Retrospective-Prospective Differential Inclusions and Their Control by the Differential Connection Tensors of Their Evolutions: The Trendometer

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There are two different motivations of this study: retrospective-prospective differential inclusions and differential connection tensors in networks of both continuous-time nonsmooth functions and time series, and several consequences: control of dynamical systems by differential connectionist tensors, detection by a "trendometer" of all local extrema of differentiable functions, but "wild" as the sum of three sines, as well as of economic and financial time series. It provides us a "trend reversal" of the Fermat rule, using the zeros of the derivative for finding all the local extrema of any numerical function of one variable. Instead, the extrema of the primitive of a function allow us to find zeros of the function. Once detected, the trendometer measures the "jerkiness" of their trend reversals. It provides an efficient econometric tool for detecting crisis (e.g., the dot.com and subprime crises), the dates of the trend reversals, and their jerkiness, helping qualitative analysts to focus their attention on the dates when quantitative jerkiness of the extrema is high.

The differential connection matrix plays for evolutions (and discrete time series) a dynamic role analogous to the static role played by the *covariance matrix* of a family of random variables measuring the covariance entries between two random coefficients. Covariance matrices deal with random variables. Differential connection tensors deal with temporal series or continuous-time evolutions. They are therefore different and cannot be compared, since they deal with different mathematical universes.

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1. Motivations

There are two different motivations of this study.

1.1 Retrospective-Prospective Differential Inclusions

The first motivation follows the plea of Efim Galperin in [1, 2] for using "retrospective" derivatives instead of "prospective" derivatives, universally chosen since their introduction by Newton and Leibniz, at a time when physics became predictive and deterministic: the "prospective derivatives" $\overrightarrow{D}x(t)$ being (more or less weak) limits of *prospective (future) difference quotients* (on positive durations $h > 0$)

$$\overrightarrow{\nabla}_h x(t) := \frac{x(t+h) - x(t)}{h}$$

are "physically nonexistent," because they are not yet known at time t. Whereas the *retrospective (past) difference quotients*

$$\overleftarrow{\nabla}_h x(t) := \frac{x(t) - x(t-h)}{h}$$

may be known for some positive durations and should be taken into account. This has been pointed out by Jiri Buquoy, who in 1812 formulated the equation of motion of a body with variable mass, which retained only the attention of Poisson before being almost forgotten. See [3–5] among the precursors in this area.

This is an inescapable issue in life sciences, since the evolutionary engines evolve with time, under contingent and tychastic uncertainty and, in most cases, cannot be recreated (at least, for the time, since synthetic biology deals with this issue). See, for instance, [6]. Popper's recommendations are valid for physical sciences, where experimentation is possible and renewable. However, the quest of the instant (temporal window with 0 duration) has not yet been experimentally created (the smallest measured duration is of the order of the yoctosecond [10^{-24} seconds]). Furthermore, our brains deal with observations that are not instantaneous, but, in the best case, are perceived after a positive transmittal duration.

For overcoming these situations, Fermat, Newton, Leibniz, and billions of human brains have invented instants and passed to the limit when duration of temporal windows goes to 0 to reach such an instant. This is actually an approximation of reality by clever mathematical constructions of objects belonging to an ever-evolving "cultural world," actually, an inductive approximation, whereas (deductive) application refers to approximate derivatives of the idealized world by difference quotients, which are closer to the actual perception of our

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brains and capabilities of digital computers. Derivatives are not perceived, but were invented, simplifying reality by passing to the limit in a mathematical paradise.

Therefore, for differentiable functions in the classical sense, the limits of retrospective and prospective difference quotients may coincide when we pass to the limit. But this is no longer the case when evolutions are no longer differentiable in the classical sense, but derivatives may still exist for "weaker" limits, such as limits in the sense of distributions or graphical limits in set-valued analysis (see [7, Section 18.9, p. 769]). Even if we restrict our analysis to Lipschitz functions, Rademacher's theorem states that Lipschitz maps from one finite-dimensional vector space to another are only almost everywhere differentiable. Although small, the set of elements that are not differentiable is interesting because Lipschitz maps always have set-valued graphical derivatives. Hence we have to make a detour by recalling what are meant by retrospective and prospective graphical derivatives of maps as well as set-valued maps and nondifferentiable (single-valued) maps.

Therefore, we devote the first part of this study to a certain class of viable evolutions governed by functional (or history-dependent) differential inclusions

$$x'(t) \in G\left(t, x(t), \overleftarrow{D}x(t)\right)$$

where $\overleftarrow{D}x(t)$ is the *retrospective derivative* (or derivative from the left since, at this stage, we consider evolutions defined on $\mathbb{R}$). Retrospective-prospective differential inclusions $x'(t) \in G\left(t, x(t), \overleftarrow{D}x(t)\right)$ describe predictions of evolutions based on the state and on the known retrospective velocity at each chronological time. As delayed differential equations or inclusions, they are particular cases of *functional* (or *historical*, *path-dependent*, etc.) differential equations (see [8–11], summarized in [12, Chapter 12], [13, 14], etc.). As for second-order differential equations, initial conditions $x(t_0)$ at time t_0 must be provided, as well as (retrospective) initial velocities for selecting evolutions governed by retrospective-prospective differential equations and inclusions. Differentiable evolutions governed by such systems boil down to solutions to implicit differential equations and inclusions of the form $x'(t) \in G(t, x(t), x'(t))$.

1.2 Differential Connection Tensors in Networks

The second motivation emerged from the study of propagation through "junctions of a network," such as crossroads in road networks, banks in financial networks, synapses in neural networks, and others.

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1.2.1 Neural Network: The Hebbian Rule

If we accept that in formal neuron networks, "(evolving) knowledge" is coded as "synaptic weights" at each synapse, their collection defines a "synaptic matrix" that evolves and thus becomes the "state of the network."

Recall that the tensor product $p \otimes q$ of two vectors $p := (p_i)_i \in \mathbb{R}^\ell$ and $q := (q_j)_j \in \mathbb{R}^\ell$ is the rank-one linear operator

$$p \otimes q \in \mathcal{L}\left(\mathbb{R}^\ell, \mathbb{R}^\ell\right) : x \mapsto \langle p, x \rangle q,$$

the entries of which (in the canonical basis) are equal to $\left(p_i\, q_j\right)_{i,j}$. Donald Hebb introduced in 1949 [15] the *Hebbian learning rule* prescribing that the velocity of the synaptic matrix is proportional to the tensor product of the "presynaptic activity" and "postsynaptic activity" described by the propagation of nervous influx in the neurons.

Hence, denoting the synaptic matrix W of synaptic weights, the basic question was to minimize a "matrix function" $W \in \mathcal{L}(X, X) \mapsto E(\mathrm{W}x)$ where $x \in X := \mathbb{R}^\ell$ and $E : X \mapsto \mathbb{R}$, a differentiable function, are given. Remembering (see [16, Proposition 2.4.1, p. 37 and Chapter 2]) that the gradient with respect to W is equal to the tensor product $E'(Wx) \otimes x$, the gradient method leads to a differential equation of the form

$$W'(t) = -\alpha\, E'(W(t)\, x) \otimes x, \tag{1}$$

which governs the evolution of the synaptic matrix (the synapse x is fixed and does not evolve). Connectionist matrices and tensors have also been used for defining the complexity of a connectionist matrix (by its distance to the identity matrix) and its connections with decentralization issues in [17], and in [18] and [19] for studying the regulation of the evolution of the architecture of a network by connectionist tensors operating on coalitions of actors.

1.2.2 Differential Connection Tensors

However, we take into account the evolution $t \mapsto x(t) \in X$ of the propagation in networks (such as the propagation of nervous influx, traffic, financial product, etc.). If the evolution is Lipschitz, retrospective and prospective derivatives exist at all times, so that we can define the tensor product $\overleftarrow{D}\, x(t) \otimes \overrightarrow{D}\, x(t)$ of their retrospective and prospective velocities: we shall call it the *differential connection tensor* of the evolution $x(\cdot)$ at time t.

It plays the role of a *trendometer* measuring the *trend reversal* (or *monotonicity reversals*) at junctions: the differential connection tensor describes the trend reversal between the retrospective and prospective

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trends when they are strictly negative, the *monotonicity congruence* when they are strictly positive, and the *inactivity* when they vanish. In neural networks, for instance, this is an inhibitory effect or trend reversal in the first case, an excitatory or trend congruence in the second case, and inactivity of a synapse: in this case, one at least of the propagations of the nervous influx stops. The absolute value of this product measures in some sense the jerkiness of the trend reversal at a junction of the network.

For individual evolutions (continuous- or discrete-time numerical functions), the trendometer detects all their local extrema. Instead, the extrema of the primitive of a function allow us to find zeros of the function. Once detected, the trendometer measures the "jerkiness" of their trend reversals. It provides an efficient econometric tool for detecting crisis (the dot.com and subprime crises), the dates of the trend reversals, and their jerkiness, helping qualitative analysts to focus their attention on the dates when quantitative jerkiness is high.

We are also tempted to control (pilot, regulate, etc.) the evolution of propagation in the network governed by a system

$$x'(t) = g(x(t), u(t)) \text{ where } u(t) \in U\left(\overleftarrow{D}\, x(t) \otimes \overrightarrow{D}\, x(t)\right) \tag{2}$$

controlled by differential connectionist tensors at junctions of the network. We recall that the evolutions governed by (Marchaud) controlled systems are Lipschitz under the standard assumption, but not necessarily differentiable. For example, in order to govern the viability of the propagation in terms of the inhibitory, excitatory, and stopping behavior at the junctions of the network, some constraints are imposed on the evolution of the differential connection tensors. Examples of retrospective-retrospective differential equations are provided by tracking or controlling differential connection tensors of the evolutions, requiring that evolutions governed by differential equations $x'(t) = f(t, x(t))$ satisfy constraints of the form $\overleftarrow{D}\, x(t) \otimes \overrightarrow{D}\, x(t) \in C(t, x(t))$. These control systems are examples of retrospective-prospective differential inclusions.

