 \Longrightarrow |\Lambda_{\text{hpt}}| \mapsto 2^{n\,H(\mathcal{S})}. \tag{21}$$

Thus, the cardinality of the set of high-probability random walks converges to $2^{n\,H(\mathcal{S})}$. This result can be interpreted as the fact that system dynamics encode the most probable evolution of the system with entropy $H(\mathcal{S}) = -\sum_{i=1}^{m} p_i \log p_i$, where p_i is the probability that the system chooses S_i as the next state of the random walk.

An astute observer will notice that this derivation follows the proof of Shannon's source coding theorem.

5. Conclusions

In many instances, a quantitative characterization of system complexity turns out to be an elusive task. In this paper, we study the dynamics of a restricted class of systems, fully connected finite state machines (FSM) subject to a special evolution pattern, the random walks; such systems reflect our informal view of complexity. We show that the cardinality of the subset of high-probability random walks $\left|\Lambda_{\text{hpt}}\right|$ converges to $2^{n\,H(S)}$—in other words, that highly probable trajectories dominate the system dynamics. It is expected that the entropy of the source that controls the random walk affects the behavior of the system and this result confirms our expectations.

This result confirms our intuition that random walks in a fully connected system expose complex system dynamics. It also justifies our conjecture that the ratio of high-probability to low-probability trajectories in a phase space could be used as a quantitative characterization of system complexity.

An open question is if this result can be extended to other topologies and if the dominance of high-probability trajectories could be used to assess ergodicity.

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