These considerations extend to "multiple synapses" when we associate with each subset S of branches j meeting at a junction the tensor products $\otimes_{j \in S} x_j'(t)$ of the velocities at the junction (see [20] and the literature on $\Sigma - \Pi$ neural systems, [7, Section 12.2], as well as [18, 21, 22] and the literature on the regulation of networks).

1.3 Organization of the Study

Section 2 defines retrospective and prospective (graphical) derivatives of tubes and evolutions, their *differential connection tensor* (Definition 1). They are the ingredients for introducing retrospective-

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prospective differential inclusions. The viability theorem (Theorem 1) is adapted for characterizing viable tubes under such differential inclusions, using characterizations linking the retrospective and prospective derivatives of the tube. When these conditions are not satisfied, we restore the viability by introducing the *retrospective-prospective viability kernel* of the tube under the retrospective-prospective differential inclusion (Section 2.3).

Section 3 studies the regulation of viable evolutions on tubes by imposing constraints on their differential connectionist tensors.

Section 4 explains how the trendometer can be used for detecting minima and extrema of differentiable functions and thus, zeros of their derivatives, including those functions that display a "wild behavior," such as the sum of three sines.

Section 5 illustrates how the tensor trendometer was used for detecting the dynamic correlations between time series, for example, the time series of the prices of the 40 assets of the CAC 40 stock market index (see [23, Chapter 2] for a more detailed study).

Section 6 defines differential connectionist tensors of set-valued maps (Section 6.1) and gathers some other classes of differential connection tensors besides the ones of the evolutions $t \mapsto x(t)$ or tubes $t \rightsquigarrow K(t)$ from $\mathbb{R}$ to $\mathbb{R}^\ell$, which provided the first source of motivations for studying differential connection tensors. Other specific examples are the differential connection tensors of numerical functions $V : \mathbb{R}^\ell \mapsto \mathbb{R}$ (Section 6.2) and tangential connection tensors of retrospective and prospective tangents (Section 6.3). These issues are the topics of forthcoming studies.

2. Retrospective-Prospective Differential Inclusions

2.1 Prospective and Retrospective Derivatives of Tubes and Evolutions

A tube is the nickname of a set-valued map $K : t \in \mathbb{R} \rightsquigarrow K(t) \subset X$. Since there are only two directions $+1$ and -1 in $\mathbb{R}$, the prospective (left) and retrospective (right) derivatives of a tube K at a point (t, x) of its graph are defined by

$$\begin{cases} v \in \overrightarrow{D} K(t, x) \text{ if and only if } \liminf_{h \to 0+} d\left(v, \dfrac{K(t+h) - x}{h}\right) = 0, \\ v \in \overleftarrow{D} K(t, x) \text{ if and only if } \liminf_{h \to 0+} d\left(v, \dfrac{x - K(t-h)}{h}\right) = 0. \end{cases} \tag{3}$$

Actually, there is a third one, 0, where $\overleftarrow{D}K(t,x)(0)$ and $\overrightarrow{D}K(t,x)(0)$ are the retrospective and prospective tangent cones studied in Section 6.3 (see Definition 3 in the general case).

Definition 1. The differential connection tensor of a tube $K(\cdot)$ at $x \in K(t)$ is defined by

$$\begin{aligned}&\forall\, \overleftarrow{v} \in \overleftarrow{D}K(t,x), \qquad \forall\, \overrightarrow{v} \in \overrightarrow{D}K(t,x),\\ &\quad \mathbf{a}_K(t,x)\left[\overleftarrow{v}, \overrightarrow{v}\right] := \overleftarrow{v} \otimes \overrightarrow{v}.\end{aligned} \tag{4}$$

In particular, an evolution $x(\cdot)$ is a single-valued tube defined by $K(t) := \{x(t)\}$, so that we can define its graphical prospective derivative $\overrightarrow{D}x(t)$ (from the right) and retrospective derivative $\overleftarrow{D}x(t)$ (from the left), respectively (see illustrations in Section 5).

2.2 Retrospective-Prospective Differential Inclusions

Recall that whenever an evolution $t \mapsto x(t)$ is viable on a neighborhood of t_0 on a tube $K(t)$, then $\overleftarrow{D}x(t_0) \in \overleftarrow{D}K(t_0, t_0)$ and $\overrightarrow{D}x(t_0) \in \overrightarrow{D}K(t_0, t_0)$.

Since we know only retrospective derivatives, forecasting future evolution can be governed by prospective differential inclusion $\overrightarrow{D}x(t) \in F(t, x(t))$ depending only on time and state, but also by the particular case of history-dependent evolutions $\overrightarrow{D}x(t) \in G\left(t, x(t), \overleftarrow{D}x(t)\right)$ depending on time, state, and the retrospective derivatives. This could be the case for system-controlling the differential connectionist tensors of the evolutions, for instance (see Section 3).

Theorem 1. Let us assume that the map $(t, x, v) \in \mathbb{R} \times X \times X \rightsquigarrow G(t, x, v) \subset X$ is Marchaud (closed graph, convex valued, and linear growth) and that the tube $t \rightsquigarrow K(t)$ is closed. Then the "tangential condition"

$$\forall\, \overleftarrow{v} \in \overleftarrow{D}K(t,x), \qquad G\left(t, x, \overleftarrow{v}\right) \cap \overrightarrow{D}K(t,x) \neq \emptyset \tag{5}$$

is equivalent to the "viability property": from any initial state $x_0 \in K(t_0)$ and initial retrospective velocity $\overleftarrow{v}_0 \in \overleftarrow{D}K(t_0, x_0)$ there exists at least one evolution $x(\cdot)$ governed by the retrospective-prospective differential inclusion $\overrightarrow{D}x(t) \in G\left(t, x(t), \overleftarrow{D}x(t)\right)$ satisfying $x(t_0) = x_0$ and $\overleftarrow{D}x(t_0) = \overleftarrow{v}_0$ and viable in the tube $K(\cdot)$.

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Proof. The proof is an adaptation of the proof of the viability Theorem 19.4.2 [7, p. 782, based on Theorems 11.2.7, p. 447 and 19.3.3, p. 777]. We just indicate the modifications to be made.

We construct approximate solutions by modifying Euler's method to take into account the viability constraints, then deduce from available estimates that a subsequence of these solutions converges in some sense to a limit, and finally, check that this limit is a viable solution to the retrospective-prospective differential inclusion ($\overrightarrow{D}x(t) \in G\big(t, x(t), \overleftarrow{D}x(t)\big)$).

1. By assumption, there exists $r > 0$ such that the neighborhood $\mathcal{K}_r := \mathrm{Graph}(K) \bigcap ((t_0, x_0) + r([-1, +1] \times B))$ of the initial condition (t_0, x_0) is compact. Since G is Marchaud, the set

   $$C_r := \left\{F\left(t, x, \overleftarrow{v}\right)\right\} + B, \text{ and } T := \frac{r}{\|C_r\|}$$

   is also compact. We next associate with any h the Euler approximation

   $$v_j^h := \frac{x_{j+1}^h - x_j^h}{h} \in G\big(j\,h, x_j^h, v_{j-1}^h\big) \text{ where } v_{j-1}^h := \frac{x_j^h - x_{j-1}^h}{h} \tag{6}$$

   starting from $\left(t_0, x_0, \overleftarrow{v}_0\right)$.

2. [7, Theorem 11.2.7, p. 447] implies that for all $\varepsilon > 0$,

   $$\begin{cases} \exists\ \eta(\varepsilon) > 0 \text{ such that } \forall\ (t, x) \in \mathcal{K}_r,\ \forall\ h \in [0, \eta(\varepsilon)], \\ x_j^h + hG\big(jh, x_j^h, v_{j-1}^h\big) \in K\big(jh, x_j^h\big) + \varepsilon B. \end{cases} \tag{7}$$

   Since

   $$\left\|x_j^h - x_0\right\| \le \sum_{i=0}^{i=j-1} \left\|x_{i+1}^h - x_i^h\right\| \le \sum_{i=0}^{i=J^h-1} h\left\|v_j^h\right\| \le \|C_k\|,$$

   the discrete evolution is viable in $\mathcal{K}_r$ on the interval $[0, T]$. Denoting by x^h, $\overleftarrow{v}^h$, and $\vec{v}_h$ the linear interpolations of the sequences x_j^h, $\overleftarrow{v}_j^h$, and $\vec{v}_j^h$, we infer that there exists a constant $\alpha > 0$ such that

   $$\begin{cases} \left(t^h, x^h, \overleftarrow{v}^h, \vec{v}\right) \in \mathrm{Graph}(G) + \varepsilon\alpha, \\ \left(t^h, x^h\right) \in \mathrm{Graph}(K) + \varepsilon\alpha, \end{cases} \tag{8}$$

   and that there exists a constant $\beta > 0$ such that the *a priori* estimates

   $$\max\left(\|x^h\|_\infty, \|\overleftarrow{\nabla}^h x^h\|_\infty, \|\overrightarrow{\nabla}^h x^h\|_\infty\right) \le \beta \tag{9}$$

   are satisfied.

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3. They imply the *a priori* estimates of the convergence theorem [7, Theorem 19.3.3, p. 777], which states the limit of a converging subsequence is a solution to the retrospective-prospective differential inclusion, viable in Graph(K). □

2.3 Retrospective-Prospective Viability Kernels

Naturally, the "tangential assumption" (5) is not necessarily satisfied, so that we have to adapt the concept of a viability kernel to the retrospective-prospective case.

Definition 2. The viability kernel of the tube $K(\cdot)$ is the set of initial conditions $\left(t_0, x_0, \overleftarrow{v}_0\right) \in \mathbb{R} \times K(t_0) \times \overleftarrow{D} K(t_0, x_0)$ from which starts at least one viable evolution $t \mapsto x(t) \in K(t)$ to the retrospective-prospective differential inclusion in the sense that

$$\begin{cases} (i) & \overrightarrow{D} x(t) \in G\left(t, x(t), \overleftarrow{D} x(t)\right) \\ (ii) & \overleftarrow{D} x(t) \in \overleftarrow{D} K(t, x(t)) \text{ and } \overrightarrow{D} x(t) \in \overrightarrow{D} K(t, x(t)). \end{cases} \tag{10}$$

We provide a viability characterization of retrospective-prospective viability kernel tubes:

Proposition 1. Let us consider the control system

$$\begin{cases} (i) & \tau'(t) = 1 \\ (ii) & x'(t) \in G\left(\tau(t), x(t), \overleftarrow{v}(t)\right) \\ (iii) & \left\|\overleftarrow{v}'(t)\right\| \le c \left\|G\left(t, x, \overleftarrow{v}\right)\right\| \\ & \text{where } \overleftarrow{v}(t) \in \overline{\text{co}}\left(\overleftarrow{D} K(\tau(t), x(t))\right). \end{cases} \tag{11}$$

Then the viability kernel of the graph $\text{Graph}\left(\overleftarrow{D}K(\cdot)\right)$ of the derivative tube $K(\cdot)$ coincides with the retrospective-prospective viability kernel of the tube.

Proof. The viability kernel of the control system (11) is the set of initial triples $\left(t_0, x_0, \overleftarrow{v}_0\right)$ such that $x_0 \in K(t_0)$ and $\overleftarrow{v}_0 \in \overleftarrow{D} K(t_0, x_0)$, from which starts an evolution $t \mapsto \left(t_0 + t, x(t), \overleftarrow{v}(t)\right)$ of the control system such that $x(t) \in K(\tau(t))$ and $\overleftarrow{v}(t) \in \overline{\text{co}}\left(\overleftarrow{D} K(\tau(t), x(t))\right)$. Setting $x_\star(t) := x(t - t_0)$ and $\overleftarrow{v}_\star(t) := \overleftarrow{v}(t - t_0)$, we observe that $x_\star(t) \in G\left(t, x_\star(t), \overleftarrow{v}_\star(t)\right)$, $\overleftarrow{v}_\star(t) \in \overleftarrow{D} K(t, x_\star(t))$, and $x_\star(t) \in K(t)$. We thus infer that $\overrightarrow{D} x_\star(t) \in \overrightarrow{D} K(t, x_\star(t))$. Since $x_\star(t)$ is viable in the tube, we also infer that $\overleftarrow{D} x_{\star}(t)$ actually belongs to $\overleftarrow{D} K(t, x_{\star}(t))$. Hence $\left(t_0, x_0, \overleftarrow{v}_0\right)$ belongs to the retrospective-prospective viability kernel of the tube $K(\cdot)$. □

When the control system is Marchaud, we obtain the following consequences.

Theorem 2. Let us assume that the set-valued map $G : \left(t, x, \overleftarrow{v}\right) \rightsquigarrow G\left(t, x, \overleftarrow{v}\right)$ is Marchaud and that the graph of $\overline{\mathrm{co}}\left(\overleftarrow{D} K(t, x)\right)$ is closed. Then the retrospective-prospective viability kernel of the tube $K(\cdot)$ under the $\overleftarrow{D} x(t) \in G\left(t, x(t), \overleftarrow{D} x(t)\right)$ is closed and inherits properties of viability kernels.

3. Control By Differential Connectionist Tensors

We study the tracking at each date t of the differential connection tensor $\overleftarrow{D} x(t) \otimes \overrightarrow{D} x(t)$ of evolutions governed by a differential inclusion $x'(t) \in F(t, x(t))$.

For that purpose, we introduce a connection map $(t, x) \rightsquigarrow C(t, x) \subset \mathcal{L}(X, X)$. We are looking for evolutions $x(\cdot)$ governed by the differential inclusion satisfying the constraints on the differential connection tensors

$$\forall\, t \geq 0,\ \overleftarrow{D} x(t) \otimes \overrightarrow{D} x(t) \in C(t, x(t)). \tag{12}$$

This is a problem analogous to the search of the slow evolutions governed by control systems (solutions governed by controls of the regulation map with minimal norm): see [24] or [12, Theorem 6.6.3, p. 229].

We follow the same strategy by introducing the set-valued map $S_K(t, x, v)$ defined by

$$G\left(t, x, \overleftarrow{v}\right) := \left\{w \in F(t, x) \text{ such that } \overleftarrow{v} \otimes w \in C(t, x)\right\}. \tag{13}$$

Theorem 3. We assume that F is Marchaud, that the tube $K(\cdot)$ is closed, and that

$$\begin{cases} (i) & \text{the graph of } (t, x) \rightsquigarrow C(t, x) \subset \\ & \quad \mathcal{L}(X, X) \text{ is closed and its images are convex} \\ (ii) & \forall\, (t, x) \in \mathrm{Graph}(K),\ \forall\, \overleftarrow{v} \in \overleftarrow{D} K(t, x), \\ & \exists\, w \in F(t, x) \cap \overrightarrow{D} K(t, x) \text{ such that } \overleftarrow{v} \otimes w \in C(t, x). \end{cases} \tag{14}$$

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For any t_0, for any $x_0 \in K(t_0)$, for any $\overleftarrow{v}_0 \in \overleftarrow{D} K(t_0, x_0)$, there exists at least an evolution $x(\cdot)$ governed by the differential inclusion $x'(t) \in F(t, x(t))$ starting at x_0 viable in the tube $K(\cdot)$ such that $\overleftarrow{v}_0 \otimes \overrightarrow{D} x(t_0) \in C(t_0, x_0)$ and satisfying the differential connection tensor constraints

$$\forall\, t \geq t_0, \quad \overleftarrow{D} x(t) \otimes \overrightarrow{D} x(t) \in C(t, x(t)) \tag{15}$$

and the retrospective-prospective viability property

$$\forall\, t \geq t_0, \quad \overleftarrow{D} x(t) \otimes \overrightarrow{D} x(t) \in \overleftarrow{D} K(t, x\,(t)) \otimes \overrightarrow{D} K(t, x(t)). \tag{16}$$

Proof. The set-valued map G satisfies the assumptions of Theorem 1 in such a way that there exists one evolution $x(\cdot)$ governed by $\overrightarrow{D} x(t) \in G\left(t, x(t), \overleftarrow{D} x(t)\right)$ viable in the tube $K(\cdot)$. Therefore, $\overrightarrow{D} x(t) \in \overrightarrow{D} K(t, x(t))$ for all $t \geq t_0$. Consequently,

$$\overleftarrow{D} x(t) \otimes \overrightarrow{D} x(t) \in C(t, x(t)), \tag{17}$$

and since the evolution is viable in the tube $K(\cdot)$, that

$$\overleftarrow{D} x(t) \in \overleftarrow{D} K(t, x\,(t)) \text{ and } \overrightarrow{D} x(t) \in \overrightarrow{D} K(t, x(t)).$$

The theorem ensues. □

For instance, we can choose

$$C\left(t, x, \overleftarrow{v}\right) := \left\{\overrightarrow{v} \text{ such that } \sup_{w \in F\,(t,x)} \sup_{(i,j)} \overleftarrow{v}_i\left(\overrightarrow{v}_j - w_j\right) \leq 0\right\}. \tag{18}$$

In other words, the entries $\overleftarrow{v}_i \overrightarrow{v}_j$ minimize the entries $\overleftarrow{v}_i w_j$ of the differential connection tensors when the velocities $w \in F(t, x)$.

[25, Proposition 6.5.4, p. 226] implies that the connection constraint map has a closed graph and convex values whenever the set-valued map F is lower semicontinuous with convex compact images.

We could just as well require that the entries of the differential connection tensor maximize the entries $\hat{\overleftarrow{v}}_i \hat{\overrightarrow{v}}_j$ and minimize the entries $\overleftarrow{v}_i w_j$ of the differential connection tensors when the velocities $w \in F(t, x)$ range over the right-hand side of the differential inclusion. Or require that, for some pairs (i, j), the entries $\hat{\overleftarrow{v}}_i \hat{\overrightarrow{v}}_j$ minimize $\overleftarrow{v}_i w_j$ and for the other pairs, that they maximize $\overleftarrow{v}_i w_j$ when the velocities $w \in F(t, x)$ range over the right-hand side.

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4. Detecting Extrema and Measuring Their Jerkiness

The question arises whether it is possible to detect the connection dates when the monotonicity of one evolution of a finite family of evolutions is followed by the reverse (opposite) monotonicity of other evolutions, in order to detect the influence of each evolution on the dynamic behavior of other evolutions. When the two evolutions are the same, we obtain their reversal dates when the evolutions achieve their extrema. The *differential connection tensor* measures the *jerkiness* between two functions, smooth or not smooth (temporal series) providing the trend reversal dates of the differential connection tensor.

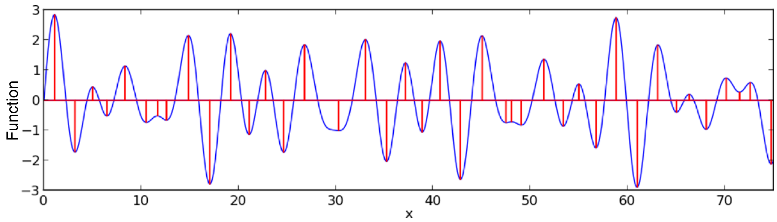
The differential connection matrix plays for evolutions (and discrete time series) a dynamic role analogous to the static role played by the covariance matrix of a family of random variables measuring the covariance entries between two random coefficients. Covariance matrices deal with random variables. Differential connection tensors deal with temporal series or continuous-time evolutions. They are therefore different and cannot be compared, since they deal with different mathematical universes. This study is devoted to evolutions governed by differential inclusions.

In other words, we replace in our analysis the dependence of random variables on random events by the dependence of evolutions on time.

The differential connection tensor software provides at each date the coefficients of the differential connection tensor.

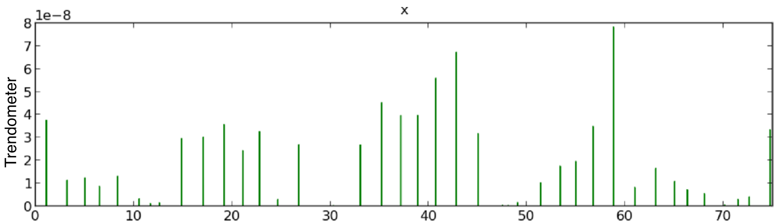
For instance, for individual evolutions, the trendometer detects the local extrema of numerical functions (Figure 1).

For the sake of comparison with the example of [26], we display the trendometer applied to this function on the interval $[0, 250]$ (Figure 2).



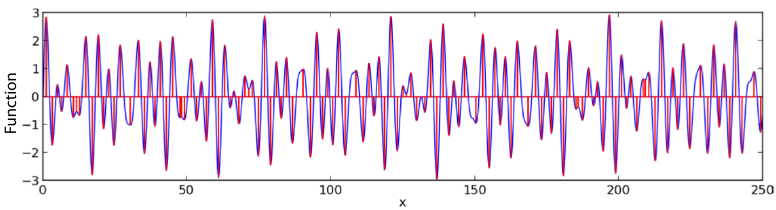
(a)

https://doi.org/10.25088/ComplexSystems.23.2.117

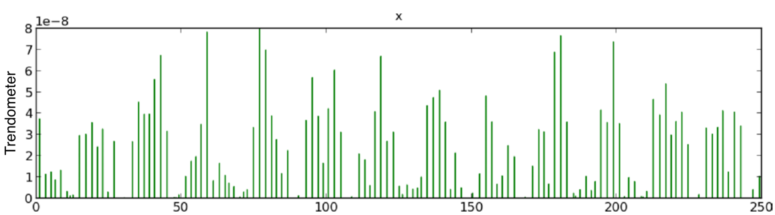


(b)

Figure 1. The trendometer can be applied to detect and measure the strength of minima and maxima of differentiable functions, such as the sum $t \in [0, 75] \mapsto \sin(x) + \sin\left(\sqrt{2}\ x\right) + \sin\left(\sqrt{3}\ x\right)$ of three trigonometric functions, as suggested in [26, p. 146], displaying two regularly spaced families. They thus detect the zeros of its derivative $t \mapsto \cos(x) + \sqrt{2}\ \cos\left(\sqrt{2}\ x\right) + \sqrt{3}\ \cos\left(\sqrt{3}\ x\right)$. (a) Displays the graph of this function and the vertical bars indicate the values at which the function reaches its extrema. (b) Displays the jerkiness of the extrema at the dates when they are reached.



(a)



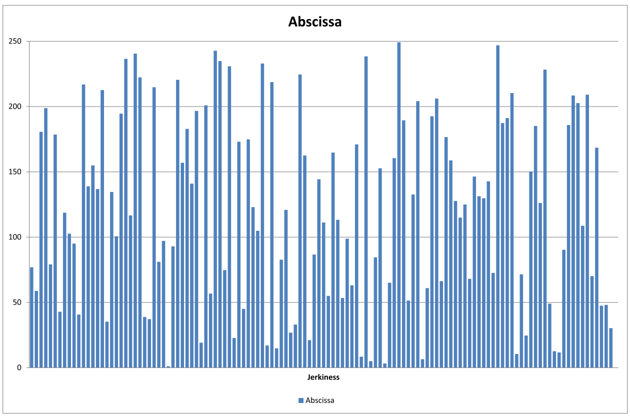
(b)

Figure 2.

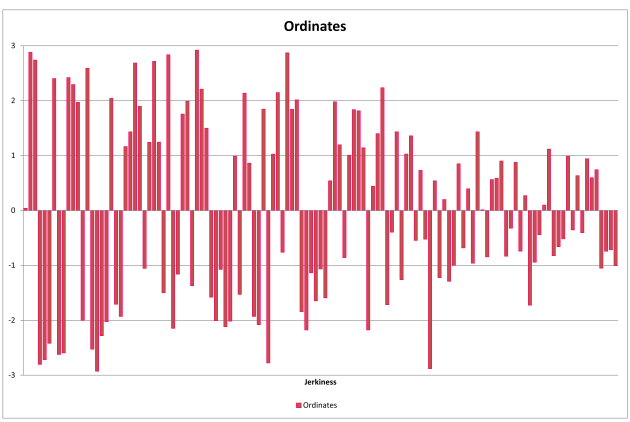
Figure 3 displays the abscissa and ordinates of the function in terms of decreasing jerkiness of their extrema.

By using a piecewise interpolation between the extrema, we obtain a "trend skeleton" summarizing the function (Figure 4).

Stephen Wolfram states [26]: "Among all the mathematical functions defined, say, in *Mathematica*, it turns out that there are also a few—not traditionally common in natural sciences—which yield complex curves which do not appear to have any explicit dependence on representations of individual numbers." This complexity, such as chaos produced by iterated maps, is linked to the fact that viability kernels of compact spaces under disconnecting maps (inverses of Hutchinson maps) are uncountable Cantor sets (see [7, Theorem 2.9.10, p. 80]).



(a)



(b)

Figure 3.

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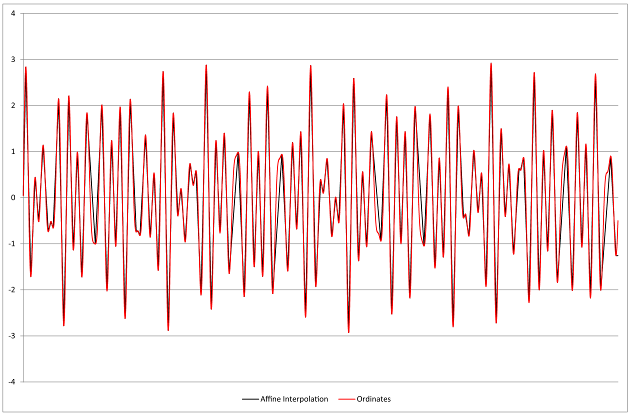

Figure 4.

The trendometer provides us a "trend reversal" of the Fermat rule. Instead of using the zeros of the derivative for finding all the local extrema of any numerical function of one variable, the extrema of the primitive of a function detected by the trendometer allow us to find the zeros of the function.

5. Applications of the Trendometer to Time Series

Since we can define the retrospective and prospective derivatives of tubes, and thus, of evolutions, we do not need to assume that the evolutions depend upon continuous time, but on discrete time: the evolutions become (discrete) time series, discussed in the "technical analysis" of economic and financial time series.

As an illustration, we use the tensor trendometer for detecting the dynamic correlations between (see [23]):

1. the two time series of prices and volumes of wheat
2. the 40 price series of the CAC 40 from 1990 to 2013

5.1 Differential Connection Matrix between Prices and Volumes

We describe the results obtained when we consider only two series for displaying meaningful figures.

The entry of the first row and the first column is the jerkiness of the trend reversal of the price; the first row and second column, the monotonicity jerkiness between price and volume; the second row and first column, the monotonicity jerkiness volume and price; and the second row and second column, the jerkiness of the trend reversal of the volume.

The selected series (Figure 5) are those of an asset price and daily volume, which is the number of units of commodities, or of shares traded of commodities, or securities exchanged during a daily session (not to be confused with the value of their transactions). The volume is believed to be an important activity indicator because it measures the interest of investors.

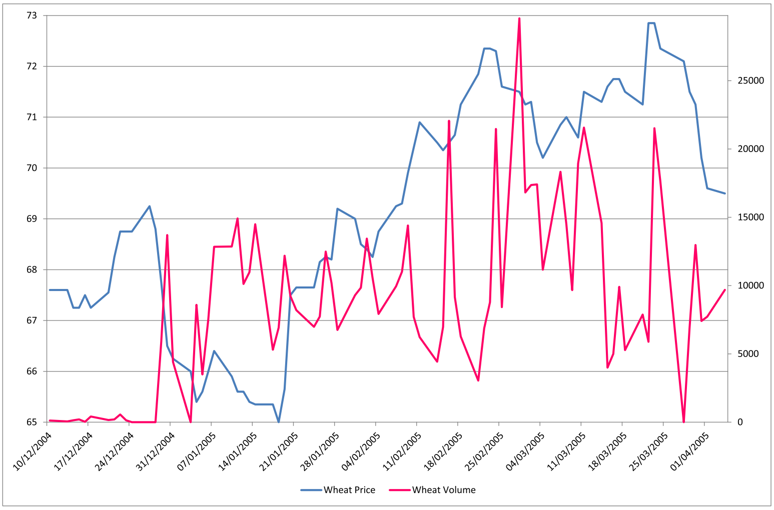

Figure 5. This figure displays the series of "settlement prices" of wheat and the volume of exchanges on the London Commodity Market from December 19, 2004 to April 4, 2005 around the date of January 10, 2005, when an important discontinuity of the volume happened (from 7534 to 12842 units). The number of dates is reduced for the visibility of this graphical representation of the series of differential connection tensors.

At each date, the connection tensor displays the jerkiness measures among and between the two series. For instance, on December 7, 2004, three weeks before the big discontinuity, all four coefficients of the differential connection tensor are different from zero:

$$\begin{pmatrix} 0,39 & 33 \\ 1,80 & 153 \end{pmatrix}. \tag{19}$$

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At the discontinuity date, a small decrease of prices was followed by a large increase in volume, as indicated by the differential connection:

$$\begin{pmatrix} 0,2 & 0 \\ 2654 & 0 \end{pmatrix}. \tag{20}$$

Figure 6 displays the dates at which at least the monotonicity of a series is followed by the reversal of itself and/or another series.

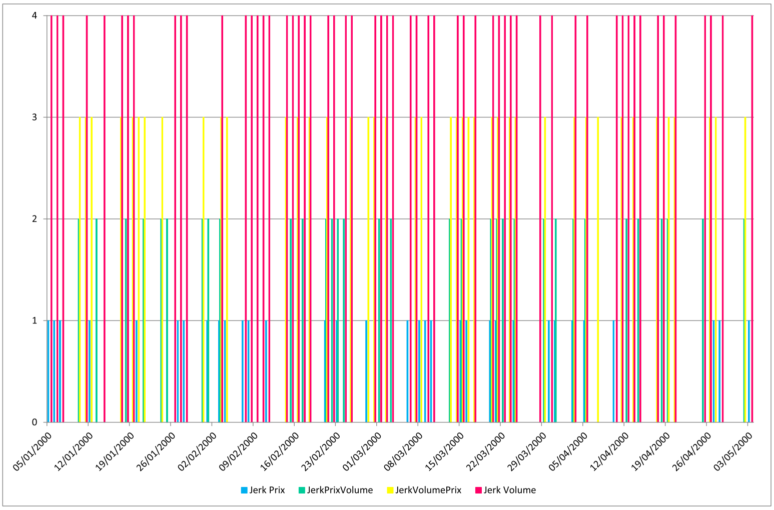

Figure 6. In order to represent the detection of the different entries of the differential connection tensor between the price and volume series at each date of the temporal window, we indicate by vertical bars between 0 and 1 the trend reversal dates of the price series and by vertical bars between 0 and 4 the trend reversal dates of the volume series, which occupy the diagonal of the differential connection tensor. The vertical bars between 0 and 2 detect the dates when the monotonicity behavior of the price precedes the monotonicity behavior of the volume, whereas vertical bars between 0 and 3 detect the dates when the monotonicity behavior of the volume is followed by the monotonicity behavior of the price.

A statistical study over the period from 05/01/2000 to 30/09/2013 shows the proportions between the following dates:

1. trend reversal dates of the price series: 26%
2. trend reversal dates between price and volume series: 24%
3. trend reversal dates between volume and price series: 22%
4. trend reversal dates of the price series: 28%

5.2 Explaining the 2000 and 2008 Financial Crisis with the Trendometer

In this section we analyze the 40 prices of the CAC 40 index and the trendometer providing dynamic correlations between these prices. For instance, on August 6, 2010, the prices are displayed in Figure 7.

Cours du CAC 40 *06/08/2012* *Qualitative Tensor Trendometer*

AC.PA	AI.PA	ALO.PA	MT.PA	CS.PA	BNP.PA	EN.PA	CAP.PA	CA.PA	ACA.PA
26,03	90,84	27,02	18,85	9,95	32,13	19,67	29,72	15,21	3,73
BN.PA	EAD.PA	EDF.PA	GTO.PA	EI.PA	GSZ.PA	LG.PA	LR.PA	OR.PA	MC.PA
48,22	30,38	15,78	64,50	68,82	17,02	37,30	26,26	99,88	126,16
ML.PA	ORA.PA	RI.PA	PUB.PA	PP.PA	RNO.PA	SGO.PA	SAN.PA	SU.PA	GLE.PA
55,58	9,97	86,10	40,35	123,02	35,42	24,14	21,58	49,64	18,66
UG.PA	STM.PA	TEC.PA	FP.PA	UL.PA	VK.PA	VIE.PA	DG.PA	VIV.PA	
6,37	4,12	85,45	35,15	79,32	35,75	7,74	32,24	14,59	

Figure 7.

At each date, it provides the 40×40 matrix displaying the qualitative jerkiness for each pair of series when the trend of the first one is followed by the opposite trend of the second one. Figure 8 provides the first 10×10 jerkiness entries, for the sake of readability.

	AC.PA	AI.PA	ALO.PA	MT.PA	CS.PA	BNP.PA	EN.PA	CAP.PA	CA.PA	ACA.PA
AC.PA	0,000	0,008	0,001	0,000	0,001	0,002	0,002	0,003	0,000	0,000
AI.PA	0,182	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,111	0,004
ALO.PA	0,023	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,014	0,001
MT.PA	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
CS.PA	0,038	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,023	0,001
BNP.PA	0,041	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,025	0,001
EN.PA	0,054	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,033	0,001
CAP.PA	0,000	0,060	0,004	0,000	0,011	0,016	0,012	0,024	0,000	0,000
CA.PA	0,045	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,028	0,001
ACA.PA	0,024	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,015	0,001

Figure 8.

The diagonal entries detect the trend reversal of each temporal series.

In order to analyze further the evolutionary behavior of the CAC 40, we present the analysis of the CAC 40 index only, but over the period from 03/01, 1990 to 09/25, 2013. Figure 9 displays the series of the CAC 40 indexes (closing prices). The vertical bars indicate the reversal dates, and their height displays their jerkiness.

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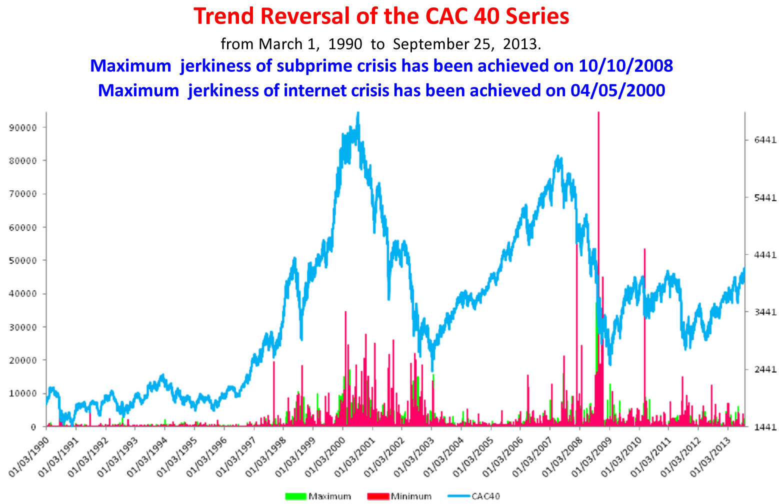
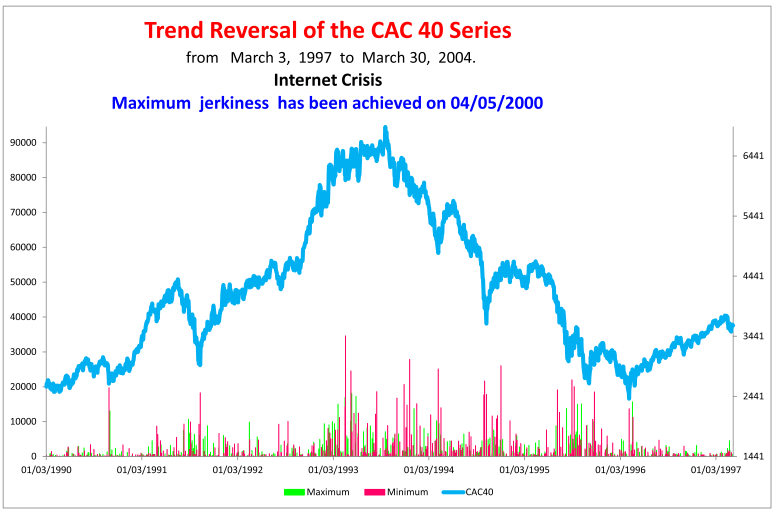
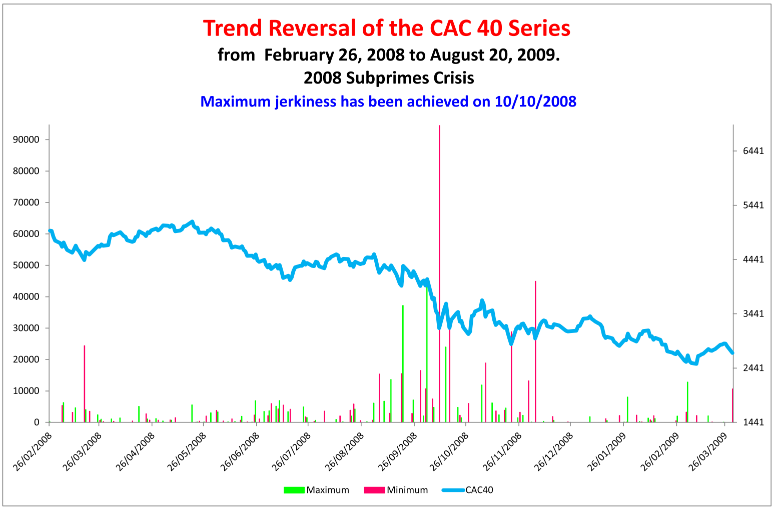


Figure 9.

The 2000 internet crisis (around May 4, 2000) and the 2008 "subprime" crisis (around October 10, 2008) are detected and measured by the trendometer (Figure 10).



(a)



(b)

Figure 10.

Figure 11 displays the velocities of the jerkiness between two consecutive trend reversal dates, a ratio involving the variation of the jerkiness and the duration of the congruence period (bull and bear).

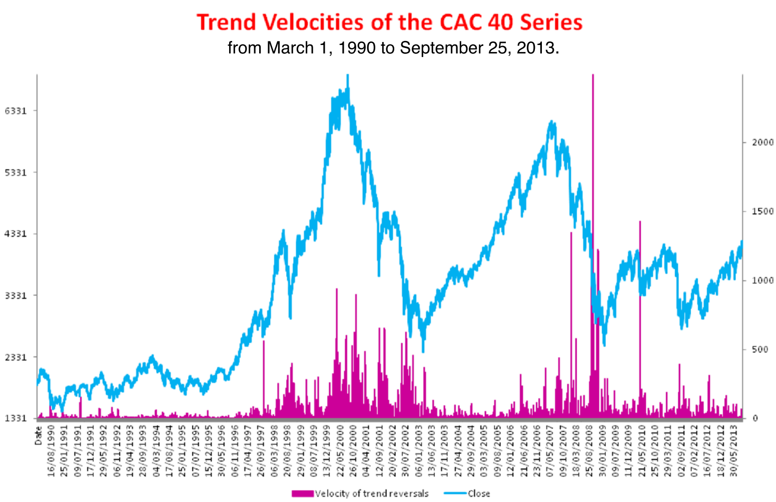
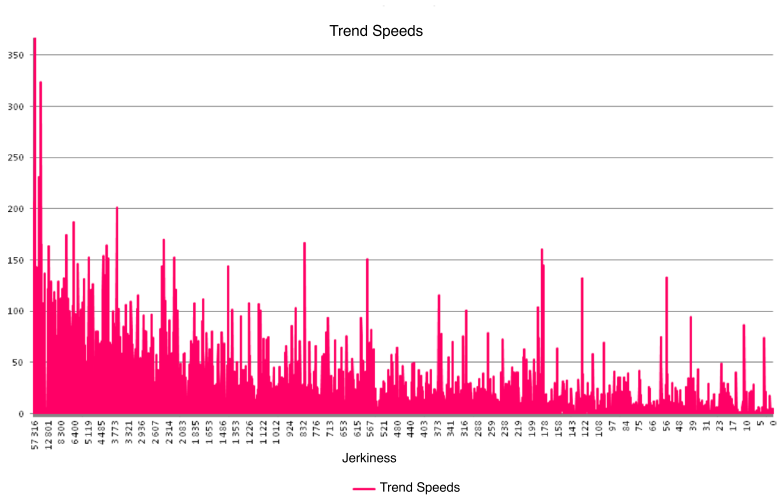
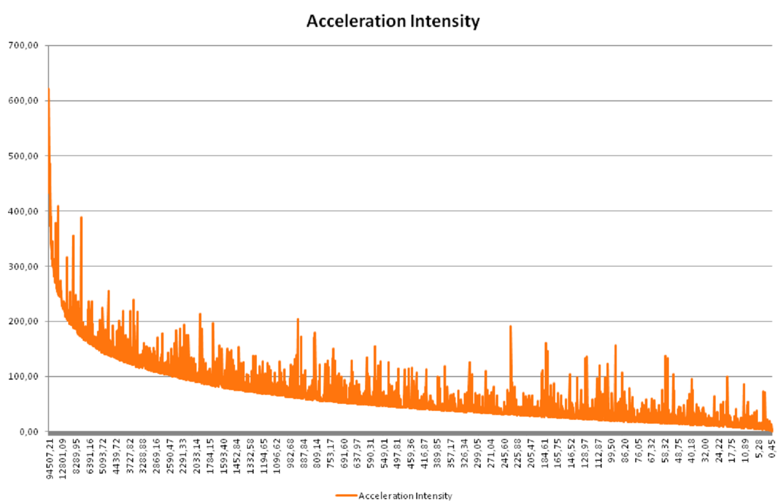


Figure 11.

Figure 12 displays the classification of trend speeds and absolute value of the accelerations by decreasing jerkiness.



(a)



(b)

Figure 12.

The analysis of this series shows that often the jerkiness at minima (bear periods) is higher than the ones at maxima (bull periods). For the CAC 40, the proportion of "bear jerkiness" (57%) is higher than "bull jerkiness" (43%). A possible explanation is a mimetic one: the fear of bear periods propagates and amplifies selling of shares, whereas investors may wait to regain confidence in bull phases.

Table 1 provides the first dates by decreasing jerkiness. The most violent are those of the subprime crisis (in bold), then the ones of the year 2006, and next, the dates of the internet crisis (in italics).

Remark 1. The trendometer has no predictive power, but is an attempt to analyze the past. For predictions, the VIMADES Extrapolator has been developed for forecasting temporal series. This extrapolator is a discrete version of historical differential inclusion (or functional, path-dependent, etc.), which depends on the history of the evolution and its derivatives, and not only on the past evolution (see [9–11] summarized in [12, 13]). For instance, the velocity at each instant depends also on the velocity, acceleration, and jerk of the past evolution for taking into account its trends. In the absence of a consensual criterion for comparing different forecasting mechanisms, there is no claim that this extrapolator provides better or worse results than other ones. It has been used in economics and finance whenever the future has to be taken into account (see [23, 27]).

6. Open Problems: Differential Connection Tensors of Maps

The core of this study was the study of connection tensors of evolutions, that is, maps from $\mathbb{R}$ to $\mathbb{R}^n$. But the concept of differential connection tensors can be extended to maps, single-valued or set-valued, from $\mathbb{R}^p$ to $\mathbb{R}^n$. They may be useful for investigating other problems using these maps.

6.1 Prospective and Retrospective Derivatives of Set-Valued Maps

We summarize the concept of graphical derivatives.

Definition 3. Consider a set-valued map $F : X \rightsquigarrow Y$ from a finite-dimensional vector space X to another one, Y. Let $(x, y) \in \mathrm{Graph}(F)$ be an element of its graph. We denote in this study by

1. *retrospective derivative* $\overleftarrow{D} F(x, y) : X \rightsquigarrow Y$ associating with any direction $u \in X$ the set of elements $v \in Y$ satisfying

$$\liminf_{h \mapsto 0+, u_h \mapsto u} d\left(v, \frac{y - F(x - h\, u_h)}{h}\right) = 0 \tag{21}$$

2. *prospective derivative* $\overrightarrow{D} F(x, y) : X \rightsquigarrow Y$ associating with any direction $u \in X$ the set of elements $v \in Y$ satisfying

$$\liminf_{h \mapsto 0+, u_h \mapsto u} d\left(v, \frac{F(x + h\, u_h) - y}{h}\right) = 0. \tag{22}$$

Date	Jerkiness	*Date*	Jerkiness	*Date*	Jerkiness
10/10/**2008**	94507,21	03/01/*2001*	15153,31	17/02/*2000*	10025,57
23/01/**2008**	57315,90	11/09/*2002*	15111,43	28/10/*2002*	9962,69
07/05/2010	53585,50	10/03/*2000*	15055,45	01/09/1998	9917,22
05/12/**2008**	44927,23	10/08/2011	15011,24	15/02/**2008**	9905,51
03/10/**2008**	43319,41	27/08/*2002*	14958,41	19/04/1999	9887,67
19/09/**2008**	37200,13	22/11/*2000*	14768,91	26/10/*2001*	9556,17
05/04/*2000*	34609,80	03/04/*2000*	14280,35	29/06/*2000*	9470,44
21/01/**2008**	34130,42	03/04/*2001*	14003,47	25/02/*2000*	9438,07
16/10/**2008**	29794,42	18/07/*2002*	13813,67	27/03/*2001*	9436,84
21/11/**2008**	28840,69	19/12/*2000*	13743,01	15/05/*2000*	9411,84
04/12/*2000*	27861,03	12/03/2003	13707,93	04/10/2011	9409,14
12/11/*2001*	26039,07	12/09/**2008**	13682,85	17/01/*2000*	9398,39
22/03/*2001*	25128,11	01/12/**2008**	13207,66	11/08/1998	9320,83
27/04/*2000*	24577,70	29/10/1997	13085,95	20/11/**2007**	9291,91
17/03/**2008**	24416,22	04/03/2009	12845,84	05/10/1998	9277,96
14/10/**2008**	24007,60	14/03/**2007**	12801,09	29/07/1999	9253,97
05/08/*2002*	22021,61	24/06/*2002*	12658,98	04/12/**2007**	9200,48
14/09/*2001*	21658,15	02/08/2012	12628,14	04/02/*2000*	9093,25
10/08/**2007**	21252,50	24/05/*2000*	12456,94	02/10/*2002*	8959,94
13/11/*2000*	20662,32	10/05/*2000*	12411,27	13/09/*2000*	8897,37
22/01/**2008**	20184,96	28/07/*2000*	12145,83	10/05/2010	8877,39
14/08/*2002*	20052,16	23/02/*2001*	11960,59	30/09/*2002*	8845,61
28/10/1997	19720,61	04/11/**2008**	11904,50	04/11/1998	8843,75
14/06/*2002*	19114,56	08/06/*2006*	11773,65	09/08/2011	8833,20
06/11/**2008**	18900,51	30/10/*2001*	11733,86	11/06/*2002*	8832,22
03/08/*2000*	18621,37	15/10/*2001*	11630,50	07/07/*2000*	8797,60
29/10/*2002*	18550,19	24/03/2003	11294,44	16/01/*2001*	8778,74
08/10/1998	18307,12	15/03/*2000*	11232,52	27/04/1998	8721,52
02/05/*2000*	18087,38	17/09/**2007**	10948,51	19/02/**2008**	8327,20
21/09/*2001*	17771,78	13/08/**2007**	10933,30	20/11/*2000*	8299,90
11/09/*2001*	17660,69	25/10/*2001*	10809,42	03/07/*2002*	8289,95
16/08/**2007**	17398,86	02/10/**2008**	10720,31	28/06/*2000*	8258,67
16/05/*2000*	17228,62	23/10/*2002*	10675,86	28/06/2010	8137,05
04/04/*2000*	16958,95	25/08/1998	10673,02	31/01/*2000*	8093,58
18/10/*2000*	16761,07	30/03/2009	10672,64	21/11/*2000*	8074,23
29/09/**2008**	16502,34	24/01/**2008**	10352,96	28/01/2009	8049,26
08/08/**2007**	16048,09	20/03/*2001*	10294,67	26/02/**2007**	8038,76
21/03/2003	15703,11	14/12/*2001*	10253,40	31/01/*2001*	8033,95
18/09/**2008**	15506,17	31/07/**2007**	10134,80	26/11/*2002*	7933,90
22/05/*2006*	15470,19	26/04/*2000*	10093,65	08/08/2011	7821,87
05/09/**2008**	15406,87	02/09/1999	10080,12	18/05/2010	7793,80

Table 1.

The retrospective and prospective difference quotients of F at $(x, y) \in \text{Graph}(F)$ are defined by

$$\overleftarrow{\nabla}_h F(x, y)\,(\overleftarrow{u}) := \frac{y - F\big(x - h\,\overleftarrow{u}\big)}{h}$$

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and

$$\overrightarrow{\nabla}_h F(x, y)(\overleftarrow{u}) := \frac{F\left(x + h\overleftarrow{u}\right) - y}{h}.$$

Whenever the set-valued map F is Lipschitz, the retrospective and prospective difference quotients are bounded, and thus, a relatively compact set, since the dimension of the vector spaces is finite. In this case, the derivatives are not empty.

We can reformulate the definition of the (contingent) derivative by saying that it is the *upper Painlevé–Kuratowski limit* of the difference quotients,

$$\forall\, \overleftarrow{u},\ \overleftarrow{D} F(x, y)(\overleftarrow{u}) = \mathrm{Limsup}_{h \mapsto 0+, u_h \to \overleftarrow{u}} \overleftarrow{\nabla}_h F(x, y)(u_h); \tag{23}$$

that is, the retrospective (resp. prospective) derivatives are the limits $\overleftarrow{v}$ of $\overleftarrow{v}_h \in \overrightarrow{\nabla}_h F(x, y)(u_h)$ (resp. of i.e., the limits of $\overrightarrow{v}_h \in \overrightarrow{\nabla}_h F(x, y)(u_h)$).

Taking the tensor product of both the retrospective and prospective derivatives allows us to define the differential connection matrix.

Definition 4. The differential connection tensor $\mathbf{a}_F(x, y)\left[\left(\overleftarrow{u}, \overrightarrow{u}\right), \left(\overleftarrow{v}, \overrightarrow{v}\right)\right]$ of retrospective and prospective derivatives of F at $(x, y) \in \mathrm{Graph}(F)$ is defined by

$$\begin{cases} \forall\, \left(\overleftarrow{u}, \overrightarrow{u}\right),\ \overleftarrow{v} \in \overleftarrow{D} F(x, y)(\overleftarrow{u}),\ \overrightarrow{v} \in \overrightarrow{D} F(x, y)(\overrightarrow{u}), \\ \mathbf{a}_F(x, y)\left[\left(\overleftarrow{u}, \overrightarrow{u}\right), \left(\overleftarrow{v}, \overrightarrow{v}\right)\right] := \overleftarrow{v} \otimes \overrightarrow{v}. \end{cases} \tag{24}$$

Remark 2. A normalized version of the differential connectionist tensor is defined by

$$\begin{cases} \forall\, \left(\overleftarrow{u}, \overrightarrow{u}\right),\ \overleftarrow{v} \in \overleftarrow{D} F(x, y)(\overleftarrow{u}),\ \overrightarrow{v} \in \overrightarrow{D} F(x, y)(\overrightarrow{u}), \\ \mathbf{a}_F(x, y)\left[\left(\overleftarrow{u}, \overrightarrow{u}\right), \left(\overleftarrow{v}, \overrightarrow{v}\right)\right] := \dfrac{\overleftarrow{v} \otimes \overrightarrow{v}}{\|\overleftarrow{v}\|\, \|\overrightarrow{v}\|}. \end{cases} \tag{25}$$

The normalized version is not that useful whenever we are interested in the signs of the entries of the connection matrix.

Remark 3. The prospective difference quotient $\overrightarrow{\nabla}_h F(x, y)(\overrightarrow{u})$ and retrospective difference quotient $\overleftarrow{\nabla}_h F(x, y)(\overrightarrow{u})$ define their second-order difference quotients

$$\nabla^2 F(x, y)\left(\overleftarrow{u}, \overrightarrow{u}\right) := \frac{\overrightarrow{\nabla}_h F(x, y)\,(\overrightarrow{u}) - \overleftarrow{\nabla}_h F(x, y)\,(\overrightarrow{u})}{h} = \frac{F\left(x + h\,\overrightarrow{u}\right) + F\left(x - h\,\overleftarrow{u}\right) - 2\,y}{h^2}. \tag{26}$$

The Painlevé–Kuratowski upper limit of $\nabla^2 F(x, y)\left(\overleftarrow{u}, \overrightarrow{u}\right)$ defines the retrospective-prospective second-order graphical derivative of F at $(x, y) \in \mathrm{Graph}(F)$ by:

$$D^2 F(x, y)\left(\overleftarrow{u}, \overrightarrow{u}\right) := \mathrm{Limsup}_{h\mapsto 0+, \overleftarrow{u}_h \to \overleftarrow{u}, \overrightarrow{u}_h \to \overrightarrow{u}} \nabla^2 F(x, y)\left(\overleftarrow{u}_h, \overrightarrow{u}_h\right). \tag{27}$$

The differential connectionist tensor replaces the difference between the retrospective and prospective derivatives by their tensor products. We refer to [25, Section 5.6, p. 315] for other approaches of higher-order graphical derivatives to set-valued maps.

Remark 4. In 1884, Giuseppe Peano proved in [28] that continuous derivatives are the limits

$$\lim_{h\to 0} \frac{x(t+h) - x(t-h)}{2\,h} = \frac{1}{2} \left(\lim_{h\to 0+} \frac{x(t) - x(t-h)}{h} + \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} \right)$$

of both the retrospective and prospective average velocities (difference quotients) at time t. We follow his suggestion by taking the average of the prospective difference quotient $\overrightarrow{\nabla}_h F(x, y)\,(\overrightarrow{u})$ and retrospective difference quotient $\overleftarrow{\nabla}_h F(x, y)\,(\overrightarrow{u})$

$$\frac{\overrightarrow{\nabla}_{2\,h} F(x, y)\,(\overrightarrow{u}) + \overleftarrow{\nabla}_h F(x, y)\,(\overrightarrow{u})}{2\,h}, \tag{28}$$

and taking their Painlevé–Kuratowski limits

$$\mathrm{Limsup}_{h\mapsto 0+, \overrightarrow{u}_h \to \overrightarrow{u}} \overrightarrow{\nabla}_h F(x, y)\,(\overrightarrow{u}_h) + \mathrm{Limsup}_{h\mapsto 0+, \overleftarrow{u}_h \to \overleftarrow{u}} \overleftarrow{\nabla}_h F(x, y)\,(\overleftarrow{u}_h) \tag{29}$$

in order to define *Peano graphical derivatives* of F at $(x, y) \in \mathrm{Graph}(F)$ depending on *pairs* $\left(\overleftarrow{u}, \overrightarrow{u}\right)$ *of directions*.

6.2 Differential Connections Tensors of Numerical Functions

When $V : x \in X \mapsto V(x) \in \{-\infty\} \bigcup \mathbb{R} \bigcup \{+\infty\}$ is an extended numerical function on $\mathbb{R}$, it can also be regarded as a set-valued map (again denoted by) $V : X \rightsquigarrow \mathbb{R}$ defined by

$$V(x) := \begin{cases} \{V(x)\} & \text{if} \quad V(x) \in \mathbb{R} \text{ (i.e., } x \in \mathrm{Dom}(V)) \\ \emptyset & \text{if not.} \end{cases} \tag{30}$$

A slight modification of [25, Theorem 6.1.6, p. 230] states that

$$\begin{cases} \overrightarrow{D} V(x)(\vec{u}) = \left[\overrightarrow{D}_{\uparrow} V(x)(\vec{u}), \overrightarrow{D}_{\downarrow} V(x)(\vec{u})\right] \\ \overleftarrow{D} V(x)(\overleftarrow{u}) = \left[\overleftarrow{D}_{\uparrow} V(x)(\overleftarrow{u}), \overleftarrow{D}_{\downarrow} V(x)(\overleftarrow{u})\right] \end{cases} \tag{31}$$

where

$$\begin{cases} \overrightarrow{D}_{\uparrow} V(x)(\vec{u}) := \\ \quad \liminf\limits_{h \to 0+} \dfrac{V\left(x + h\vec{u}\right) - V(x)}{h} \text{ (epiderivative of } V) \\ \overrightarrow{D}_{\downarrow} V(x)(\vec{u}) := \\ \quad \limsup\limits_{h \to 0+} \dfrac{V\left(x + h\vec{u}\right) - V(x)}{h} \text{ (hypoderivative of } V) \\ \overleftarrow{D}_{\uparrow} V(x)(\overleftarrow{u}) := \liminf\limits_{h \to 0+} \dfrac{V(x) - V\left(x - h\overleftarrow{u}\right)}{h} = -\overrightarrow{D}_{\downarrow} V(x)\left(-\overleftarrow{u}\right) \\ \overleftarrow{D}_{\downarrow} V(x)(\overleftarrow{u}) := \limsup\limits_{h \to 0+} \dfrac{V(x) - V\left(x - h\overleftarrow{u}\right)}{h} = -\overrightarrow{D}_{\uparrow} V(x)\left(-\overleftarrow{u}\right). \end{cases} \tag{32}$$

Definition 4 implies that

$$\begin{cases} \forall \left(\overleftarrow{u}, \vec{u}\right), \ \overleftarrow{v} \in \overleftarrow{D} V(x)(\overleftarrow{u}), \ \vec{v} \in \overrightarrow{D} V(x)(\vec{u}), \\ \mathbf{a}_V(x, y)\left[\left(\overleftarrow{u}, \vec{u}\right), \left(\overleftarrow{v}, \vec{v}\right)\right] := \overleftarrow{v}\,\vec{v} \end{cases} \tag{33}$$

since tensor products of real numbers boil down to their multiplication.

Therefore, for any pair $\left(\overleftarrow{u}, \vec{u}\right)$, the subset of differential connection tensors of retrospective and prospective directions is equal to

$$\begin{cases} \overleftarrow{D} V(x)(\overleftarrow{u}) \otimes \overrightarrow{D} V(x)(\vec{u}) := \\ \left\{\overleftarrow{v}\,\vec{v}\right\}_{\left(\overleftarrow{v}, \vec{v}\right) \in \left[\overleftarrow{D}_{\uparrow} V(x)(\overleftarrow{u}), \overleftarrow{D}_{\downarrow} V(x)(\overleftarrow{u})\right] \times \left[\overrightarrow{D}_{\uparrow} V(x)(\vec{u}), \overrightarrow{D}_{\downarrow} V(x)(\vec{u})\right]}. \end{cases} \tag{34}$$

Definition 5. A pair $\left(\overleftarrow{u}, \overrightarrow{u}\right)$ of directions $\overleftarrow{u} \in X$ and $\overrightarrow{u} \in X$ is a reversal direction pair of V at $x \in \mathrm{Dom}(V)$ if

$$\overleftarrow{D}_{\uparrow}(x)\,(\overleftarrow{u})\,\overrightarrow{D}_{\uparrow}(x)\,(\overrightarrow{u}) = \overleftarrow{D}_{\downarrow}(x)\left(-\overrightarrow{u}\right)\overrightarrow{D}_{\downarrow}\,V(x)\left(-\overleftarrow{u}\right) < 0. \tag{35}$$

A direction $u \in X$ is a reversal direction of V at x if the diagonal pair (u, u) is a reversal direction pair.

This means that a positive (resp. negative) retrospective epiderivative of V at x in the direction $\overleftarrow{u}$ is followed by a negative (resp. positive) prospective epiderivative in the direction $\overrightarrow{u}$, or respectively, that a positive (resp. negative) retrospective hypoderivative in the direction $-\overrightarrow{u}$ is followed by a negative (resp. positive) prospective hypoderivative in the direction $-\overleftarrow{u}$.

Recall that if V achieves a local minimum at x, the Fermat rule states that

$$\forall\,\overrightarrow{u} \in X,\ \overrightarrow{D}_{\uparrow}\,V(x)\,(\overrightarrow{u}) \geq 0 \text{ and } \forall\,\overleftarrow{u} \in X,\ \overleftarrow{D}_{\downarrow}\,V(x)\,(\overleftarrow{u}) \leq 0, \tag{36}$$

and if it achieves a local maximum at x, that

$$\forall\,\overrightarrow{u} \in X,\ \overrightarrow{D}_{\downarrow}\,V(x)\,(\overrightarrow{u}) \leq 0 \text{ and } \forall\,\overleftarrow{u} \in X,\ \overleftarrow{D}_{\uparrow}\,V(x)\,(\overleftarrow{u}) \geq 0. \tag{37}$$

These conditions are not sufficient for characterizing local extrema: convexity or many second-order conditions provide sufficient conditions.

Recall that the prospective epidifferential (or prospective epidifferential subdifferential) $\overrightarrow{\partial}_{\uparrow}\,V(x)$ of a function V at x is the set of elements $\overrightarrow{p}_{\uparrow} \in X^{\star}$ such that for any $v \in X$, $\left\langle\overrightarrow{p}_{\uparrow}, v\right\rangle \leq \overrightarrow{D}_{\uparrow}\,V(x)\,(v)$. In the same way, we define the retrospective epidifferential (or retrospective epidifferential subdifferential) $\overleftarrow{\partial}_{\uparrow}\,V(x)$ of a function V at x as the set of elements $\overleftarrow{p}_{\uparrow} \in X^{\star}$ such that for any $v \in X$, $\left\langle\overleftarrow{p}_{\uparrow}, v\right\rangle \leq \overleftarrow{D}_{\uparrow}\,V(x)\,(v)$. It is equal to prospective hypodifferential (or prospective superdifferential) $\overrightarrow{\partial}_{\downarrow}\,V(x)$, the set of elements $\overrightarrow{p}_{\downarrow} \in X^{\star}$ such that for any $v \in X$, $\left\langle\overrightarrow{p}_{\downarrow}, v\right\rangle \geq \overrightarrow{D}_{\downarrow}\,V(x)\,(v)$.

6.3 Tangential Connection Tensors

The tangent spaces to differentiable manifolds being vector spaces, directions arriving at a point (we may call them *retrospective*) and directions starting from this point (*prospective*) belong to the same vector space. This is no longer the case when the subset is any (closed) subset

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$K \subset X$ of a finite-dimensional vector space X. However, we may replace vector spaces by cones.

We are indebted to the historical studies [29] (in which the authors quote Maurice Fréchet stating that "Cette théorie des "contingents et paratingents" dont l'utilité a été signalée d'abord par M. Beppo Levi, puis par M. Severi, mais dont on doit à M. Bouligand et ses élèves d'en avoir entrepris l'étude systématique.") and [30]. Since Francesco Severi and Georges Bouligand, a whole menagerie of tangent cones, the definitions of which depend upon the limiting process, have been proposed (among many monographs, see [25] and [31], for instance). At some points, the tangent cones are not vector spaces, and the opposite of some tangent directions may no longer be tangent.

We suggest regarding the (contingent) tangent cone as the *prospective tangent cone* to K at $x \in K$ defined by the Painlevé–Kuratowski upper limits

$$\begin{aligned} &\overrightarrow{T}_K(x) := \operatorname{Limsup}_{h \mapsto 0+} \frac{K - x}{h} := \\ &\left\{ \overrightarrow{v} \in X \text{ such that } \liminf_{h \mapsto 0+} \frac{d_K\left(x + h\,\overrightarrow{v}\right)}{h} = 0 \right\}, \end{aligned} \tag{38}$$

with which we associate the *retrospective tangent cone* (backward evolutions and negative tangents have been introduced in [32, 33] for characterizing lower semicontinuous (viscosity) solutions to Hamilton–Jacobi–Bellman equations)

$$\begin{aligned} &\overleftarrow{T}_K(x) := \operatorname{Limsup}_{h \mapsto 0+} \frac{x - K}{h} := \\ &\left\{ \overleftarrow{v} \in X \text{ such that } \liminf_{h \mapsto 0+} \frac{d_K\left(x - h\,\overleftarrow{v}\right)}{h} = 0 \right\} \end{aligned} \tag{39}$$

satisfying $\overleftarrow{T}_K(x) := -\overleftarrow{T}_K(x)$. It is natural to consider their tensor product $\left(x - h\,\overleftarrow{v}\right) \otimes \left(x + h\,\overrightarrow{v}\right)$. The signs of its entries detect the "blunt" and "sharp" elements of the boundary in the same directions (*trend congruence*) or in opposite directions (*trend reversal*).

6.4 Toward Cellular Automata

Can connections between differential inclusions and cellular automata be made?

Rules in cellular automata are indeed discrete dynamical systems, specifying the successor of any given state, and its inverse provides the states of its preceding states (it may be set-valued). The (topological) vector space structure, as well as the "mutational structure" of metric spaces (see for instance [34, 35]), allows not only defining successors of a state (flows or semi-groups), but also comparing the state and its successor through the concept of (average) velocity. Hence, differential equations provide semi-groups; the converse being true under severe assumptions (linear or monotone infinitesimal generators, which can be extended to mutational spaces).

Since the concept of velocity is missing in cellular automata (to our knowledge), it is difficult to "transfer" the ideas of this study that use retrospective and prospective velocities for defining retrospective-prospective differential inclusions: they provide systems with history $x_{n+1} = f(x_n, x_{n-1})$. The concept of differential connection tensors requires the vector space structure and tensorial algebra.

The question boils down to the definition of velocities in cellular automata, that is, a way to compare and measure pairs (x_{n-1}, x_n) of two successive states of an evolution independently of any rule (velocities do that) and to define order relations on "velocity" pairs for defining monotonic behavior and trend reversals.

Can we define velocities in cellular automata? As far as the authors know, this question remains open.

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